NUMBER OF PRIMES OF GIVEN LINEAR FORMS

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1. *Introduction.*—The object of this paper is to give the results of actually counting the numbers of primes of certain linear forms, and to compare the numbers counted with a certain formula for the same.

Let N = total number of primes (p) within a given "range" (say R), of the natural numbers.

Let p denote a prime of the linear form $p = n\varpi + a$ $(a < n)$: n is hereinafter styled the "modulus" of the prime-form.

Let M_a = the number of primes (p) of that form within that range (R).

Let $\phi(n)$ denote the totient of n.

2. *Approximate Formula for* M_a.—Considering the form of p with respect to the "modulus" n, it is seen that a must be prime to n (in order that p may be *prime*): so that the set of values possible for a is the whole set of integers prime to n, and $< n$; and the number of such values (of a) is therefore $= \phi(n)$, the totient of n.

Hence, unless there be some reason for certain of the $\phi(n)$ values of a yielding more primes than other values of a yield, it would follow that the numbers (M_a) of primes of the forms

$$p = n\varpi + a_1, \quad n\varpi + a_2, \quad \ldots, \quad n\varpi + a_r \quad (a \text{ prime to } n, \text{ and } < n)$$

in any one *large* range of the natural numbers should be nearly equal; so that

$$M = \frac{1}{\phi(n)} N \text{ approximately (for each value of } a). \qquad (1)$$

This rule has now been tested by the author by actually counting*

* The labour of such counts is very great: the author had, however, a number of printed and MS. tables ready to hand, specially suited to this work. The risk of error (especially of missing one or two primes in counting) is considerable; all the counts have been done by two independent counters.

the numbers (M) of primes p of a considerable number of forms ($n\varpi+a$) within various ranges.

3. *Form* $p = n\varpi+1$. *Count of* M_1.—The particular form

$$p = (n\varpi+1)$$

(in which $a = 1$, always) seems the most interesting on account of its connexion with Fermat's theorem for bases a (prime to p), viz., whether

$$a^{(p-1)\div n} \equiv \mp 1 \text{ or } \not\equiv \mp 1 \pmod p, \quad [p = n\varpi+1]. \tag{2}$$

The main Table (on p. 251) shows the number (M) of primes of form $p = (n\varpi+1)$ actually connected with certain ranges ($R = 1$ to 10^4, 10^5, or 5.10^5) for a number (61) of values of the modulus (n). The short abstract below shows the total* number (N) of primes in each range (R), and the number of cases (*i.e.*, of different values of n) available for discussion in each range.

Values of n.	Cases.	Total number (N) of primes in Range (R)		
		$R = 1$ to 10^4	$R = 1$ to 10^5	$R = 1$ to 5.10^5
All *even* numbers, 2 to 60	30	$N = 1228$	$N = 9591$	
Certain *even* numbers, 64 to 210	15	$N = 1228$	$N = 9591$	
Numbers $n = 8q$ (q prime = 101 to 241)	16		$N = 9591$	$N = 41537$

The Table (on p. 251) shows also the totient $\phi(n)$ of each value of n, and the (computed) value of $N\div\phi(n)$, for comparison with the (counted) M.

An examination of the table shows at once that the formula (1) really is a *very close approximation* to the counted number (M), but also discloses the remarkable result that—

The counted number (M) of primes of form $p = (n\varpi+1)$ is (almost always) *less than*, and in many cases *markedly less than*, the computed average, $N\div\phi(n)$. (3)

The exceptions to this rule, *i.e.*, the number of cases of $M > N\div\phi(n)$, are very few (see detail in the abstract below): and the excess of M over $N\div\phi(n)$ is, in all those cases, *trifling* compared with the relatively large deficiency in many of the other cases.

* The primes 1 and 2 are excluded—throughout this paper—from the totals denoted by N.

Range.	Values of n yielding $M > N \div \phi(n)$.	Cases.
1 to 10^4	22, 28, 34, 44, 50, 54	6 in 45
1 to 10^5	34, 50, 54, 58, 100, 130, 720, 768	8 in 61
1 to 5.10^5	432, 552, 720, 768	4 in 16

This renders it probable that the deficiency of the number (M) of primes of the forms $p = (n\varpi + 1)$ below the average number $N \div \phi(n)$ above disclosed really is a property of primes of that form, when taken through a *large range* of the natural numbers. The cause of this property has yet to be sought.

p																
	$n =$	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
	$\phi(n) =$	1	2	2	4	4	4	6	8	6	8	10	8	12	12	8
$\ngtr 10^4$	$N \div \phi =$	1228	614	614	307	307	307	204	153	204	153	123	153	102	102	153
	$M =$	1228	609	611	295	306	300	203	144	203	152	126	143	99	103	152
$\ngtr 10^5$	$N \div \phi =$	9591	4795	4795	2398	2398	2398	1598	1199	1598	1199	959	1199	799	799	1199
	$M =$	9591	4783	4784	2384	2387	2374	1593	1188	1592	1181	945	1181	798	787	1189

p																
	$n =$	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60
	$\phi(n) =$	16	16	12	18	16	12	20	22	16	20	24	18	24	28	16
$\ngtr 10^4$	$N \div \phi =$	77	77	102	68	77	102	61	56	77	61	51	68	51	44	77
	$M =$	73	78	101	64	71	98	63	55	69	63	50	70	50	43	76
$\ngtr 10^5$	$N \div \phi =$	600	599	799	533	599	799	480	436	599	480	399	533	399	342	599
	$M =$	599	603	797	525	582	787	466	429	581	486	392	535	385	345	585

p																
	$n =$	64	70	72	80	88	90	96	100	104	110	120	130	140	150	210
	$\phi(n) =$	32	24	24	32	40	24	32	40	48	40	32	48	48	40	48
$\ngtr 10^4$	$N \div \phi =$	38	51	51	38	30	51	38	31	26	31	38	25	25	30	26
	$M =$	38	48	47	35	29	50	35	30	22	29	35	25	24	29	23
$\ngtr 10^5$	$N \div \phi =$	300	399	399	299	240	399	299	240	200	300	299	200	200	300	200
	$M =$	300	392	397	293	228	388	292	243	194	231	286	202	183	235	192

p																		
	$n =$	2	8.101	8.103	8.107	8.109	8.137	8.139	8.149	8.157	8.167	8.173	8.181	8.193	8.199	8.227	8.229	8.241
	$\phi(n) =$	1	400	408	424	432	544	552	592	624	664	688	720	768	792	904	912	960
$\ngtr 10^5$	$N \div \phi =$	9591	24	24	23	22	18	17	16	15	15	14	13	13	12	11	10	10
	$M =$	9591	21	18	18	16	17	17	16	11	15	14	15	14	6	11	8	10
$\ngtr 5.10^5$	$N \div \phi =$	41537	104	102	98	96	76	75	70	67	63	60	58	54	57	46	46	43
	$M =$	41537	104	99	96	101	75	77	69	64	63	60	61	58	48	44	43	41

4. *Form* $p = n\varpi + a$. *Count of* M_a.—The above result carries with it the property that the number (M_a) of primes of the form $p = (n\varpi + a)$ (with $a > 1$) must *exceed the average number* of the formula for some

values of a (> 1); and that, in particular, some M_a must be $> M_1$; all the M_a being, of course, within the same (large) range (R).

To test this further property, the numbers (M_a) of primes of each of the following forms

$$p = 4\varpi \pm 1;\ 6\varpi \pm 1;\ 8\varpi \pm 1, 3;\ 10\varpi \pm 1, 3;\ 12\varpi \pm 1, 5$$

have now been counted within the same range $R = 1$ to 10^5.

The table below—drawn up very similarly to the preceding—shows under each modulus ($n = 4, 6, 8, 10, 12$) the totient $\phi(n)$ of n, the computed average number $N \div \phi(n)$ of primes of each form $p = (n\varpi + a)$, and lastly the actual counted number (M) of primes of that form.

$n =$	4		6		8				10				12			
$\phi(n) =$	2		2		4				4				4			
N* $=$	9591		9590		9591				9590				9590			
$N \div \phi(n) =$	4795		4795		2398				2398				2398			
$p =$	$4\varpi+1$	$4\varpi+3$	$6\varpi+1$	$6\varpi+5$	$8\varpi+1$	$8\varpi+3$	$8\varpi+5$	$8\varpi+7$	$10\varpi+1$	$10\varpi+3$	$10\varpi+7$	$10\varpi+9$	$12\varpi+1$	$12\varpi+5$	$12\varpi+7$	$12\varpi+11$
$M_a =$	4783	4808	4784	4806	2384	2409	2399	2399	2387	2402	2411	2391	2374	2409	2410	2397

An examination of this table shows the following somewhat remarkable relations between the numbers (M_a) of primes of the $\phi(n)$ forms $p = (n\varpi + a)$ *with the same modulus* (n), all taken, of course, through the same range R ($= 1$ to 10^5 in this case):—

The numbers M_a are *approximately equal*, so that formula (1) is a *good approximation*. (4)

The number M_1 is the *least of all* the M_a. (5)

The number M_{n-1} is the *next least* of all the M_a. (6)

[It should be stated that the particular result for the modulus $n = 4$, viz., that the number M_1 is $< M_3$, has been *proved generally true* by† Tchebycheff.]

The two results (5), (6) together involve the following:—

The number of primes (within a given large range), of which 2, 3, 5 are 2-ic residues, is *markedly less*‡ *than* the number of which these bases are 2-ic non-residues. (7)

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* The primes 1 and 2 are excluded from all these totals (N): the prime 3 is excluded when $n = 6$ and 12, and 5 is excluded when $n = 10$.

† In a letter to M. Fuss, published in the *Bull. de l'Acad. de St. Pétersbourg*, 1853; quoted in Glaisher's *Factor Table for the Fourth Million*, London, 1879, Introduction. p. 33.

‡ This property as to 2-ic residuacity is believed to be true for other bases; and a similar property is believed to be true for higher orders of residuacity. It is hoped to make these properties the subject of a further communication: the data are well in hand.

5. *Author's Previous Work.*—A certain number of the counts (M_a) now reported had been previously published by the author on p. xx of the Introduction to his *Tables of Quadratic Partitions*, London, 1904; viz.,

8 cases of M_1 (*i.e.*, for $p = n\varpi+1$), viz., for $n = 4, 6, 8, 10, 12, 16, 24, 30$.

2 cases of M_{n-1}, viz., for $p = (4\varpi-1)$ and $(6\varpi-1)$.

Slight errors in those counts have been found, viz.,

Under $p =$	$(6\varpi+1)$;	$(6\varpi-1)$;	$(12\varpi+1)$;	$24\varpi+1$;
Read	611 (not 612);	4806 (not 4807);	2374 (not 2373);	1181 (not 1180).