LVII. On the summation of slowly converging and diverging infinite series

J.R. Young

rise to new decompositions, and in that way only, causing obstruction to the passage of the electric current, I was freed from the necessity of considering the peculiar effects described by that philosopher. I was the more willing to avoid for the present touching upon these, as I must at the same time have entered into the views of Sir Humphry Davy upon the same subject*, and also those of Marianini† and Ritter‡, which are connected with it.

[To be concluded in the next number.]

LVII. *On the Summation of slowly converging and diverging Infinite Series.* By J. R. Young, *Professor of Mathematics in Belfast College.*

COMMODIOUS methods for approximating to the sum of a slowly converging infinite series are very valuable in many departments of physical science. Philosophical inquiries of the highest interest and importance frequently terminate in series of this kind, which would be practically useless, on account of the impossibility of the actual summation, did we not possess the means of transforming them to others of such rapid convergency that the sum of a moderate number of the leading terms may in each case afford a near approximation to that of the entire series. Of all such methods of transformation that furnished by the well-known *Differential Theorem* is, perhaps, the most extensively applicable; and it is, therefore, in one form or other, generally employed for this purpose. In the application, however, of this theorem, as well as in that of all other practical formulæ intended to abridge numerical labour, there is room for the exercise of some ingenuity as to the most advantageous arrangement of the arithmetical process; for if this arrangement be not such as to render the amount of calculation by the proposed formulæ a minimum, it is plain that we do not derive from that formula all the advantage, as a facilitating principle, which it is capable of affording.

In the present paper it is my wish, first to give a short and easy investigation of the differential theorem, and, by deducing it in a somewhat more complete form than that in which it usually appears, to show that it is capable of furnishing, not only a near approximation, but also very close superior and inferior limits, to the sum of a slowly converging or diverging

* Philosophical Transactions, 1826, p. 413. [or Phil. Mag. and Annals, N.S., vol. i. p. 193.—Edit.]

† *Annales de Chimie* [*et de Physique*], tom. xxxiii. pp. 117, 119, &c.

‡ *Journal de Physique*, tom. lvii. pp. 349, 350.

series; and lastly, to exhibit its numerical application in a more commodious form than any in which I have yet seen it.

Let $$a-b\,x+c\,x^2-d\,x^3+\&c. = S$$

$$\therefore\ -b\,x+c\,x^2-d\,x^3+\&c. = S-a$$

$$-b+c\,x-d\,x^2+e\,x^3-\&c. = \frac{S-a}{x}$$

$$\therefore\ \overline{-b-\Delta x+\Delta' x^2-\Delta'' x^3+\&c.} = \frac{\overline{x+1}}{x}(S-a)$$

$$\therefore\ -b\,x-\Delta x^2+\Delta' x^3-\Delta'' x^4+\&c. = (x+1)(S-a)=S'$$

$$\therefore\ S = \frac{S'}{x+1}+a;$$

that is,

$$S = a-\frac{b\,x}{x+1}-\frac{\Delta x^2}{x+1}+\frac{\Delta' x^3}{x+1}-\frac{\Delta'' x^4}{x+1}+\&c.$$

$$= a-\frac{b\,x}{x+1}+\frac{x}{x+1}\left\{0-\Delta x+\Delta' x^2-\Delta'' x^3+\&c.\right\}$$

Hence, treating the series within the brackets as we have treated the original, to which it is similar in form, we have

$$S = a-\frac{b\,x}{x+1}-\frac{\Delta x^2}{(x+1)^2}-\frac{\Delta^2 x^3}{(x+1)^2}+\frac{\Delta^{2\prime} x^4}{(x+1)^2}-\frac{\Delta^{2\prime\prime} x^5}{(x+1)^2}+\&c.$$

Similarly,

$$S = a-\frac{b\,x}{x+1}-\frac{\Delta x^2}{(x+1)^2}-\frac{\Delta^2 x^3}{(x+1)^3}-\frac{\Delta^3 x^4}{(x+1)^3}+\frac{\Delta^{3\prime} x^5}{(x+1)^3}-\&c.$$

$$\vdots$$

$$S = a-\frac{b\,x}{x+1}-\frac{\Delta x^2}{(x+1)^2}-\frac{\Delta^2 x^3}{(x+1)^3}\cdots\cdots-\frac{\Delta^n x^{n+1}}{(x+1)^{n+1}}$$

$$-\frac{\Delta^{n+1} x^{n+2}}{(x+1)^{n+1}}+\&c. \qquad \text{(A.)}$$

These several expressions for S may be regarded as so many differential theorems, but the last is that which corresponds most nearly to the form usually given: it is, however, more efficient, as it shows that if we stop at the term

$$\frac{\Delta^n x^{n+1}}{(x+1)^{n+1}},$$

we get one limit to the sum S, and if we stop at the immediately following term

$$\frac{\Delta^{n+1} x^{n+2}}{(x+1)^{n+1}},$$

we get another limit, in the contrary sense. These limits, as we shall presently see, may be easily narrowed. By considering the series to terminate at the former of the above terms, it will be the same as that investigated by the late Mr. Baron Maseres, in his *Scriptores Logarithmici*, vol. iii. p. 219, and which the Baron considered to be different from that deduced by Simpson, from a formula still more general, in his Mathematical Dissertations, p. 62. The two forms are, however, mutually interchangeable, regard being paid to the signs of the differences.

Suppose in the series S that $a = 0$; then, dividing both sides of (A.) by $-x$, we shall have

$$S = b - cx + dx^2 - ex^3 + \&c. =$$

$$\frac{b}{x+1} + \frac{\Delta x}{(x+1)^2} + \frac{\Delta^2 x^2}{(x+1)^3} \cdots\cdots + \frac{\Delta^n x^n}{(x+1)^{n+1}} + \frac{\Delta^{n+1} x^{n+1}}{(x+1)^{n+1}}$$

$$- \&c. \ \ldots\ldots\ldots\ldots\ldots\ldots \quad (B.)$$

which when $x = 1$ becomes

$$S = b - c + d - e + \&c. =$$

$$\frac{b}{2} + \frac{\Delta}{4} + \frac{\Delta^2}{8} \cdots\cdots + \frac{\Delta^n}{2^{n+1}} + \frac{\Delta^{n+1}}{2^{n+1}} - \&c. \ \ldots \quad (C.)$$

This series, supposing it to terminate at $\frac{\Delta^n}{2^{n+1}}$, furnishes the method proposed by Dr. Hutton (Mathemat. Tracts, vol. i. p. 176). In Dr. Hutton's process the value of S is approached by determining in succession the values of

$$\frac{b}{2},\ \frac{b}{2} + \frac{\Delta}{4},\ \frac{b}{2} + \frac{\Delta}{4} + \frac{\Delta^2}{8},\ \&c.,$$

which is done by finding first the successive sums of a few of the leading terms in the proposed series, taking the arithmetical means between each pair of consecutive results, then the means between each pair of these means, and so on. But simple as this operation undoubtedly is, yet, when the numbers consist of several places of figures, it cannot safely be performed mentally; so that Dr. Hutton's numerical examples, which exhibit only the results of these operations, do not fairly present the whole amount of numerical labour. Still his method is, upon the whole, more easy and convenient than any which I have elsewhere seen, although, like all the other methods, it leaves us in doubt about the accuracy of the final decimal or two in the results determined by it, as indeed, every process of approximation must do which does not furnish limits both above and below the value sought.

In the series above marked (B.) it is plain that if we stop at the term involving Δ^n, we shall obtain an inferior limit to the sum S, and if we take in the term immediately following we shall get a superior limit.

Now, to narrow these limits, we must observe, that since

$$\Delta^{n+1} = \Delta^n - \Delta^{n'},$$

a superior limit will be obtained by adding to the inferior limit the quantity

$$\frac{\Delta^n x^{n+1}}{(x+1)^{n+1}}.$$

Moreover, the inferior limit cannot differ from the entire sum by so much as

$$\frac{\Delta^{n+1} x^{n+1}}{(x+1)^{n+1}},$$

whereas the superior limit differs from the truth by more than

$$\frac{\Delta^{n'} x^{n+1}}{(x+1)^{n+1}},$$

in as much as the succeeding term in the series (B.) is also negative. The inferior limit will therefore be nearer to the truth than the superior, since $\Delta^{n'} > \Delta^{n+1}$ and, consequently, half the sum of the two limits will be a superior limit still nearer. We may conclude, therefore, that if we multiply the final term in the inferior limit by $\frac{x}{2}$, and add the product to that limit, we shall thus obtain a near superior limit.

It is scarcely necessary to observe here, that in what has been hitherto said, the coefficients in the proposed series S are supposed continually to diminish, as also the several series of differences, which supposition is conformable to what usually occurs in practice when S is convergent. When the series is divergent, the formula (B.) or (A.) is still applicable, regard being paid to the signs of the differences.

1. As a first example, let there be proposed the slowly converging series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \&c.,$$

which expresses the length of the quadrantal arc of a circle whose diameter is 1. It will be advisable to actually sum up a few of the leading terms, and to apply the formula (B.), or rather (C.), to the remaining part. The work may be arranged as follows:

+ 1					
− ·3					
+ ·2					
− ·142857					
+ ·111111					
− ·090909	·744012 = sum of first six terms.				
+ ·076923	10256				
− ·066667	7844	2412			
+ ·058823	6191	1653	759	284	
− ·052632	5013	1178	475		
+ ·047619					

The numbers ·076923, 10256, 2412, and 284 are the respective values of b, Δ, Δ^2, Δ^3, and Δ^4 in (C.), and to deduce from these the value of S, it will be necessary merely to add half the last number 284 to the preceding, half the sum to the next, and so on, to the last, adding half the final sum to the number ·744012 previously found. The remainder of the operation, therefore, is as follows:

·076923	10256	2412	759	284
·082767	11688	2864	901	142
·041384	5844	1432	452	
·744012				

·785396 = inferior limit.

$$4 = \frac{142}{2^5}$$

·785400 = superior limit.

$$\therefore\ \cdot 785398 = \frac{\pi}{4} \text{ very nearly.}$$

This value of $\frac{\pi}{4}$ we know from the small interval between the limits cannot possibly differ from the truth by more than a unit in the final decimal. It is, in fact, true even in the last decimal. We have separated the two parts of the process in this example for the purpose of clearer illustration; but they may be combined as in the following.

2. To find the value of the converging series

$$1 - \frac{1}{2^2} + \frac{3^2}{2^2 . 4^2} - \frac{3^2 . 5^2}{2^2 . 4^2 . 6^2} + \frac{3^2 . 5^2 . 7^2}{2^2 . 4^2 . 6^2 . 8^2} - \&c.,$$

which occurs in the expression for the time of vibration of a pendulum in a circular arc,

+	1				
−	·25				
+	·140625				
−	97656				
+	74768				
−	60562	·807175 = sum of first six terms.			
+	50889	7010	1696	548	212
−	43879	5314	1148	336	106
+	38565	4166	812	654	
−	34399	3354	2023	327	
+	31045	8022	1012		
	54900	4011			
	27450				
	·807175				

·834625 = inferior limit.

$3 = \frac{106}{2^5}$

834628 = superior limit.

∴ 834626 = S very nearly.

A value which cannot differ from the truth by more than a unit in the last decimal. It is, in fact, true in all its places.

3. Let it be required to find the value of the diverging series

$$\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \&c.$$

+	·5				
−	·666666				
+	·75				
−	·8				
+	·833333				
−	·857143	−·240476 = sum of first six terms.			
+	875000	−13889	−2778	−758	−253
−	888889	11111	2020	505	126
+	9	09091	1515	884	
−	909091	07576	3220	442	
+	916667	14499	1610		
	867251	7749			
	433625				
	−240476				

·193149 = superior limit.

$4 = \frac{126}{2^5}$

·193145 = inferior limit. Hence ·193147 = S very nearly, which is true in the last decimal.

4. As a last example let the series

$$1 - 2 + 4 - 8 + \&c.$$

be proposed. Then, comparing with the formula (B.), we have $x = 2$ and b, c, &c. each $= 1$,

$$\therefore \text{S} = \frac{b}{x+1} = \frac{1}{3}.$$

In a similar way may the sum of $1-4+9-16$ be readily found.

March 9, 1835. J. R. Young.

[To be continued.]

LVIII. *Experiments on the Action of Metals in determining Gaseous Combination.* By William Charles Henry, *M.D., F.R.S.**

The property, first discovered by Döbereiner† in spongy platina of inducing gaseous union, has been recently shown by Dr. Faraday ‡ to exist also in compact plates of that metal, as well as in plates of palladium and of gold, and hence to be independent of fineness of division or porosity of structure. This important result, while it demonstrated the inadequacy of all theories of the action of platina that had been before proposed, suggested to Dr. Faraday the idea, that gaseous combination, thus induced, may be due partly to the statical relations subsisting between elastic fluids and the solid surfaces by which they are bounded, and partly to an attractive force, exerted at insensible distances, and probably belonging to all bodies. By the joint influence of these two conditions, the gases, he imagines, are so far condensed on the metallic surface "as to be brought within the action of their mutual affinities at the existing temperature." This ingenious theory, though mainly resting on the fact that perfect purity from foreign matter is the only condition in the metallic surface essential to its activity, is further supported by an extensive induction of analogous actions.

Receiving the theory of Dr. Faraday, in so far as it has been developed by him, as correctly representing the nature of these phænomena, it still remains to assign a cause for the

* Read before the Literary and Philosophical Society of Manchester; and communicated by the Author.

† [An account of Prof. Döbereiner's experiments, and of those of other chemists, on the same subject, will be found in Phil. Mag. (First Series), vol. lxii. p. 282—292: see also vol. lxiii. p. 71, and vol. lxiv. p. 3.—Edit.]

‡ [An abstract of Mr. Faraday's Sixth Series of Experimental Researches in Electricity, containing his observations on this subject, was given in Lond. and Edinb. Phil. Mag., vol. iv. p. 291.—Edit.]