Lamé's Differential Equation. *By* A. G. Greenhill.

[*Read May* 10*th*, 1888.]

1. This differential equation, in the form

$$\frac{1}{y}\frac{d^2y}{dx^2} = n(n+1)k^2\,\mathrm{sn}^2x + h,$$

employing Jacobi's notation of the elliptic functions, has been solved completely by M. Hermite (*Sur quelques applications des fonctions elliptiques*, Paris, 1885, a collection of articles from the *Comptes Rendus*).

But the advantage of the notation of Weierstrass has been pointed out by M. Halphen (*Mémoire sur la réduction des équations différentielles linéaires aux formes intégrables*, Paris, 1884); the differential equation of Lamé then takes the form

$$\frac{1}{y}\frac{d^2y}{dx^2} = n(n+1)\wp x + h \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

where $\wp x$ is Weierstrass's elliptic function; and the solution of Hermite then takes the form

$$y = CF(x) + C'F(-x) \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where $F(x)$ is a *doubly periodic function of the second kind* (*fonction doublement périodique de seconde espèce*), which, according to Hermite, can be expressed in the form

$$F(x) = D_x^{n-1}\phi(x) - A_1 D_x^{n-3}\phi(x) + A_2 D_x^{n-5}\phi(x) - \ldots \quad \ldots\ldots (3),$$

where $\phi(x)$, called the simple element, expressed by Weierstrass's σ and ζ functions, is of the form

$$\phi(x) = \frac{\sigma(x+\omega)}{\sigma x\,\sigma\omega}\exp(\lambda - \zeta\omega)x \;\ldots\ldots\ldots\ldots\; (4);$$

and Halphen has shown (*Fonctions elliptiques*) that, when $\lambda = 0$, $\phi(x)$ satisfies the differential equation

$$\frac{1}{y}\frac{d^2y}{dx^2} = \frac{\phi'' x}{\phi x} = 2\wp x + \wp\omega \;\ldots\ldots\ldots\ldots\ldots\ldots\; (5),$$

Lamé's differential equation for $n = 1$.

2. In order to obtain the coefficients A_1, A_2, ... in (3), the functions $F(x)$ and $\wp x$ are expanded in powers of x in the neighbourhood of $x = 0$, in the form

$$(-1)^{n-1} F(x) = \frac{(n-1)!}{x^n} - \frac{A_1 (n-3)!}{x^{n-2}} + \frac{A_2 (n-5)!}{x^{n-4}} - \ldots\ldots \quad (6),$$

$$\wp x = \frac{1}{x^2} + \frac{g_2 x^2}{20} + \frac{g_3 x^4}{28} + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$\phi(x) = \frac{1}{x} + \lambda + (\lambda^2 + P_2)\frac{x}{2!} + (\lambda^3 + 3P_2\lambda + P_3)\frac{x^3}{3!} + \ldots$$

(Halphen, *Fonctions elliptiques*, I., p. 231), and then, substituting in the differential equation, we find

$$A_1 = \frac{(n-1)(n-2)}{2(2n-1)} h,$$

$$A_2 = \frac{(n-1)(n-2)(n-3)(n-4)}{8(2n-1)(2n-3)} \left\{ h^2 - \frac{n(n+1)(2n-1)}{10} g_2 \right\},$$

$$A_3 = \ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

3. But, if we suppose that a particular solution $\Phi(x)$ of (1) is of the form

$$\Phi(x) = \frac{\sigma(x+a_1)\,\sigma(x+a_2)\ldots\sigma(x+a_n)}{\sigma a_1\,\sigma a_2\ldots\sigma a_n\,(\sigma x)^n} \exp(-\zeta a_1 - \zeta a_2 - \ldots - \zeta a_n)\,x,$$

the product of n simple elements, each of the form

$$\phi(x) = \frac{\sigma(x+a)}{\sigma a\,\sigma x} \exp(-\zeta a)\,x,$$

and if we seek to satisfy the differential equation (1), we shall have, putting $y = \Phi(x)$,

$$\frac{1}{y}\frac{dy}{dx} = \Sigma\left\{\zeta(x+a) - \zeta x - \zeta a\right\}$$

$$= \Sigma \tfrac{1}{2} \frac{\wp' x - \wp' a}{\wp x - \wp a},$$

$$\frac{1}{y}\frac{d^2 y}{dx^2} = \Sigma\left\{\wp x - \wp(x+a)\right\} + \left(\frac{1}{y}\frac{dy}{dx}\right)^2$$

$$= \Sigma\left\{\wp x - \wp(x+a)\right\} + \tfrac{1}{4}\Sigma\left(\frac{\wp' x - \wp' a}{\wp x - \wp a}\right)^2$$

$$+ \tfrac{1}{2}\Sigma\left(\frac{\wp' x - \wp' a_r}{\wp x - \wp a_r}\right)\left(\frac{\wp' x - \wp' a_s}{\wp x - \wp a_s}\right)$$

$$= 2n\wp x + \Sigma\wp a + \tfrac{1}{2}\Sigma(\;)(\;)$$

$$= n(n+1)\wp x + (2n-1)\Sigma\wp a,$$

provided that we can make

$$\tfrac{1}{2}\Sigma\left(\frac{\wp' x-\wp' a_r}{\wp x-\wp a_r}\right)\left(\frac{\wp' x-\wp' a_s}{\wp x-\wp a_s}\right)=n(n-1)\wp x+(2n-2)\Sigma\wp a,$$

and then $$h=(2n-1)\Sigma\wp a.$$

The necessary conditions for the above relation to hold are

$$\Sigma\wp' a=0,\quad \Sigma\wp a\wp' a=0,\quad \Sigma(\wp a)^2\wp' a=0,\ \ldots\ \Sigma(\wp a)^{n-2}\wp' a=0,$$

as Brioschi has demonstrated (*Comptes Rendus*, XCII., p. 325); then

$$\Sigma(\wp a)^{n-1}\wp' a=C=f'(\rho)\wp' v,$$

putting, with Brioschi, $h=n(2n-1)\rho$, and then $\rho=\wp v$.

Here $f(\wp x)$ denotes the product $\Phi(x)\Phi(-x)$ of two particular solutions $\Phi(x)$ and $\Phi(-x)$ of (1); and thus

$$f(\wp x)=\Pi(\wp v-\wp a),$$

and $$\Sigma\wp a=n\rho=n\wp v,$$

$$h=n(2n-1)\wp v.$$

These conditions of Brioschi may be replaced by

$$\Sigma\wp' a=0,\quad \Sigma\wp''' a=0,\ \ldots\ \Sigma\wp^{(2n-3)}a=0,$$

and $$\Sigma\wp^{(2n-1)}a=Nf'(\rho)\wp' v.$$

4. In order to compare the two solutions $F(x)$ and $\Phi(x)$ of the differential equation (1), it will be necessary to decompose $\Phi(x)$ into simple elements of the form $\phi(x)$ of (4), and of its derivatives (Halphen, *Fonctions elliptiques*, I., p. 228), and then we shall find

$$a_1+a_2+a_3+\ldots+a_n=\omega,$$

$$\zeta\omega-\zeta a_1-\zeta a_2-\ldots-\zeta a_n=\lambda.$$

Differentiating $\Phi(x)$ logarithmically,

$$\frac{\Phi'(x)}{\Phi(x)}=\Sigma\left\{\zeta(x+a)-\zeta x-\zeta a\right\}$$

$$=-\frac{n}{x}+B_2\frac{x}{1!}+B_3\frac{x^2}{2!}+B_4\frac{x^3}{3!}+\ldots$$

where
$$B_2 = -\Sigma \wp a = -n \wp v,$$
$$B_3 = -\Sigma \wp' a = 0,$$
$$B_4 = -\Sigma \wp'' a + \frac{n g_2}{10}, \quad B_5 = 0, \ldots$$

and we have
$$\Phi(x) = \frac{1}{x^n}\left(1 + P_2 \frac{x^2}{2!} + P_4 \frac{x^4}{4!} + \ldots\right),$$

where
$$P_2 = B_2, \quad P_4 = B_4 + 3B_2^2,$$
$$P_6 = B_6 + 15B_2B_4 + 15B_2^3, \ldots.$$

Consequently the decomposition of $\Phi(x)$ into simple elements is of the form
$$(-1)^{n-1}\Phi(x) = \frac{D_x^{n-1}\phi(x)}{(n-1)!} + \frac{P_2}{2!}\frac{D_x^{n-3}\phi(x)}{(n-3)!} + \frac{P_4}{4!}\frac{D_x^{n-5}\phi(x)}{(n-5)!} + \ldots,$$

and
$$F(x) = (n-1)!\,(-1)^{n-1}\Phi(x),$$
$$P_2 = -n\wp v,$$
$$P_4 = \frac{3n}{2n-3}\left\{n(2n-1)\wp^2 v - \frac{1}{10}(n+1)g_2\right\}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

whence
$$\Sigma \wp a = n \wp v,$$
$$\Sigma \wp'' a = -\frac{n^2 \wp'' v}{2n-3},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$\Sigma \wp^{(2n)} a$ being, in general, an integral function of $\wp v$.

5. The differential equation for $Y = f(\wp x)$, the product of $\Phi(x)$ and $\Phi(-x)$, two solutions of Lamé's equation, is easily formed (Halphen, *Fonctions elliptiques*, II., p. 498).

For, if we take the linear differential equation of the second order in its canonical form,
$$\frac{1}{y}\frac{d^2 y}{dx^2} = I,$$

and if y and z are two particular solutions, so that
$$Y = yz,$$

then, denoting differentiation by accents,
$$Y' = y'z + yz',$$
$$Y'' = y''z + 2y'z' + yz''$$
$$= 2Iyz + 2y'z',$$

or
$$Y''-2IY = 2y'z',$$
and
$$Y'''-2IY'-2I'Y = 2y''z'+2y'z''$$
$$= 2I(yz'+y'z) = 2IY',$$
or
$$Y'''-4IY'-2I'Y = 0,$$
a differential equation of the third order for Y, the general solution of which is
$$ay^2+2byz+cz^2,$$
with y^2, yz, and z^2 for particular solutions.

6. In Lamé's differential equation
$$I = n(n+1)\wp x+h,$$
$$Y'''-4\{n(n+1)\wp x+h\}Y'-2n(n+1)\wp' xY = 0,$$
and this equation has, as a particular solution, a rational integral function of $\wp x$ of the n^{th} degree, which is
$$Y = f(\wp x) = \Pi(\wp x-\wp a).$$
Then
$$\frac{Y'}{Y} = \frac{y'}{y}+\frac{z'}{z} = \Sigma\frac{\wp' x}{\wp x-\wp a};$$
also, since
$$y''z-yz'' = 0,$$
therefore
$$y'z-yz' = C,$$
or
$$\frac{y'}{y}-\frac{z'}{z} = \frac{C}{Y} = \frac{C}{\Pi(\wp x-\wp a)},$$
and this, according to Brioschi, may, when resolved into partial fractions, be replaced by
$$\Sigma\frac{\wp' a}{\wp x-\wp a}.$$
Thus
$$\frac{y'}{y} = \Sigma\tfrac{1}{2}\frac{\wp' x+\wp' a}{\wp x-\wp a},$$
$$\frac{z'}{z} = \Sigma'\tfrac{1}{2}\frac{\wp' x-\wp' a}{\wp x-\wp a},$$
leading to the solution given above in § 3.

7. For certain particular values of h we obtain the solutions originally considered by Lamé (Ferrers, *Spherical Harmonics*, Chap. VI.), which

are rational integral functions of $\wp x$ and $\wp' x$; but these have recently been shown by Halphen (*Fonctions elliptiques*, II., p. 273) to be identical with binary forms Z, which are identical with the covariant

$$\Phi_{22} Z_{11} - 2\Phi_{12} Z_{12} + \Phi_{11} Z_{22},$$

composed of Z and a form Φ of the fourth degree.

8. Consider the particular cases of $n = 1$, 2, and 3.

Case I., $n = 1$. The differential equation is then

$$\frac{1}{y}\frac{d^2 y}{dx^2} = 2\wp x + \wp v,$$

the solution of which is

$$y = CF(x) + C'F(-x),$$

where
$$F(x) = \frac{\sigma(x+v)}{\sigma x \, \sigma v} \exp(-x\zeta v)$$

(Halphen, p. 235).

Case II., $n = 2$. The differential equation is then

$$\frac{1}{y}\frac{d^2 y}{dx^2} = 6\wp x + 6\wp v;$$

and then

$$F(x) = D_x \frac{\sigma(x+\omega)}{\sigma x \, \sigma \omega} \exp(\lambda - \zeta\omega) x,$$

$$= -\frac{\sigma(x+a_1)\,\sigma(x+a_2)}{\sigma a_1 \, \sigma a_2 \, (\sigma x)^2} \exp(-\zeta a_1 - \zeta a_2) x,$$

where

$$a_1 + a_2 = \omega,$$

$$\lambda = \zeta\omega - \zeta a_1 - \zeta a_2,$$

$$\wp a_1 + \wp a_2 = 2\wp v,$$

$$\wp' a_1 + \wp' a_2 = 0,$$

$$\wp v - \wp\omega = \frac{\wp'^2 v}{2\wp'' v},$$

$$\lambda = \tfrac{1}{2}\frac{\wp'\omega}{\wp v - \wp\omega} = \zeta\omega - \tfrac{1}{2}\zeta(\omega + v) - \tfrac{1}{2}\zeta(\omega - v) = \frac{\wp'\omega\,\wp'' v}{\wp'^2 v},$$

$$\lambda^2 - 3\wp\omega = 2(\wp v - \wp\omega) = \frac{\wp'\omega}{\lambda}, \text{ \&c.}$$

Case III., $n = 3$. The differential equation is

$$\frac{1}{y}\frac{d^2 y}{dx^2} = 12\wp x + 15\wp v,$$

and then $$F(x) = D_x^2 \frac{\sigma(x+\omega)}{\sigma x \sigma \omega} \exp(\lambda - \zeta\omega) x$$

$$-3\wp v \frac{\sigma(x+\omega)}{\sigma x \sigma \omega} \exp(\lambda - \zeta\omega) x,$$

$$= 2 \frac{\sigma(x+a_1)\sigma(x+a_2)\sigma(x+a_3)}{\sigma a_1 \sigma a_2 \sigma a_3 (\sigma x)^3} \exp(-\zeta a_1 - \zeta a_2 - \zeta a_3) x,$$

where $$a_1 + a_2 + a_3 = \omega,$$

$$\lambda = \zeta\omega - \zeta a_1 - \zeta a_2 - \zeta a_3,$$

$$\wp a_1 + \wp a_2 + \wp a_3 = 3\wp v,$$

$$\wp' a_1 + \wp' a_2 + \wp' a_3 = 0,$$

$$\Sigma \wp a \wp' a = 0,$$

$$\frac{\wp'\omega}{\lambda} = \lambda^2 - 3\wp\omega - 9\wp v$$

$$= 2(\wp v - \wp\omega) - \frac{3\wp'^2 v}{2\wp'' v}, \text{ \&c.}$$

In interpreting the results of M. Hermite (*Sur quelques applications*, &c., pp. 124–129) in this notation, we must take his

$$\Omega = \wp\omega, \quad \Omega_1 = \tfrac{1}{2}\wp'\omega,$$

$$\Omega_2 = \wp^2\omega - \tfrac{1}{6}g_2, \quad \Omega_3 = \tfrac{1}{2}\wp\omega\,\wp'\omega, \ \ldots$$

$$h = -5l = 15\wp v,$$

and generally $$h = n(2n-1)\wp v,$$

$$a = \tfrac{3}{4}g_2, \quad b = \tfrac{27}{4}g_3, \quad 4a^3 - b^2 = \tfrac{27}{16}\Delta, \ \ldots .$$

The cases of $n = 4$ and $n = 5$ are also investigated by Halphen in his *Fonctions elliptiques*, II., p. 529, but the complexity increases very rapidly.

9. The origin of Lamé's differential equation in connection with physical problems relating to confocal quadric surfaces was explained in *Proc. Lond. Math. Soc.*, XVIII., p. 275, employing the notation of Weierstrass.

Putting, in the usual notation,

$$a^2 + \lambda = \wp u - e_1, \quad b^2 + \lambda = \wp u - e_2, \quad c^2 + \lambda = \wp u - e_3,$$

$$a^2 + \mu = \wp v - e_1, \quad b^2 + \mu = \wp v - e_2, \quad c^2 + \mu = \wp v - e_3,$$

$$a^2 + \nu = \wp w - e_1, \quad b^2 + \nu = \wp w - e_2, \quad c^2 + \nu = \wp w - e_3,$$

then Poisson's equation becomes

$$(\wp v-\wp w)\frac{\partial^2\phi}{\partial u^2}+(\wp w-\wp u)\frac{\partial^2\phi}{\partial v^2}+(\wp u-\wp v)\frac{\partial^2\phi}{\partial w^2}=0,$$

and supposing that ϕ may be decomposed into terms of the form UVW, where U is a function of u, V of v, and W of w only, then

$$(\wp v-\wp w)\frac{d^2 U}{U\,du^2}+(\wp w-\wp u)\frac{d^2 V}{V\,dv^2}+(\wp u-\wp v)\frac{d^2 W}{W\,dw^2}=0,$$

equivalent to

$$\frac{1}{U}\frac{d^2 U}{du^2}=g\wp u+h,$$

$$\frac{1}{V}\frac{d^2 V}{dv^2}=g\wp v+h,$$

$$\frac{1}{W}\frac{d^2 W}{dw^2}=g\wp w+h;$$

and g must be put equal to $n(n+1)$ for the solution of these equations to be a *uniform* function.

It is usual to take $e_1 > e_2 > e_3$, so that we must suppose a^2, b^2, c^2 to be in ascending order of magnitude.

10. In dealing with spheroidal harmonics, two of these three quantities are equal.

For oblate spheroids, $b^2 = c^2$, and $e_2 = e_3$; and we can choose the constants so that

$$\wp u-e_1=\cot^2 u,\quad \wp u-e_3=\operatorname{cosec}^2 u,$$

by making $\quad e_1=\frac{2}{3},\quad e_2=e_3=-\frac{1}{3}.$

For prolate spheroids, $a^2=b^2$, and $e_1=e_2$, and then, by making

$$e_1=e_2=\tfrac{1}{3},\quad e_3=-\tfrac{2}{3},$$

$$\wp u-e_3=\coth^2 u,\quad \wp u-e_1=\operatorname{cosech}^2 u.$$

The corresponding Lamé equations are then of the form

$$\frac{1}{y}\frac{d^2 y}{du^2}=n(n+1)\operatorname{cosec}^2 u+h,$$

or

$$=n(n+1)\operatorname{cosech}^2 u+h,$$

the solution of which can be expressed in Hermite's manner by the corresponding degenerate circular or hyperbolic functions.

11. For instance, the solution in general being written

$$y = CF(x) + C'F(-x),$$

for the particular case of $(n = 1)$

$$\frac{1}{y}\frac{d^2y}{dx^2} = 2\operatorname{cosec}^2 x + \cot^2 a$$

we have $$F(x) = \frac{\sin(x+a)}{\sin x \sin a}e^{-x\cot a} = (\cot x + \cot a)\,e^{-x\cot a};$$

and for $(n = 2)$ $$\frac{1}{y}\frac{d^2y}{dx^2} = 6\operatorname{cosec}^2 x + \cot^2 a,$$

$$F(x) = \frac{d}{dx}\left\{\frac{\sin(x+b)}{\sin x \sin b}e^{-x\cot a}\right\},$$

where $$\cot b = \tfrac{1}{3}\cot a - \tfrac{2}{3}\tan a;$$

with corresponding expressions when the circular functions are replaced by hyperbolic functions; and so on for other particular cases which can be indefinitely multiplied.

12. A still more degenerate case is obtained by supposing that

$$e_1 = e_2 = e_3 = 0;$$

then $$a^2 = b^2 = c^2,$$

and $$\wp u = \frac{1}{u^2},$$

and we obtain the ordinary spherical harmonics as the solution of Laplace's equation.

Then Lamé's equation degenerates into

$$\frac{1}{y}\frac{d^2y}{dx^2} = \frac{n(n+1)}{x^2} + h,$$

the differential equation discussed in Boole's *Differential Equations*, p. 424; Forsyth's *Differential Equations*, p. 176; also by Glaisher.

Thus, if we take $n = 1$ and $h = q^2$, we have

$$y = CF(x) + C'F(-x),$$

the solution of $$\frac{1}{y}\frac{d^2y}{dx^2} = \frac{2}{x^2} + q^2,$$

where $$F(x) = \left(\frac{1}{x} + q\right)e^{-qx};$$

while $y = C\left(\frac{1}{x}\cos qx + q\sin qx\right) + C'\left(\frac{1}{x}\sin qx - q\cos qx\right)$

is the solution of $$\frac{1}{y}\frac{d^2y}{dx^2} = \frac{2}{x^2} - q^2.$$

13. The differential equation for the propagation of an impulsive jerk T along a uniform chain lying in a curve on a smooth table is

$$\frac{1}{T}\frac{d^2T}{ds^2} = \frac{1}{\rho^2},$$

and is therefore soluble in the manner explained above for curves in which the intrinsic equation

$$\frac{1}{\rho^2} = I = n(n+1)\wp s + h;$$

but these curves do not appear to possess any simple properties.

14. Consider the differential equation

$$\frac{1}{y}\frac{d^2y}{dx^2} = \tfrac{1}{4}\operatorname{sech}^2 x;$$

this is the form assumed by the differential equation for K and K', given in Cayley's *Elliptic Functions*, p. 51, when we put

$$k^2 = \frac{1}{1+e^{2x}}, \quad k'^2 = \frac{1}{1+e^{-2x}};$$

or $$k^2 = \tfrac{1}{2}(1-\tanh x), \quad k'^2 = \tfrac{1}{2}(1+\tanh x),$$

so that its solution is $$y = CK + C'K';$$

or $$T = CK + C'K'$$

is the solution of $$\frac{1}{T}\frac{d^2T}{ds^2} = \frac{1}{\rho^2},$$

if $$\frac{1}{\rho^2} = \frac{1}{4c^2}\operatorname{sech}^2\frac{s}{c},$$

and then $$k^2 = \frac{1}{1+e^{2s/c}}, \quad k'^2 = \frac{1}{1+e^{-2s/c}},$$

or $$\cos 2\theta = k'^2 - k^2 = \tanh s/c,$$

θ denoting the modular angle, so that

$$\tfrac{1}{2}\pi - 2\theta = \operatorname{gd} s/c.$$

15. In this case the equation of the curve in which the chain lies may be evaluated; for

$$\frac{1}{\rho} = \frac{d\psi}{ds} = -\frac{\operatorname{sech} s/c}{2c},$$

taking the negative sign; and then

$$2\psi = \sin^{-1} \operatorname{sech} s/c,$$

$$\sin 2\psi = \operatorname{sech} s/c,$$

$$\cos 2\psi = \tanh s/c,$$

so that we find $\theta = \psi.$

Then
$$\frac{dx}{ds} = \cos\psi = \surd\{\tfrac{1}{2}(1+\cos 2\psi)\}$$

$$= \surd\{\tfrac{1}{2}(1+\tanh s/c)\} = \frac{e^{s/c}}{\surd(e^{2s/c}+1)},$$

$$\frac{dy}{ds} = \sin\psi = \frac{e^{-s/c}}{\surd(1+e^{-2s/c})};$$

and integrating, from $s = 0$,

$$x/c = \sinh^{-1} e^{s/c} - \sinh^{-1} 1,$$

$$y/c = \sinh^{-1} 1 - \sinh^{-1} e^{-s/c};$$

or, putting $\sinh^{-1} 1 = a = \cosh^{-1}\surd 2 = \log(\surd 2+1)$,

$$e^{s/c} = \sinh(x/c+a),$$

$$e^{-s/c} = \sinh(a-y/c),$$

so that
$$\sinh(x/c+a)\sinh(a-y/c) = 1,$$

the Cartesian equation of the curve of the chain, a catenary in which the linear density varies as $e^{-s/c}$.

16. We see that

$$x/c+a = 0 \quad\text{and}\quad a-y/c = 0$$

are asymptotes; and, changing to them for coordinate axes,

$$\sinh x/c \sinh y/c = 1.$$

This may be written, $\sinh y/c = \operatorname{cosech} x/c,$

$$\cosh y/c = \coth x/c,$$

$$e^{y/c} = \coth x/c + \operatorname{cosech} x/c$$

$$= \coth \tfrac{1}{2}x/c,$$

or $$y/c = \log \coth \tfrac{1}{2} x/c,$$

or $$x/c = \log \coth \tfrac{1}{2} y/c.$$

17. Lamé's equation has received considerable attention of recent years, and has led to the discovery of a large class of differential equations, also soluble by elliptic functions, for which Halphen's Chapter XIII., t. II., *Fonctions elliptiques*, may be consulted.

Besides the references already given, the following articles may be consulted :—Brioschi, *Annali di Matematica*, IX., p. 11 ; Fuchs, *Annali di Matematica*, IX., p. 25 ; Brioschi, *Annali di Matematica*, X., pp. 1 and 74 ; Mittag-Leffler, *Annali di Matematica*, XI., p. 65 ; K. Henn, *Math. Annalen*, XXXI. and XXXIII. ; A. Pick, *Wiener Sitz.*, Nov., 1887.

Thursday, April 11*th*, 1889.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

Mr. C. E. Haselfoot was admitted into the Society.

The following communications were made :—

On the Free Vibrations of an Infinite Plate of Homogeneous Isotropic Elastic Matter : Lord Rayleigh, Sec. R.S.

Ueber die constanten Factoren der Thetareihen im allgemeinen Falle $p = 3$: von Felix Klein in Gottingen.

On the generalised Equations of Elasticity, and their application to the Theory of Light : Prof. K. Pearson.

On the Reduction of a complex Quadratic Surd to a Periodic Continued Fraction : Prof. G. B. Mathews.

Construction du Centre de Courbure de la développée de la Courbe de Contour apparent d'une surface que l'on projette orthogonalement sur un plan : Prof. Mannheim.

The President made a few remarks "On an unsymmetric quadrinomial form of the general plane cubic, for which the fundamental invariants are both binomial only."

The Treasurer also made a brief impromptu communication.

The following presents were received :—

"Proceedings of the Royal Society," Vol. XLV., No. 277.

"Proceedings of the Physical Society of London," Vol. X., Part I.

"The Educational Times," for April.