In 1869 we had "The Practical Application of Reciprocal Figures to the Calculation of Strains on Framework," in which he exemplified in a very clear manner the mode of applying to important *statical* questions a beautiful principle, due in part to Rankine but mainly to Clerk-Maxwell.

The paper for which the award of the Keith Prize is now made is more thoroughly original, and may be roughly described as an extension of Maxwell's principle to the *kinetics* of machinery, where all parts move in one plane. It is entitled "The Application of Graphic Methods to the Determination of the Efficiency of Machinery." The first part was read to the Society in 1877, and the second in the following year. All three of these papers are in our Transactions.

Among his other contributions may be mentioned his application (in conjunction with Professor Ewing) of the Phonograph records to the "Harmonic Analysis of certain Vowel Sounds." This is an ingenious and elaborate piece of work, and shows us (among other things) within what wide limits the components of a sound may vary while it is still recognised by the ear as having a definite vowel quality.

Professor Jenkin, in handing you this medal I express, I am sure, the feelings of all the Fellows of the Society, when I say that we thank you heartily for the valuable contributions you have already sent to our Transactions, and that we look with confidence for an additional series.

Professor Jenkin then took the Chair.

The following communications were read:—

1. Non-Euclidean Geometry. By Professor Chrystal. (Plate XX.)

When I had the honour of being asked by the Council of the Royal Society to give the following address, I chose the subject partly because it had been brought under the notice of the fellows by my predecessor, Professor Kelland. His memoir was written comparatively early in the history of the subject; and he seems to have been but little acquainted with what others had done even up to the time at which he wrote. Accordingly, although the subject is treated very ably in his paper, it is treated from only one point of view; and, indeed, one side of it is left out of sight altogether. The relation of the whole theory to the question of the origin and mutual independence of the axioms of geometry has been made much clearer of late, and I believed that some account of the more modern views might be of interest.

I am particularly desirous of bringing *pangeometrical* speculations under the notice of those engaged in the teaching of geometry. In discussing with schoolmasters the difficult problem of the reform of geometrical teaching, I have met with much enlightened and some unenlightened criticism. The former kind of criticism has convinced me that many teachers of mathematics will be glad to have this subject made more accessible; and I believe that a knowledge of what great mathematicians have thought on the subject would destroy criticism of the latter kind altogether.

It will not be supposed that I advocate the introduction of pangeometry as a school subject; it is for the teacher that I advocate such a study. It is a great mistake to suppose that it is sufficient for the teacher of an elementary subject to be just ahead of his pupils. No one can be a good elementary teacher who cannot handle his subject with the grasp of a master. Geometrical insight and wealth of geometrical ideas, either natural or acquired, are essential to a good teacher of geometry; and I know of no better way of cultivating them than by studying pangeometry.

The following sketch is addressed to those already familiar with Euclid's geometry. I have made no attempt to give a detailed account of modern researches, or to build up a systematic treatise. I have simply tried to give in a synthetic way a general idea of what is known in a certain department of a now very widely developed subject. In so doing I have used the materials and methods of Euclid as much as I consistently could, at some sacrifice of elegance, no doubt, but with obvious practical advantage.

I have not attempted to give any bibliographical details, for the simple reason that any one who wants them will find nearly all that can be desired in two papers by Mr Halsted in the first volume of the "American Journal of Mathematics."

On Pangeometry.

I know of no question possessing more interest for a thinker, and none of more importance for a mathematician, than the well-worn one of the origin of the axioms of geometry.

Passing over the discussions of mental philosophers, which, so far as I am acquainted with them, are of little mathematical or physical interest, we find two great modern contributions to this interesting subject; one by the mathematicians headed by Gauss, Lobatschewsky, Bolyai, and Riemann; the other by the physiologists represented by Helmholtz.

The mathematical investigators may be taken as representing the subjective side of the subject, the physiologists as representing the objective; although, in point of fact, Helmholtz, the personal representative of the latter, is a happy union of both classes of philosopher.

Any purely abstract science starts with certain data called definitions and axioms;* and of these materials reason builds the fabric of the science.

I do not intend to take up the question of the origin of axioms directly. On the contrary, I shall lay down axioms, and the only argument against me, so far, will be to prove the inconsistency of my conclusions with my premises, or with one another.

The absence of such inconsistency is what I mean by conceivability. I do not deny that other meanings may be attached to this word, and that the question of the conceivability of axioms might be profitably discussed from other points of view. We might discuss it as a purely personal question, each man to be judge and jury, or it might be granted, as I, for the most part in what follows, take it to be, that any axioms that can be made the foundation of a consistent reasoned system are given *à priori*. I suspect that this would be

* In Euclid's Geometry the functions of definition and axiom are not always clearly separated; at all events, some of his definitions serve purposes for which others are unfit, and this must be kept in view in what follows. With postulates I have at present nothing to do, as I am concerned solely with geometrical theorems. The *mixture* of problems with theorems is a peculiarity of Euclid's method for which there is no absolute necessity, and which is certainly inconvenient in an elementary text-book. Geometrical constructions are in a sense the applications of geometrical theory, and ought to be kept by themselves. The Society for the Improvement of Geometrical Knowledge have acted wisely, I think, in following this arrangement in their syllabus.

allowed by most of those who have considered the question of axioms in what I believe to be by far the most useful and effective way, viz., by examining and pushing the conclusions to be drawn from them to the utmost; and by investigating what change on these conclusions would be induced by varying one or more of the axioms themselves.

The question might also be approached from the side of experience. I take, for the sake of illustration, an instance which brings me at once to my subject. We have, by generalisation from experience, ideas more or less refined according to our individual physical education of a geometrical straight line, and of a geometrical point. Let us think, then, of two straight lines intersecting at a point, and let us ask ourselves, Can two such lines intersect again? Our first impulse is to answer no; but due consideration will show us that, in point of fact, experience does not settle the question. All we can say is that no one starting from the point of intersection of two straight lines has ever followed them by physical (say optical) observation to a second intersection. But then we must admit that, on our usual assumption that space is of infinite extent, and straight lines of infinite length, the distance through which any one has so followed them is, after all, relatively speaking, but an infinitely little way. Our assertion, therefore, that two straight lines never intersect again is merely an assumption, accordant, no doubt, with our limited experience, but otherwise unfounded, and certainly not of necessity involved in our idea of straightness, though we may superadd it thereto if we please. I recommend those who doubt this statement to begin by defining a straight line by a single geometrical property, which is not verbally equivalent to the assertion in question, and to attempt to prove it.

It may be well to remark here that the discussion of the properties of tridimensional space in reality divides itself into two parts:—first, what may the properties of space be conceived to be? *conceive* being understood in the sense above explained; second, what are the properties of space as we know, or think we know, them? The former question is a purely mathematical one; the latter is one in the main for the physicist or the mental philosopher, and the function of the mathematician in connection with it is to make clear what the question exactly is, and what alternatives are open for us. What the bearing of modern mathematical research on this point appears to be, I shall endeavour to explain later on.

With these preliminary remarks in explanation, I now proceed briefly to sketch a system of geometry which, as to its foundations, differs from that of Euclid only in the alteration of one (or at most two) axioms. Its conclusions will be found to differ very materially from his, although this difference is merely in the way of wider generality, Euclid's geometry being contained as a particular case in what I shall, for distinction's sake, call Pangeometry.

The space which I shall consider is to be tridimensional. I appeal to the ordinary conceptions of

Point, Line or curve, Surface, Solid ;

and, for the sake of the words, state that a point has no extension, a line is once extended, a surface twice, a solid thrice.

As a test of these distinctions, the idea of motion may be introduced. I cannot stop now to justify this, but merely remark that nothing is to be predicated concerning time.

Farther, space is to be uniform, in the double sense that it has no properties depending either on position or direction.

The great test of this last statement is congruency,* which I mention thus early, because it is the touchstone of geometry. Thus the statement that space has no properties depending on position, simply means that congruent figures exist, *e.g.*, that a solid of a certain size and shape can be carried from one part of space to another without alteration in either respect ; and that two congruent figures can be conceived as separately existing in different parts of space. It is evident that all space measurement rests on congruency.

It is essential to be careful with our definition of a *straight line*, for it will be found that virtually the properties of the straight line determine the nature of space.

Our definition shall be that two points *in general* determine a straight line, or that in general a straight line cannot be made to pass through *three* given points.

It is important to notice the force of the phrase *in general*. This

* Two figures are said to be congruent when one can be placed on the other, so that every point of one shall coincide with a point of the other, and *vice versa*. The phrase *equal in every respect* is used in the same sense in most English editions of Euclid.

will be best understood from an illustration. We all know from the case of a three legged stool, if not from any more scientific source, that three points determine a plane. Yet not any three points; for, if the third foot were put in line with the other two, the one stool would be as unsafe a seat as the proverbial two. Yet again, and very near indeed to our case, two points on a sphere in general determine a great circle on it. But there are exceptions; a point and the diametrically opposite point do not determine a great circle, and yet it would be a good definition of a great circle to call it that line on a sphere which is in general determined when two of its points are given, no other condition being assigned.*

We recognise therefore that, although in general, any two points being taken, a line will thereby be determined, yet it may happen that, one point being taken, another point may exist which along with the first does not determine a straight line. The necessity for this admission appears when we consider space in which two straight lines have more than one point of intersection.

Here let it be mentioned, to avoid misconception, that it follows from our definition of a straight line, and from the uniformity of space (the test being congruency), that space is symmetrical round every straight line. This is at once an answer to those who say that pangeometry is merely an analogy drawn from the theory of surfaces of constant curvature.

A plane may be defined as Euclid defines it, and the conclusions drawn, that two intersecting lines, a point and a line, or a line passing through a given point and moving perpendicular to a given line, all in general determine a plane. The last form of definition of course presupposes the definition of a right angle.

Farther, we adopt all Euclid's definitions up to the definition of an

* It is interesting to notice that any curve already conditioned a number of times less by two than the whole number of conditions that completely determine it, fulfils in many respects the definition of a straight line, for any two points completely determine the curve. A very interesting particular case is that of a series of circles which always pass through a given fixed point. Such a series of circles may take the place of straight lines in many of Euclid's propositions. Most of the propositions as to congruency hold for them. The sum of the three angles of a triangle formed by three such circles is two right angles; the perpendiculars from the vertices of such a triangle on the opposite sides are concurrent; and so on, as is otherwise evident by the theory of inversion.

acute angled triangle, but reject in the meantime, at all events, all that follow in the first book.

Next we adopt Euclid's propositions concerning angles at a point, viz., I. 13, 14, 15; also the propositions as to congruency I. 4, 5, 6, 8, and the first part of 26, with a protest to the effect that in many cases his demonstrations are needlessly circuitous and difficult. All that is wanted for the demonstration of these propositions is the defining property of the straight line and the ordinary axioms and definitions as to equality.

Different Kinds of Space.

Before going farther, we must distinguish the different cases that may arise when we consider two intersecting straight lines.

1. They may never intersect again and be of infinite length (*i.e.*, each is non-re-entrant). Space which has this characteristic is called, for the present, hyperbolic space. We shall see, however, by and by that another case must be distinguished under this head, that, viz., of homaloidal or Euclidean space.

2. They may intersect again. Space having this characteristic is called elliptic space.

The simplest space of this kind is that in which a straight line returns into itself, so that the next point in which two straight lines intersect is the point in which they first intersected. In this kind of space, which I shall call single elliptic space, two straight lines intersect in only one point; and there is no exception to the statement that two points determine a straight line.

The next simplest case would be that in which two straight lines intersect a second time in a distinct point, and then re-enter at the next point of intersection which coincides with the original one. This might be called double elliptical space. I am not yet certain* whether the symmetry of space will allow us to carry this multiplicity

* I have not been able to find a definite settlement of this question by any of the great authorities on hyper space. Frischauf takes double elliptic space as the representative of elliptic space, and seems to hold that this is the only possible kind. Klein ("Mathematische Annalen," vi. 125) takes single elliptic space, and criticises Frischauf's view ("Fortschritte der Mathematik," viii. 313, 1876). Newcomb (Borchardt's Journ., lxxxiii. p. 293) professes himself unable to settle the question. If the notion of double elliptic space cannot be shown to be self-contradictory, then it would appear that the question becomes simply one of the choice of axioms. See note below, p. 661.

of elliptical space farther. In the meantime, I may remark that in a space of this second kind we must, as already explained, admit exceptions to the statement that two points determine a straight line.

In what follows I take single elliptical space as the representative of elliptical space generally, although on account of the non-existence of a closed surface of uniform positive curvature, on which a pair of geodetics intersect only once, the conclusions of the geometry of single elliptical space appear in some respects more bizarre than those of double elliptical space, whose planimetry is mirrored by the geodesy of a sphere.

It is obvious that Euclidean, or homaloidal, space is included in hyperbolic space as above defined. We shall afterwards show, however, that it may be regarded as a limiting case of elliptic space. It is therefore the transition case lying between the other two.

Sketch of the Geometry of Hyperbolic (Infinite) Space.

From the definition of this kind of space it is clearly infinite. Here I must insist on the distinction between infinite and unbounded, a distinction first brought into notice by Riemann. The uniformity of space necessarily involves the notion that it is unbounded, but by no means necessitates that it shall be infinite in extent; in fact, I shall point out directly that a single elliptical space is necessarily of finite extent.*

After the propositions relating to congruency already proved, the next fundamental proposition to be established is the following:—

In hyperbolic space the sum of the three angles of a rectilineal triangle cannot exceed two right angles.

The following proof of this proposition is due in substance to Bolyai. Legendre had given another, but he failed to see exactly the nature of the assumptions on which he founded.

ABC (fig. 1) is any triangle, O the middle point of BC, OD = OA; so that CD falls within the angle BCL. (Here we assume that a straight line is non-re-entrant, and that a pair of straight lines never intersect twice.) Then DOC ≃ †AOB; and ADC is equal in area to ABC, and

* An ellipse and a circle are unbounded but finite lines; a hyperbola is both unbounded and infinite.

† I adopt the sign ≃ used by continental writers for *congruent to*, or *equal in every respect to.*

has the sum of its angles the same, while the sum of A and D = BAC. Of these angles one is $\ngtr$, and the other $\nless$ than $\frac{1}{2}$ A. Taking the least of them, and bisecting the opposite side, we derive as before from ADC a triangle, still having the same area, and the same sum of all the angles, but in which the sum of two of the angles $\ngtr \frac{1}{2}$ A.

By a similar process we derive another triangle, still having the area and the sum of its angles unaltered, but in which the sum of two angles $\ngtr \frac{1}{2^2}$A.

At last we get a triangle, in which the area is the same as at first, and the sum of the angles the same, but the sum of two of them $\ngtr \frac{1}{2^n}$A, where n may be as great as we please; that is, in which the sum of two angles is as small as we please.

But the third angle can never be greater than 2R, hence the sum of the angles of the original triangle cannot be $>$ 2R.

It is to be noticed that this demonstration would fail if a straight line were re-entrant, or if two straight lines had more than one point of intersection.

Corollary.—If C′ be the external angle at C of the triangle ABC, then, since

$$A + B + C = 2R - \delta,$$

where R stands for a right angle, and δ is either zero or essentially positive, and

$$C + C' = 2R,$$

we have

$$C' = A + B + \delta;$$

That is, *the exterior angle of any triangle is not less than the sum of the two interior opposite angles.*

Of course it follows that *the exterior angle of any triangle is greater than either of the interior opposite angles; and that the sum of any two angles of a triangle is less than two right angles.*

We can now prove for hyperbolic space :—

That the greater side of every triangle has the greater angle opposite, and conversely.

That any two sides of a triangle are together greater than the third side.

Also *Euclid I.* 21.

Euclid I. 24 *and* 25.

Euclid I. 26 (*the second part*).

Also the usual propositions concerning the perpendicular and the obliques drawn from a given point to a given straight line.

The amount by which the sum of the three angles of a triangle falls short of 2R is called the *defect* of the triangle. This is the same as the excess of the sum of its exterior angles over 4R. If we take the latter statement of the definition, we may talk of the defect of any plane rectilineal figure. In forming the external angles of figures generally, we must go round, producing all the sides in the direction of our progress, assigning the positive or negative sign according as the angle is not or is re-entrant.

Thus in figure 2 the defect is

$$\alpha + \beta - \gamma + \delta + \epsilon - 4R.$$

Defining defect in this way, it is easy to prove that

The defect of any rectilineal figure is equal to the sum of the defects of any rectilineal figures of which it may be supposed to be composed.

Cor. Hence if one rectilineal figure lie wholly within another the defect of the former is not greater than that of the latter.

Hence follows at once the following important proposition:—

If the defect of any triangle whose sides are finite be zero, then the defect of every finite triangle must be zero.

For if ABC (fig. 3) be a triangle whose defect is zero, then, by applying to its sides three triangles, each congruent with itself, as shown in the figure, we evidently construct a triangle A′B′C′, having the same angles as ABC, and hence zero defect, each of whose sides is double a corresponding side in ABC. We may repeat this process with A′B′C′, and so on. Hence we may construct a triangle, having zero defect, large enough to contain within it any finite triangle whatever. But the defect of any triangle cannot be greater than that of a triangle within which it is contained, and the defect cannot be less than zero; hence the defect of every finite triangle must be zero, if the defect of any one finite triangle be zero.

Thus in *hyperbolic space*, as defined above, we are shut up to one or other of two alternatives. *Either the defect of a triangle is always positive or it is always zero.*

If we take the latter alternative, we get Euclidean or homaloidal space; and, from the defining property by which we have characterised it, we can prove Euclid's parallel axiom, and develop Euclid's geometry in his or any other equivalent manner.

Having separated out homaloidal space, let us now consider more closely hyperbolic space proper, in which the defect is always positive.

The fundamental proposition to be proved is the following.

The defect of a triangle (and consequently the defect of any plane rectilineal figure) is proportional to its area.

Various proofs of this proposition might be given. I select that which depends on the properties of the curves of equidistance from a straight line, because the intermediate propositions are the analogues in hyperbolic space to the propositions regarding parallels and parallelograms that are given in the latter part of Euclid's first book.

If in any plane perpendiculars of constant length be erected upon a given straight line, their extremities generate two curves which I shall call the equidistants, the two equidistants corresponding to a given length of the perpendicular may be called conjugate equidistants.

The equidistant is a self congruent line.

For if we take any piece AB (fig. 4) of the given line, and LM the corresponding piece of the equidistant, and if also $A'B' = AB$ and $L'M'$ be corresponding points to A' and B', then, if we place $A'B'$ on AB, L' and M' will coincide with L and M, and, if $A'P' = AP$, Q' will coincide with Q, and so on. Hence the piece $L'M'$ is congruent with the piece LM.

The equidistant is at every point at right angles to the generating perpendicular.

This is at once evident by considering two equal pieces (fig. 5) LP and LQ of the equidistant on either side of L, and the corresponding points A and B on the straight line, so that $OA = OB$. We have $LOAP \simeq LOBQ$, hence $\angle OLP = \angle OLQ$, each $= R$.

The equidistant in hyperbolic space is a curved line, concave towards the given line.

Let LQM (fig. 6) be a piece of the equidistant, LM a straight line cutting the perpendicular through P, the middle point of AB, in R. Then $LRPA \simeq MRPB$. Hence $\angle PRL = \angle PRM = R$, and the angles at P are each $= R$, therefore $ALR < R$.

But ∠ QLA = R, therefore LQ falls above LRQ, however small the distance AB may be; in other words, LQM is concave towards AB.

Every straight line terminated by a pair of conjugate equidistants to a given straight line is bisected by the given straight line, and makes equal alternate angles with the equidistants, &c.

If AB (fig. 7) be the given straight line, XP and YQ the equidistants, POQ the line terminated by the equidistants, then the proposition follows at once by observing that, if AP and BQ be perpendiculars to AB, then AOP ≃ BOQ.

The common perpendicular to two conjugate equidistants is the least distance between them, the oblique distances are greater according as the angle they make with the perpendicular is greater, and the length of an oblique can be increased without limit.

It will be seen that conjugate equidistants are analogous to Euclidean parallels. The analogy may be carried much farther.

If equal arcs of two conjugate equidistants be joined towards the same parts by two straight lines, the figure so formed may be called a *hyperbolic parallelogram.*

A mixed triangle whose base is the arc of an equidistant, whose two remaining sides are straight lines, and whose vertex lies on the conjugate equidistant, may be called a *hyperbolic triangle.* The following propositions are then very easily proved.

The sum of the three angles of a hyperbolic triangle is 2R.

The opposite straight sides of a hyperbolic parallelogram are equal to one another; its diagonals bisect one another in a point on the straight line to which the equidistants that form its curved sides belong; and each diagonal divides it into two congruent hyperbolic triangles.

A series of propositions analogous to those of Euclid, Book I., 35–41, may be proved very easily; we have only to substitute hyperbolic parallelograms and triangles for ordinary parallelograms and triangles, and conjugate equidistants for parallels. In particular, we see (fig. 8) that

Two hyperbolic triangles CAOB, DAOB, which have for common base the arc AOB of an equidistant (and consequently have their vertices on the conjugate equidistant) are equal in area.

Hence follows at once that—

The rectilinear triangles CAB, DAB on the same chord of an equidistant, whose vertices lie on the conjugate equidistant, are equal in area and defect.

N.B.—the defect is 2 ∠ OAB in both cases. It is obvious that, if we join the middle points of the sides of any triangle, the extremities of its base lie on an equidistant to the line so drawn, and the vertex lies on the conjugate equidistant. Bearing this in mind, the properties of equidistants enable us to establish the following propositions :—

We can always construct an isosceles triangle whose base is equal to one side of a given triangle, and whose area and defect are the same as those of the given triangle.

*Given two triangles, we can always transform one or other of them into another of equal area and defect which has one of its sides equal to one of the sides of the remaining triangle.**

Hence two triangles that have the same area must have the same defect, and conversely, for we can transform them into a pair of isosceles triangles on the same base without altering either area or defect. It is obvious that two such triangles must be congruent if they are equal in area, and hence they must be equal in defect; and from what I have proved concerning the defect of composite figures, the converse follows with equal ease.

Hence *the area of a triangle is proportional to its defect.* Hence, ρ being a certain linear constant, characteristic of a hyperbolic space, and A the area of a rectilineal triangle of defect δ, we have

$$A = \rho^2 \delta .$$

A great variety of very important conclusions can at once be drawn from this formula. I mention some of the most interesting.

Since $\delta = \frac{A}{\rho^2}$, if ρ be infinite, then $\delta = 0$ for every triangle of finite area; in other words, homaloidal space is simply a hyperbolic space whose linear constant is infinite. This conclusion may be looked at from another, but mathematically equivalent, point of view. Let us imagine a hyperbolic space of given linear constant ρ.

* I leave the reader to consider and settle for himself whether a simpler proposition than the above could be established. In particular he should consider the following problem in hyperbolic geometry:—"To construct an isosceles triangle of given area on a given base."

If we take a region in this space whose greatest linear dimension is an infinitely small fraction of ρ, then the defect of every triangle within that region will be infinitely small, and its geometry will not differ sensibly from that of a homaloidal space. This is often expressed by saying that hyperbolic space is homaloidal in its smallest parts.

It appears, therefore, that, even in hyperbolic space, Euclid's planimetry will apply to infinitely small figures. For instance, the ratio of the circumference of a circle to its diameter will be $\pi = 3{\cdot}14159 \ldots$ (the ordinary transcendental constant), when the diameter is made infinitely small. We may, therefore, if we please, measure our angles in radians (circular measure), and in fact use all the formulæ of homaloidal plane trigonometry, if proper restrictions be observed.

It should also be noticed that the existence of this length ρ related to the space, but not *directionally* related, suggests the possibility of explaining the properties of tridimensional space by subsuming it in a space of four or more dimensions. I have not chosen to enter into speculations of this nature, partly because their development has been entirely analytical hitherto; and partly because, so far as I can see at present, it may be justly contended that the conceivability of hyperspace of three dimensions rests on different grounds from that which we must necessarily assume when we attempt to add another dimension. In this, however, I may be but one of those whom Gauss playfully called Bœotians.*

* Before leaving this part of the subject, I may mention the curious solution of the problem of dividing a plane in hyperbolic space into a network of regular polygons.

If n be the number of sides of each polygon, p the number of polygons round a point of the network, A the area of each of the n-gons, then

$$A = n\pi\rho^2\left(1 - \frac{2}{n} - \frac{2}{p}\right),$$

with the condition $\frac{1}{n} + \frac{1}{p} < \frac{1}{2}$.

Suppose, for instance, we wish to divide a plane into squares, *i.e.*, regular four-sided figures. Then $n=4$. If $p=4$, *i.e.*, if the angles of the square be right angles, $A=0$, which does not, strictly speaking, give a solution. The next case is $p=5$, so that $A = \frac{2}{5}\pi\rho^2$ is the area of the smallest finite square with which we could pave a plane floor. Of course there are an infinite num-

Theory of Parallels.

If O (figs. 9 and 10) be any point outside a line, P any point in it to the right of the foot of the perpendicular, then the limiting position of OP, when P is moved in the direction DI to the right, without limit, is called the parallel through O to DI. The corresponding limiting line on the other side of OD is called the parallel through O to DI′.

Thus

$$OK \; // \; DI$$
$$OK' \; // \; DI'.$$

It is obvious, from the uniformity of space, that OK and OK′ make equal angles with OD. Whether they are parts of the same line or not, remains to be seen.

As P moves off along DI the angle at P diminishes without limit.

This is easily shown (fig. 10) by taking $PP_1 = OP$, $P_1P_2 = OP_1$ and so on *ad. inf.*

In homaloidal space the parallel to DI through O is the perpendicular to DO at the point O: for the sum of the three angles of the triangle DPO is always 2R, and P diminishes without limit, hence the angle at O approaches nearer to R than by any assignable quantity.

Thus in homaloidal space the two parallels OK, OK′ are parts of the same straight line, and all the lines through O cut IDI′, except the parallel, which may be said to cut it at an infinite distance. In the language of modern geometry there is but one point at infinity on the line IDI′.

In hyperbolic space there are two parallels through a given point to a given straight line.

For as we move P away from D the area of ODP, and consequently its defect, constantly increases, but the angle OPD constantly diminishes, hence the angle at O can never exceed a certain angle which is less than a right angle.

It follows, therefore, that if we take any line IDI′ and any external point O, we must classify the lines through O as follows :—(1) intersectors, (2) non-intersectors, (3) two parallels.

ber of solutions, the angles of the squares becoming less and their area greater as p increases. The area of the greatest possible square tile that we could use would be $2\pi\rho^2$, but the lengths of the sides would be infinite.

In figure 11 KOL′ and K′OL are the two parallels; all lines lying in the angles KOL, K′OL′, are non-intersectors, all those lying in KOK′, LOL′ are intersectors. The fact that in hyperbolic space there are two parallels through a given point to a given straight line is expressed in modern geometry by saying that in hyperbolic space a straight line has two distinct real points at infinity.

After what has been laid down, the following propositions either are immediately evident, or can be proved with very little trouble.

If a line is parallel to another at any point, it is so at every point of itself.

Parallelism is mutual.

Lines which are parallel to the same line are parallel to one another.

Lines that are parallel continually approach one another on the side towards which they are parallel.

Non-intersectors in the same plane have a minimum distance, which is the common perpendicular.

The angle which a parallel through O to L makes with the perpendicular on L is called the parallel angle.

The parallel angle is a function of the length of the perpendicular, increasing when the perpendicular diminishes.

If θ be the angle, p the length of the perpendicular, then it may be shown by methods which I shall presently explain that

$$\tan \tfrac{1}{2}\theta = e^{-\frac{p}{\rho}},$$

$$\text{When } p = 0,\ \theta = \frac{\pi}{2};\ \text{when } p = \infty,\ \theta = 0.$$

Geometry of Elliptic Space.

For simplicity I take single elliptic space, but there will be no difficulty in modifying what follows so as to make it apply to double elliptic space.

In single elliptic space every straight line returns into itself; and two straight lines intersect in only one point. Thus, starting from any point P, and proceeding in any direction continuously, we at last return to the point P; the length L travelled over in this process is called the length of the *complete straight line.*

It is obvious that in single (as well as in double) elliptic space two intersecting complete straight lines enclose a plane figure. Such a figure I call a *biangle.*

Two biangles are congruent when their angles are equal. All complete straight lines are of the same length, and all the straight lines emanating from the same point intersect in the same second point.

These propositions are all equivalent to one another, and are equally true for single or double elliptic space. The last of them is a mere truism for *single* elliptic space. The following demonstration, which holds good for single or double elliptic space, may help to render the matter clearer.

Let APBQA A′P′B′Q′A′ (fig. 12) be two biangles having the angles A and A′ equal. If A′B′ be placed on AB so that A lies on A′, and A′P′ along AP, then A′Q′ will lie along AQ, since the angles at A are equal; hence by the fundamental property of a straight line APB and A′P′B′ must wholly coincide, and AQB and A′Q′B′ must wholly coincide; and hence B′ must fall on B. It is to be noticed that the biangles are multiply congruent.

Next, suppose AKA′, AK′A′ (fig. 13) to be any pair of intersecting straight lines. Let AL bisect the angle A and cut the lines in J and J′. Since AJ and AJ′ are equiangular biangles, they are congruent; from this it follows at once that J and J′ must coincide with each other, and therefore each with A′. Hence the bisector of the angle A passes through A′; and it and AKA′ and AK′A′ are all of equal length. We may next bisect either of the halves of A, and so on; and we may double any of the angles thus obtained as often as we please. Hence the propositions stated above are completely proved. The length L of a complete straight line is therefore an absolute linear constant which characterises an elliptic space.

In single elliptic space the least distance between two points can never be greater than ½L, and the greatest distance can never be greater than L.

This is obvious, since the whole length of a complete straight line through the two points is L.

If we consider the plane determined by two intersecting straight lines AOA, BOB, and if we pass from O along OA through a length L, we return to O, but find ourselves on the opposite side of the plane to that from which we started, and only arrive at the same point O *on the same side as before by travelling once more through a length L.*

This curious conclusion is an immediate result of the fact that straight lines are re-entrant and intersect only once. (In double elliptical space the apparent anomaly does not occur on account of the double intersection.)

The best way of representing the thing to the mind that I can think of is to imagine a rigid body composed of three rectangular arrows Ix, Iy, Iz (fig. 14). Ix slides along OA; Iy passes through a ring which slides on OB (being long enough never to slip out); Iz is, of course, determined in position when Ix and Iy are fixed in any positions.

In starting from O, let Ix and Iy be horizontal and Iz vertical; then slide Ix along OA. Ix will at last return along A′O. The ring will return along B′O. It is obvious, therefore, that, at our first return to O, Iz must be downwards, for, since the system of arrows is rigid, one who plants himself with feet at I, head at z and looks along Ix must see y to his left as he did at starting.

It is obvious that during the journey Iy as well as Iz has rotated through 180°, a repetition of the process rotates both through 180° more, and then everything is as before.

If we cause a complete straight line of length L to revolve through 360°, always remaining perpendicular to a given line, it will sweep out the two sides of a *complete plane.*

It follows at once, therefore, that the area of a complete plane, taking into account both sides, is finite, and the same for every complete plane. This I shall call P in the meantime. We also see, in accordance with what was proved before, that the two sides of the complete plane are not distinct, since we can pass continuously upon the plane from a point on one side to the same point on the other side.

Those who find difficulty in realising this property of the plane in single elliptic space should take a ribbon of paper, twist it through 180°, and then gum the ends together. A surface is thus formed which has the property that one can trace a continuous line upon it from a point on one side to a point exactly opposite on the other side.

After what has been laid down the following propositions are obvious.

They are given by Newcomb in an extremely interesting article to which reference was made above. I arrange them in the order which best suits what has gone before.

All the perpendiculars in a given plane to a given straight line intersect in a single point, whose distance from the straight line is $\frac{1}{2}L$.

Conversely, the locus of all the points at a distance $\frac{1}{2}L$ *on straight lines passing through a given point in a given plane is a straight line perpendicular to all the radiating lines.*

The fixed point is called the *pole*, and the straight locus its *polar*.

If we cause the given plane to rotate about the polar the pole describes a straight line which may be called the conjugate of the given polar.

The relation of these two lines is mutual, every point on one being at a distance $\frac{1}{2}L$ *from every point on the other.*

Without dwelling farther upon propositions of this kind, I proceed at once to establish the fundamental proposition concerning the sum of the angles of a plane triangle. I might follow a course like that adopted for hyperbolic space, but a much simpler method suggests itself at once as applicable to finite space.

In the first place, since a complete plane is generated by the revolution of a complete straight line through 360°, it follows that the area of a biangle whose angle is A° is $\frac{A}{360}P$.

In figure 15 let ABC be any triangle. Produce the sides to form biangles. Each of the biangles departs from the vertex on the upper side of the plane and returns to the vertex on the lower side. To make this clear areas in the neighbourhood of ABC in the figure are shaded with vertical lines when reckoned on the upper and with horizontal lines when reckoned on the lower side of the plane. A glance will show that if we take the three biangles they overlap the triangle ABC thrice, and that the rest of the plane is covered every where once on one side or the other, but nowhere on both sides. Hence, Δ denoting the area of the triangle, we have

$$\frac{A}{360}P + \frac{B}{360}P + \frac{C}{360}P = \tfrac{1}{2}P + 2\Delta$$

$$\Delta = \frac{A+B+C-180}{360}P.$$

If, therefore, we define $A^\circ + B^\circ + C^\circ - 180^\circ$ as the *excess* of the triangle, we have the proposition that—

The excess of every triangle is positive, and is proportional to its area.

The conclusions drawn above (p. 650) for hyperbolic space follow here, *mutatis mutandis.* In particular, we see that we may apply Euclidean planimetry to infinitely small figures. On this remark we can, as will be done later, found a system of planimetry for elliptic space, and determine P. The result is $P = \frac{4L^2}{\pi}$. Hence, writing ρ for $\frac{L}{\pi}$, and ϵ for the radian measure of the excess, we have

$$\Delta = \rho^2 \epsilon$$

where ρ is a linear constant characteristic of the elliptic space.

It is easy after what has now been established to work out the propositions corresponding to Euclid's first book. The conclusions will, of course, be subject to certain modifications, but these are easily found. I may mention in particular that the propositions concerning the curves of equidistance already given for hyperbolic space, hold with very slight modification for elliptic space, the main difference being that the equidistants are convex instead of concave to the given straight line.

Theory of Parallels.

In elliptic space there is, of course, no such thing as a parallel, because there are no infinitely distant points on a straight line.*

If O (fig. 16) be a point outside the line IDI′; then it is easy to see that the two segments of the perpendicular from O are respectively the least and greatest distances from the given line. If OD be the least distance, then, as OP, starting from OD, revolves about O, OP continually increases, until it has rotated through 180°, and then it is at its maximum, after which it decreases again.

It can easily be shown that, as OP revolves from OD, the angle OPD decreases, until OP is perpendicular to OD, and then OPD is at its minimum value. After that, as may be easily shown by producing the line backwards through O, the angle again increases.

* In the language of modern geometry the points at infinity on a straight line in elliptic space are imaginary.

The line OI, perpendicular to OD, is all that there is in elliptic space to represent a parallel through O to the line I′DI.

General Conclusions.

If I have succeeded in my attempt to explain the results of modern research concerning the axioms of geometry, it will be apparent that, even if we overlook the possibility of space being non-uniform, in the sense of having properties depending on position and direction, it is still possible to develop three self-consistent kinds of geometry —the hyperbolic, the homaloidal, and the elliptic. It is impossible, it appears to me, to say on *à priori* grounds that any one of these is more reasonable than the others. If, therefore, *à priori* ground is to be sought for the axioms of geometry, such tests of its firmness "as the inconceivability of the opposite" and others like it are not to be relied upon. They are merely an appeal to ignorance.

If, on the other hand, we view the question from the side of experience, three alternatives are open to us. We may hold that space is homaloidal and therefore infinite. In this case we extend to the infinite part of space which we do not know the results of our experience of the finite part of it that we do know.

Again, we may hold that space is hyperbolic and therefore infinite. In this case experience teaches us that the radius of the sphere of our experience is infinitely small compared with the linear constant of space; for Lobatschewsky calculated from astronomical observations the sum of the three angles of triangles whose smallest sides were about double the distance of the earth from the sun, and found that the difference from two right angles was not greater than the probable error of observation.

Lastly, we may suppose that space is elliptic and therefore finite, in this case we must admit that our experience extends to but an infinitely small fraction of its whole extent, since no sensible excess can be found in the largest triangles with which we are acquainted.

Before leaving this subject, it may be well to illustrate with some care what is meant by the words finite and infinite as I have used them. They have, of course, a purely relative meaning. In the geometry of homaloidal space no distinction can be built on the relative dimensions of figures apart from their form. Owing to the existence of similar figures, the geometrical experience of a cheese mite in homaloidal space would not be different from that of a being one of whose habitual walking steps was from the sun to the dog star.

In hyperbolic or elliptic space the case is otherwise. In either of these two kinds of space we might divide intelligent beings into two classes according to their bodily dimensions. We might have a race of micranthropes, whose bodily dimensions and the radius of whose sphere of experience were infinitely small compared with the linear constant of space. For instance, if the space were elliptic, the world of the micranthropes would be but an infinitely small fraction of the elliptic universe. It must be noticed, however, that from the point of view of a micranthrope, his world need not be a prison-house by any means, for he would compare it not with the linear constant of universal space, of whose magnitude he must necessarily be ignorant, but with some arbitrary standard such as the length of his own arm, and so considered his world would to him be infinite, if we only suppose him small enough. Again, we might have a race of macranthropes, whose bodily dimensions were comparable with the linear constant of space. In the case of an elliptic and finite space, we could, of course, conceive one of these himself so great that there would not be room enough in the universe for another as great.

The geometry of the micranthropes would, of course, be homaloidal. The axioms of Euclid would appear to them strictly in accordance with experience, and, although they lived in part of an elliptic or hyperbolic space, their prejudices would render the conceptions of the general properties of such a space as difficult to them as they are to us. On the other hand, the geometry of the macranthropes would be elliptic or hyperbolic, as the case might be. A hyperbolic macranthrope would, of course, be familiar with the fact that the defect of a triangle diminishes as its area diminishes. If he were a mathematician he would be aware of the relation of proportionality, and might speculate concerning triangles of zero defect, much as we do about absolute zero of temperature. If Euclid's geometry were to fall into the hands of an instructed macranthrope, he would very likely regard it as the production of some macranthropic lunatic, who had meditated on the fact that the defect of a triangle diminishes with its area, until he had so far lost his wits as to commit the *ὑστερον προτερον* of discussing the construction of an equilateral triangle before proving that when two straight lines cut one another the vertically opposite angles are equal!

Appendix on the Trigonometry of Elliptic and Hyperbolic Space.

The following appears to me to be the simplest, and at the same time the most instructive way of establishing the Trigonometry of Elliptic and Hyperbolic Space.

The method might, indeed, by assuming proper axioms, be made to take the place of the preceding synthesis. As it is, I shall base it upon the results of that synthesis. What I shall want are mainly the propositions concerning the excess or defect of plane triangles, and the conclusion founded on them that homaloidal trigonometry may be applied to figures, all of whose dimensions are infinitely small compared with the linear constant of space.

Let KA and LB (fig. 17) be two straight lines in the same plane at an infinitely small distance apart. They may be either non-intersectors, whose minimum distance d is infinitely small, or intersectors which make a very small angle α with each other at their point of intersection.

Let KL, AB, CD be lines making equal angles with KA and LB; and let $KA=LB=r$, $AC=BD=dr$, $AB=D$, $CD=D+dD$, where dr is infinitely small compared with r, dD infinitely small compared with D; D of course is infinitely small compared with ρ, the linear constant of space.

Further, let $\angle LBA = \angle KAB=\frac{\pi}{2}-\theta$, and $\angle LDC = \angle KCD=\frac{\pi}{2}-\theta-d\theta$.

Since all the dimensions of ABDC are infinitely small compared with ρ, we may apply Euclidean trigonometry. Draw Bm parallel to AC. Then $\angle ABm=\frac{\pi}{2}-\theta$, $AB=Cm$, and $Dm=dD$.

$$2\sin\tfrac{1}{2}DBm=2\sin(-\theta)=\frac{dD}{dr};$$

$$\theta=-\tfrac{1}{2}\frac{dD}{dr}. \qquad (1)$$

Now the excess of ABDC $=2\left(\frac{\pi}{2}+\theta\right)+2\left(\frac{\pi}{2}-\theta-d\theta\right)-2\pi=-2d\theta$; and its area $=Ddr$. Hence

$$Ddr=\rho^2\epsilon=-2\rho^2d\theta.$$

which gives

$$2\frac{d\theta}{dr}+\frac{D}{\rho^2}=0.$$

Whence by (1)

$$\frac{d^2D}{dr^2}+\frac{D}{\rho^2}=0. \qquad (2)$$

This is the equation for Elliptic Space; that for Hyperbolic Space is of course

$$\frac{d^2D}{dr^2} - \frac{D}{\rho^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad (2')^*$$

From the equation (2) we get at once

$$D = \rho\alpha \sin\frac{r}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

r being measured from the intersection of the lines, and the constants of integration determined by the condition

$$D = \alpha r$$

when r is infinitely small compared with ρ, which of course includes the condition $D = 0$ when $r = 0$.

The corresponding formulæ for Hyperbolic Space are

$$D = \rho\alpha\left(\frac{e^{\frac{r}{\rho}} - e^{-\frac{r}{\rho}}}{2}\right)$$

$$= \rho\alpha \sin h\frac{r}{\rho} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

for a pair of intersectors; and

$$D = d\left(\frac{e^{\frac{r}{\rho}} + e^{-\frac{r}{\rho}}}{2}\right)$$

$$= d\cos h\frac{r}{\rho} \quad . \quad . \quad . \quad . \quad (5)$$

for a pair of non-intersectors, r being measured in the one case from the intersection, in the other from the points of minimum distance.

From the formulæ (3), (4), and (5) all the trigonometry of Elliptic and Hyperbolic Space can be deduced most readily. I append one or two applications, and select for my purpose important formulæ, but anything like a complete development would be out of place here.†

* The differential equations (2) and (2′) contain all the metrical properties of elliptic and hyperbolic space. (2) suggests that a pair of straight lines diverging at a small angle from a point might intersect again in distinct points any number of times. The proposition proved above for elliptic space generally, that all the lines radiating from any point intersect in the same second point, seems, however, to compel us to conclude that at the point where any line intersects another for the second time, it must return into itself; for a line can be brought by continuous rotation into coincidence with its prolongation, hence we must reach the same second point of intersection in whichever direction we proceed from the first point. I can see no way out of this at present; and if there is none, it would appear that we cannot get beyond double elliptic space, even if we can consistently get so far.

† I may refer the reader to Frischauf, "Elemente der Absolute Geometrie," Leipzig, 1876; Lobatschewsky, Crelle, xvii. p. 295; Klein, Annalen der Mathematik, iv. p. 573. vi. p. 112, &c.; Cayley, Annalen der Mathematik, v. p. 630.

Area of Complete Plane and Total Volume of Elliptic Space.

The area of a biangle having the infinitely small angle α is

$$\rho\alpha\int_0^{\pi\rho}\sin\frac{r}{\rho}\,dr=2\rho^2\alpha\,.$$

Hence
$$P=4\pi\rho^2=\frac{4L^2}{\pi}\quad . \quad . \quad . \quad . \qquad (6)$$

From this result we can deduce very easily the total volume S of elliptical space (single). The locus of the most distant points on the radii through any point of space is a plane. Suppose this plane divided up into infinitely small regular quadrilaterals (squares) of side k. The volume dS contained by four radii drawn to the vertices of one of these figures is

$$dS=k^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}\rho}\sin\frac{2r}{\rho}\,dr=\frac{\pi}{2}k^2\rho\,;$$

Hence
$$S=\frac{P}{2k^2}dS=\frac{P\rho}{4}\,,$$
$$=\pi^2\rho^3\,,$$
$$=\frac{L^3}{\pi}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

This curious result can also be obtained by calculating the volume swept out by a complete plane rotating through 180° about any line in it.

Formulæ for Right-Angled Triangles.

Let ACB (fig. 18) be a triangle right angled at C. Let B$\text{A}b=d$A , B$b=da$. CbA $=$ B $+ d$B. If Bm be perpendicular to Ab, then $bm=dc$.

We have at once by (3)

$$\sin B\,da=\rho\sin\frac{c}{\rho}\,dA\quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Also
$$dc=da\cos B\quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

Calculating the area BAb in the two different ways we get

$$\int_0^c D\,dc=\rho^2\epsilon\,.$$

Whence
$$dA\rho^2\left(1-\cos\frac{c}{\rho}\right)=\rho^2(dA+dB)$$

i.e.,
$$dB=-\cos\frac{c}{\rho}\,dA\quad . \quad . \quad . \qquad (10)$$

Whence $$dc \tan B = -\rho \tan \frac{c}{\rho} dB \quad . \quad . \quad . \quad . \quad (11)$$

From these equations we get successively:

$$\sin B \sin \frac{c}{\rho} = \sin \frac{b}{\rho} \quad . \quad . \quad . \quad . \quad \text{I.}$$

$$\cos \frac{a}{\rho} \cos \frac{b}{\rho} = \cos \frac{c}{\rho} \quad . \quad . \quad . \quad . \quad \text{II.}$$

$$\sin A \cos \frac{b}{\rho} = \cos B \quad . \quad . \quad . \quad . \quad \text{III.}$$

$$\tan \frac{b}{\rho} \cot \frac{c}{\rho} = \cos A \quad . \quad . \quad . \quad . \quad \text{IV.}$$

For hyperbolic space we get in like manner

$$\sin B \sin h \frac{c}{\rho} = \sin h \frac{b}{\rho} \quad . \quad . \quad . \quad . \quad \text{I}'.$$

$$\cos h \frac{a}{\rho} \cos h \frac{b}{\rho} = \cos h \frac{c}{\rho} \quad . \quad . \quad . \quad . \quad \text{II}'.$$

$$\sin A \cos h \frac{b}{\rho} = \cos B \quad . \quad . \quad . \quad . \quad \text{III}'.$$

$$\tan h \frac{b}{\rho} \cot h \frac{c}{\rho} = \cos A \quad . \quad . \quad . \quad . \quad \text{IV}'.$$

The reader will observe that these are simply the formulæ included in Napier's rules for right-angled spherical triangles. The only modification being that in hyperbolic space hyperbolic functions take the place of circular functions. In other words, the trigonometry of single elliptic space is identical with the geodetic trigonometry of a sphere, although it would not be correct to say that the planimetry of single elliptic space is identical with the geodesy of a sphere.

For hyperbolic space the analogue is the pseudo-spherical surface of Beltrami.

Parallels.

As an illustration of the application of the above formulæ to parallels, I shall find the parallel angle in hyperbolic space.

Taking formula IV′, if we make B move off to an infinite distance, then AB becomes the parallel to CB. A is then the parallel angle corresponding to b. Now since $c = \infty$ we have

$$\cot h \frac{c}{\rho} = 1,$$

therefore

$$\cos A = \tan h \frac{b}{\rho} = \frac{e^{\frac{b}{\rho}} - e^{-\frac{b}{\rho}}}{e^{\frac{b}{\rho}} + e^{-\frac{b}{\rho}}} \quad . \quad . \quad . \quad . \quad (12)$$

Whence $$\tan \frac{A}{2} = e^{-\frac{b}{\rho}},$$

the relation stated above (p. 653).

Non-Intersectors.

As an example of the trigonometry of non-intersectors, I select the following formulæ, the proof of which I leave to the reader.

If KA and LB be two non-intersectors, K and L the points of least distance, $KA = LB = r$, $KAB = LBA = \phi$, $KL = d$, $AB = D$.

Then $$\sin h \frac{D}{2\rho} = \sin h \frac{d}{2\rho} \cos h \frac{r}{\rho} \quad . \quad . \quad . \quad . \quad (13)$$

$$\sin \phi = \frac{\cos h \frac{d}{2\rho}}{\cos h \frac{D}{2\rho}} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The results of (6) to (11) are given by Newcomb (Borchardt, lxxxiii. p. 293) mostly without demonstration. He assumes formula (3) as one of the axioms on which he bases his synthesis. Although I have read most of the original literature on the subject, I am more immediately indebted to Newcomb and Frischauf for the materials of the foregoing sketch.

2. Note on the Theory of the "15 Puzzle." By Professor Tait.

[After this note had been laid before the Council, the new number (vol. ii. No. 4) of the "American Journal of Mathematics" reached us. In it there are exhaustive papers by Messrs Johnston and Story on the subject of this American invention. The principles they give differ only in form of statement from those at which I had independently arrived. I have, therefore, cut down my paper to the smallest dimensions consistent with intelligibility.—P. G. T.]

The essential feature of this puzzle is that the circulation of the pieces is necessarily in rectangular channels. Whether these form four-sided figures, or have any greater (*even*) number of sides, the number of squares in the channel itself is always even. (This is the same thing as saying that a rook's re-entrant path always contains an even number of squares. This follows immediately from the fact that a rook always passes through black and white squares alternately. The same thing is true of a bishop's re-entering

Fig. 1.

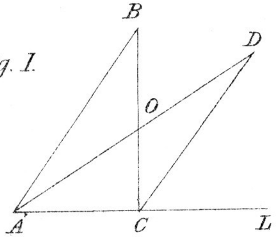


Fig. 2.

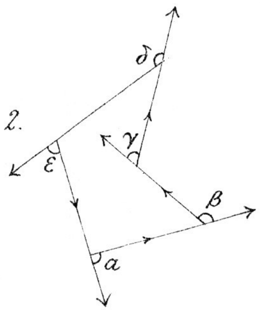


Fig. 11.

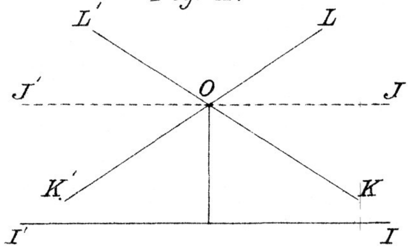


Fig. 12.

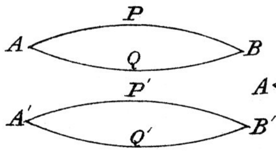


Fig. 13.

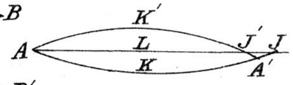


Fig. 3

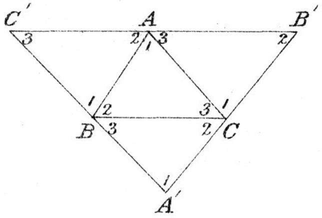


Fig. 4.

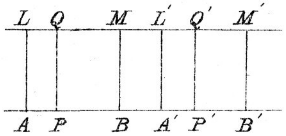


Fig. 5.

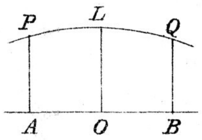


Fig. 14.

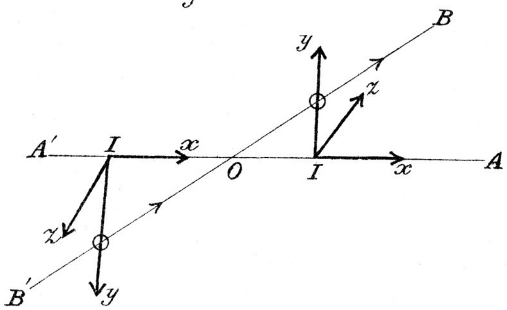


Fig. 15.

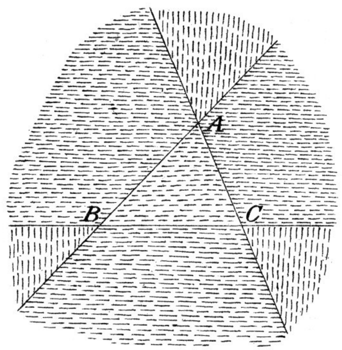


Fig. 6.

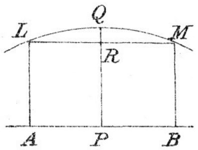


Fig. 7.

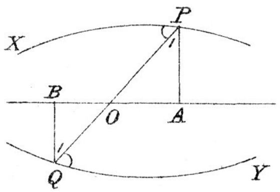


Fig. 8.

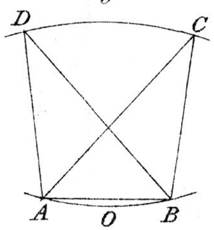


Fig. 16.

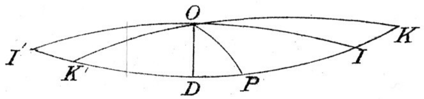


Fig. 9.

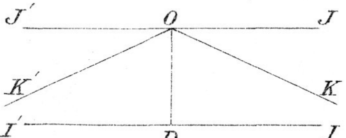


Fig. 10.

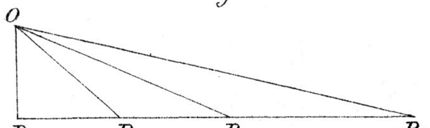


Fig. 17.

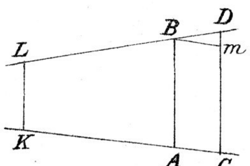


Fig. 18.

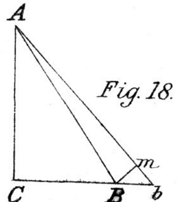
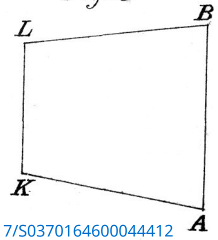


Fig. 19.



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