PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER HAVING INTEGRAL SYSTEMS FREE FROM PARTIAL QUADRATURES

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1. In his memoir* on partial differential equations and their integrals, Ampère made a selection of one aggregate of equations of the second order. The characteristic property, defining equations which are included in this aggregate, belongs to their general integrals. These integrals are to be explicitly free (or must be capable of expression in a form that is explicitly free) from partial quadratures; and they must be expressible by an equation, or by a set of equations, which occur in finite form. In the simplest cases, the integral is given by means of a single equation: more often, and in the less simple cases, the integral is given by means of three independent equations, each of which has the specified form and which, when taken together, usually may be regarded as determining the dependent variable z, and the two independent variables x and y, in terms of two parameters and two arbitrary functions of those parameters. Thus a primitive of the equation

$$r - t = 2\frac{p}{x}$$

can be exhibited in the form of the single equation

$$z = \phi(y+x) + \psi(y-x) - x\{\phi'(y+x) - \psi'(y-x)\},$$

where ϕ and ψ denote arbitrary functions; it can also be exhibited in the form of the three equations

$$\left.\begin{aligned} x &= \tfrac{1}{2}(u-v) \\ y &= \tfrac{1}{2}(u+v) \\ z &= U + V - \tfrac{1}{2}(u-v)(U' - V') \end{aligned}\right\},$$

where U and V denote arbitrary functions of u and of v respectively.

* *Journal de l'École Polytechnique*, cah. XVII. (1815), pp. 549–611: his definition of the "first class" of equations of the second order is given on p. 558 of the memoir.

A more general form of integral, implying a more extensive aggregate of equations, is suggested by Goursat*; as defined, it is given by means of three equations expressing x, y, z in terms of two parametric variables u and v, of m arbitrary functions of u connected by $m-1$ differential equations, and of n arbitrary functions of v connected by $n-1$ differential equations.

For the purposes of the present discussion, however, it will be sufficient to take the simpler form when there is only a single arbitrary function of u and when there is only a single arbitrary function of v. The object of the discussion is to obtain some of the relations between the integral and the equivalent partial equation of the second order.

Statement of the Problem, with the various Cases that can occur.

2. Accordingly, it is assumed that the equations of the second order to be considered are those characterised by the possession of a general integral given by means of three equations

$$\left.\begin{aligned} x &= f(u, U, U_1, \ldots, v, V, V_1, \ldots) \\ y &= g(u, U, U_1, \ldots, v, V, V_1, \ldots) \\ z &= h(u, U, U_1, \ldots, v, V, V_1, \ldots) \end{aligned}\right\}.$$

In these equations, f, g, h denote specific functions of their arguments; U denotes an arbitrary function of u, and U_1, ... are its successive derivatives; V similarly denotes an arbitrary function of v, and V_1, ... are its successive derivatives. The integral equations are to be finite in form; so that only a finite number of derivatives of U and a finite number of derivatives of V occur. It will be assumed that U_m is the derivative of U of highest order which occurs in the three equations, and that V_n is the derivative of V of highest order which occurs; but it is not assumed (and it is not, in fact, always the case) that U_m must occur in each of the equations or that V_n must occur in each of them.†

* *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*, t. II. (1898), p. 217.

† [In the most familiar examples of equations which have integrals of this type, such as the equation of minimal surfaces, the highest derivatives of U and of V that occur are of the second order. Transformations can be effected which make the highest derivatives appear to be of higher order; in such cases the inference is that, in an extended sense of the word, the transformed expressions are reducible. Instances of equations having integral equivalents, which are not reducible, are provided by Laplace's linear equations

$$s + ap + bq + cz = 0$$

when these are of finite rank in either variable or in both variables. An instance of an equation, Specialised forms of integrals may arise. Thus we may have a primitive of an equation given by

$$\left.\begin{aligned} x &= f(u, U, U_1, \ldots)+F(v, V, V_1, \ldots) \\ y &= g(u, U, U_1, \ldots)+G(v, V, V_1, \ldots) \\ z &= h(u, U, U_1, \ldots)+H(v, V, V_1, \ldots) \end{aligned}\right\}$$

where f, F; g, G; h, H are specific functions of their arguments; and a primitive of another equation might be given by

$$\left.\begin{aligned} x &= f(u, U, U_1, \ldots) \\ y &= g(v, V, V_1, \ldots) \\ z &= h(u, U, U_1, \ldots, v, V, V_1, \ldots) \end{aligned}\right\}.$$

Some illustrations of these two particular forms will be discussed.

But there are limitations upon the degeneration of the forms of the functions f, g, h as they occur initially in the general case. Thus the combinations

$$\left.\begin{aligned} x &= f(u, U, U_1, \ldots) \\ y &= g(u, U, U_1, \ldots) \\ z &= h(v, V, V_1, \ldots) \end{aligned}\right\}, \qquad \left.\begin{aligned} x &= f(u, U, U_1, \ldots) \\ y &= g(v, V, V_1, \ldots) \\ z &= h(u, U, U_1, \ldots) \end{aligned}\right\}$$

are not permissible; at least one of the three functions f, g, h, which represent the values x, y, z, must involve both the parametric variables u and v. We shall therefore assume that h involves u and v; if, in any set of equations, f or g (but not h) should involve u and v, a change of dependent variable can be effected whereby the expression for the new dependent variable does involve u and v.

Having made this explanation and this assumption as regards the expression for z, which is

$$z = h(u, U, U_1, \ldots, v, V, V_1, \ldots),$$

we see that there are three groups of cases, discriminated by the forms of f and of g in the equations

$$x = f, \qquad y = g.$$

which is not linear and the integral equivalent of which involves non-reducible expressions containing derivatives of order higher than the second, is Ampère's equation

$$(r-pt)^2 = q^2rt;$$

a primitive is given by

$$x = \frac{1}{u} V' + U'''', \qquad y = uV' - (u^2U'''' - 2uU''' + 2U''),$$

$$z = \tfrac{1}{3}(2v-u^3)V' - \tfrac{2}{3}V + u^4U'''' - 4u^3U''' + 12u^2U'' - 24uU' + 24U.$$

Other instances can be obtained by taking $f(\lambda)$ and $g(\mu)$, in the equations at the end of § 26, to be polynomials not consisting of a single term alone.—*Added February 8th*, 1907.]

(I.) Each of the functions f and g involves both the quantities u and v, either with or without the arbitrary functions U and V.

Next, one of the two functions involves both the quantities u and v, either with or without the arbitrary functions U and V, while the other function involves only one of these quantities, again either with or without the associated arbitrary function. There really are four cases; but, by interchange of the variables x and y and by interchange of the parameters u and v, they can be assigned to the single case :—

(II.) The function f involves both u and v, and the function g involves v only : it being understood that the associated arbitrary functions can occur.

Lastly, one of the functions may involve only one of the parameters u and v, and the other may involve only the other of those parameters: the two possible cases can be merged into one by interchange of x and y. So we have :—

(III.) The function f involves u only, and the function g involves v only : with an understanding as to the possible occurrence of U and of V respectively.

Of these three cases to be discussed in succession, it is manifest that (I.) is the most general in form. Relative formal simplifications will occur initially in the remaining cases, though (as will appear) this simplification will not persist throughout the analysis : but the initial simplifications enable us to use the results of Case (I.) for the other two cases, and therefore to discuss the latter more briefly.

Also, it will be found that relative complications are caused in different sub-cases by conditions which, at first sight, might be deemed likely to simplify the analytical results.

A Lemma.

3. A preliminary lemma is required. Several of the results are made to depend upon the elimination of two variables ξ and η between three equations of the form

$$a\xi^2+2b\xi+c=0, \qquad a'\eta^2+2b'\eta+c'=0,$$
$$k\xi\eta+l\xi+m\eta+n=0;$$

and therefore an expression for the eliminant is needed. We have

$$a\xi+b = (b^2-ac)^{\frac{1}{2}} = d, \text{ say,}$$
$$a'\eta+b' = (b'^2-a'c')^{\frac{1}{2}} = d', \text{ say;}$$

thus
$$k(d-b)(d'-b')+la'(d-b)+ma(d'-b')+naa'=0,$$

that is,
$$kdd'+Ld+Md'+N=0,$$

where $$L = la' - kb', \qquad M = ma - kb,$$
$$N = naa' - mab' - lba' + kbb'.$$

The rationalised form of the last equation is

$$(N^2 - L^2 d^2 - M^2 d'^2 + k^2 d^2 d'^2)^2 - 4\,(kN - LM)^2\, d^2 d'^2 = 0.$$

When the values of d^2 and d'^2 are substituted, and some slight reduction is effected, the eliminant becomes

$$\Theta^2 = 4\,(kn - lm)^2\,(b^2 - ac)(b'^2 - a'c'),$$

where $$\Theta = k^2 cc' - 2klcb' - 2kmbc' + 2knbb' + l^2 ca' + 2lmbb' - 2lnba'$$
$$+ m^2 ac' - 2mnab' + n^2 aa'.$$

The umbral expression of the eliminant is simpler: but it appears to be less useful.

A General Property, stated by Darboux.

4. As already stated, the main purpose of the investigation is the determination of differential equations, which possess integrals of the assigned type, as well as the construction of primitives of these equations. But when the equations, thus characterised by the kind of primitive possessed, have been determined, the construction of their primitive can sometimes be effected more directly by utilising a definite property: *every such equation is integrable by Darboux's method.** The property can be formally established as follows.

The equations $$x = f, \qquad y = g, \qquad z = h$$

involve the variables u and v, the arbitrary function U and its derivatives up to U_m, and the arbitrary function V and its derivatives up to V_n. Hence, as

$$\frac{dh}{du} = p\,\frac{dx}{du} + q\,\frac{dy}{du},$$

$$\frac{dh}{dv} = p\,\frac{dx}{dv} + q\,\frac{dy}{dv},$$

where complete derivatives with regard to u and to v are taken, the quantities p and q will generally involve derivatives of U up to order $m+1$ and derivatives of V up to order $n+1$. Similarly, r, s, t will generally

* This result is proved in another (but equivalent) form by Goursat, *l.c.*, p. 227, being derived from the consideration of the characteristics; it was first stated by Darboux himself, *Comptes Rendus*, t. LXX. (1870), p. 748. For references to Darboux's method, and for a general discussion of the process, see chapter xviii. of Part IV. of my *Theory of Differential Equations*.

involve derivatives of U up to order $m+2$ and derivatives of V up to order $n+2$; and derivatives of z of order μ will generally involve derivatives of U up to order $m+\mu$ and derivatives of V up to order $n+\mu$.

Accordingly, construct all the derivatives of z up to order μ inclusive; when these are explicitly obtained, the total number of equations, expressing x, y, z and the derivatives of z, is

$$3+2+3+\ldots+(\mu+1) = \tfrac{1}{2}\mu(\mu+3)+3.$$

These equations involve u, U, U_1, ..., $U_{m+\mu}$, that is, $m+\mu+2$ quantities dependent upon u; and they also involve v, V, V_1, ..., $V_{n+\mu}$, that is, $n+\mu+2$ quantities dependent upon v.

If, then, μ denotes the smallest value of μ for which

$$\tfrac{1}{2}\mu(\mu+3)+3 > n+\mu+2,$$

that is, for which
$$\tfrac{1}{2}\mu(\mu+1) > n-1,$$

then generally the $n+\mu+2$ quantities v, V, V_1, ..., $V_{n+\mu}$ can be eliminated: and we should have

$$\tfrac{1}{2}\mu(\mu+1)-n+1$$

equations, involving derivatives of z up to order μ, and involving also u and an arbitrary function of u with its derivatives. These equations manifestly are not derivatives of the supposed equation of the second order, because of the presence of the arbitrary function U; hence they are equations of order μ, compatible with the equation of the second order, involving one arbitrary function, and therefore derivable by Darboux's method.

Similarly, if λ denote the smallest value of λ for which

$$\tfrac{1}{2}\lambda(\lambda+3)+3 > m+\lambda+2,$$

that is, for which
$$\tfrac{1}{2}\lambda(\lambda+1) > m-1,$$

then generally the $m+\lambda+2$ quantities u, U, U_1, ..., $U_{m+\lambda}$ can be eliminated: and we should have

$$\tfrac{1}{2}\lambda(\lambda+1)-m+1$$

equations, involving derivatives of z up to order λ, and involving also v and an arbitrary function of v with its derivatives. As before, these are equations of order λ, compatible with the equation of the second order, involving one arbitrary function, and therefore derivable by Darboux's method.

But it is to be remembered that this generality is only formal. For particular equations, the result of eliminating u, U, ..., $U_{m+\mu}$ may lead to an equation that does not involve v, V, ..., $V_{n+\mu}$: or some of the equations may be evanescent: and, in such instances, it may be necessary to proceed to the construction of derivatives of order higher than the minimum which has been indicated. On the other hand, combinations of the derivatives of U and of V may disappear: then it may be unnecessary to proceed to equations of the orders indicated.

Preliminary Formulæ.

5. We proceed now to the formation of the values of the first and the second derivatives of z, denoted by p, q, r, s, t as usual: they will be constructed initially for the most general case, when

$$\left.\begin{aligned} x &= f(u, U, U_1, \ldots, U_m, v, V, V_1, \ldots, V_n) \\ y &= g(u, U, U_1, \ldots, U_m, v, V, V_1, \ldots, V_n) \\ z &= h(u, U, U_1, \ldots, U_m, v, V, V_1, \ldots, V_n) \end{aligned}\right\},$$

and the values for the restricted cases are then derivable by imposing the respective restrictions upon the forms of f, g, h. Complete derivatives with regard to u are indicated by means of a suffix 1 and with regard to v by means of a suffix 2; thus

$$\frac{df}{du} = f_1, \quad \frac{d^2f}{du^2} = f_{11}, \quad \frac{df}{dv} = f_2, \quad \frac{d^2f}{dv^2} = f_{22}, \quad \frac{d^2f}{du\,dv} = f_{12};$$

and so for other functions. With this notation, we have

$$\begin{aligned} h_1 du + h_2 dv &= dz \\ &= p\,dx + q\,dy \\ &= p(f_1 du + f_2 dv) + q(g_1 du + g_2 dv); \end{aligned}$$

and therefore $$h_1 = pf_1 + qg_1, \qquad h_2 = pf_2 + qg_2,$$

so that $$p = \frac{h_1 g_2 - h_2 g_1}{f_1 g_2 - f_2 g_1}, \qquad q = \frac{-h_1 f_2 + h_2 f_1}{f_1 g_2 - f_2 g_1}.$$

Now the quantity $f_1 g_2 - f_2 g_1$ cannot vanish: it does not vanish in virtue of the equations $x = f$, $y = g$, $z = h$; and, if it vanished identically, we should have

$$\frac{dx}{du}\frac{dy}{dv} - \frac{dx}{dv}\frac{dy}{du} = 0,$$

also identically—a relation which would mean that x and y are expressible in terms of one another by an equation otherwise involving constants alone. Similarly, the quantities $h_1 g_2 - h_2 g_1$ and $-h_1 f_2 + h_2 f_1$ do not vanish. Thus the values of p and q, as obtained, are neither zero, infinite, nor indeterminate, though (in restricted cases) it may happen that the expressions for their values acquire simplified forms.

Again, from $h_1 = pf_1 + qg_1$, we have

$$h_{11} = pf_{11} + qg_{11} + (rf_1 + sg_1) f_1 + (sf_1 + tg_1) g_1 ;$$

and so for the other derivatives of the second order. Hence

$$\left.\begin{aligned} h_{11} - pf_{11} - qg_{11} &= rf_1^2 \; + 2sf_1 g_1 + tg_1^2 \\ h_{12} - pf_{12} - qg_{12} &= rf_1 f_2 + s(f_1 g_2 + f_2 g_1) + tg_1 g_2 \\ h_{22} - pf_{22} - qg_{22} &= rf_2^2 \; + 2sf_2 g_2 + tg_2^2 \end{aligned}\right\} ;$$

the individual values of r, s, t, if required, can be obtained simply by the resolution of these equations, or from the relations

$$r = \frac{p_1 g_2 - p_2 g_1}{f_1 g_2 - f_2 g_1},$$

$$s = \frac{-p_1 f_2 + p_2 f_1}{f_1 g_2 - f_2 g_1} = \frac{q_1 g_2 - q_2 g_1}{f_1 g_2 - f_2 g_1},$$

$$t = \frac{-q_1 f_2 + q_2 f_1}{f_1 g_2 - f_2 g_1}.$$

The three equations involving r, s, t, either in the form given or in equivalent forms, together with the two equations expressing p and q, and the three original equations expressing x, y, and z, are the fundamental relations for the determination of the problem. It is from among these eight relations that the quantities u, v, U and its derivatives, V and its derivatives, have to be eliminated.

6. Again, we have

$$f_1 = \frac{\partial f}{\partial U_m} U_{m+1} + \sum_{\mu=0}^{m-1} \frac{\partial f}{\partial U_\mu} U_{\mu+1} + \frac{\partial f}{\partial u}$$

$$= \frac{\partial f}{\partial U_m} U_{m+1} + F_1,$$

say ; and, similarly,

$$g_1 = \frac{\partial g}{\partial U_m} U_{m+1} + G_1,$$

$$h_1 = \frac{\partial h}{\partial U_m} U_{m+1} + H_1,$$

where F_1, G_1, H_1 involve only derivatives of U and V which occur in x, y, z.

Also
$$f_2 = \frac{\partial f}{\partial V_n} V_{n+1} + \sum_{\mu=0}^{n-1} \frac{\partial f}{\partial V_\mu} V_{\mu+1} + \frac{\partial f}{\partial v}$$
$$= \frac{\partial f}{\partial V_n} V_{n+1} + F_2,$$
say; and, similarly,
$$g_2 = \frac{\partial g}{\partial V_n} V_{n+1} + G_2,$$
$$h_2 = \frac{\partial h}{\partial V_n} V_{n+1} + H_2,$$
where also F_2, G_2, H_2 involve only derivatives of U and V which occur in x, y, z. Let
$$\left.\begin{aligned} A &= \frac{\partial h}{\partial U_m} - p\frac{\partial f}{\partial U_m} - q\frac{\partial g}{\partial U_m} \\ B &= \frac{\partial h}{\partial V_n} - p\frac{\partial f}{\partial V_n} - q\frac{\partial g}{\partial V_n} \end{aligned}\right\};$$
then
$$\left.\begin{aligned} AU_{m+1} + H_1 - pF_1 - qG_1 &= 0 \\ BV_{n+1} + H_2 - pF_2 - qG_2 &= 0 \end{aligned}\right\}.$$
If A is different from zero, the first equation expresses U_{m+1} in terms of p, q, and of quantities that occur in x, y, z; while, if A is zero, no such expression for U_{m+1} is derivable. Similarly, when B is different from zero, the second equation expresses V_{n+1} in terms of p, q, and of quantities that occur in x, y, z; but, when B is zero, no such expression for V_{n+1} is derivable.

Again, we have
$$f_{11} = \frac{\partial f}{\partial U_m} U_{m+2} + \frac{\partial^2 f}{\partial U_m^2} U_{m+1}^2$$
$$+ \left\{ \frac{\partial f}{\partial U_{m-1}} + 2\sum_{\mu=0}^{m-1} \left(U_{\mu+1} \frac{\partial^2 f}{\partial U_\mu \partial U_m} \right) + 2\frac{\partial^2 f}{\partial u \partial U_m} \right\} U_{m+1} + F_{11},$$
where F_{11} involves only quantities that occur in x, y, z. There are similar expressions for g_{11}, h_{11}; hence, writing
$$a_0 = \frac{\partial^2 h}{\partial U_m^2} - p\frac{\partial^2 f}{\partial U_m^2} - q\frac{\partial^2 g}{\partial U_m^2},$$
$$2b_0 = \frac{\partial h}{\partial U_{m-1}} + 2\sum_{\mu=0}^{m-1} \left(U_{\mu+1} \frac{\partial^2 h}{\partial U_\mu \partial U_m} \right) + 2\frac{\partial^2 h}{\partial u \partial U_m}$$
$$-p\left\{ \frac{\partial f}{\partial U_{m-1}} + 2\sum_{\mu=0}^{m-1} \left(U_{\mu+1} \frac{\partial^2 f}{\partial U_\mu \partial U_m} \right) + 2\frac{\partial^2 f}{\partial u \partial U_m} \right\}$$
$$-q\left\{ \frac{\partial g}{\partial U_{m-1}} + 2\sum_{\mu=0}^{m-1} \left(U_{\mu+1} \frac{\partial^2 g}{\partial U_\mu \partial U_m} \right) + 2\frac{\partial^2 g}{\partial u \partial U_m} \right\},$$
$$c_0 = H_{11} - pF_{11} - qG_{11},$$

we have an expression for $h_{11}-pf_{11}-qg_{11}$. Similarly, writing

$$a_0' = \frac{\partial^2 h}{\partial V_n^2} - p\frac{\partial^2 f}{\partial V_n^2} - q\frac{\partial^2 g}{\partial V_n^2},$$

$$2b_0' = \frac{\partial h}{\partial V_{n-1}} + 2\sum_{\mu=0}^{n-1}\left(V_{\mu+1}\frac{\partial^2 h}{\partial V_\mu \partial V_n}\right) + 2\frac{\partial^2 h}{\partial v \partial V_n}$$

$$-p\left\{\frac{\partial f}{\partial V_{n-1}} + 2\sum_{\mu=0}^{n-1}\left(V_{\mu+1}\frac{\partial^2 f}{\partial V_\mu \partial V_n}\right) + 2\frac{\partial^2 f}{\partial v \partial V_n}\right\}$$

$$-q\left\{\frac{\partial g}{\partial V_{n-1}} + 2\sum_{\mu=0}^{n-1}\left(V_{\mu+1}\frac{\partial^2 g}{\partial V_\mu \partial V_n}\right) + 2\frac{\partial^2 g}{\partial v \partial V_n}\right\},$$

$$c_0' = H_{22} - pF_{22} - qG_{22},$$

we have an expression for $h_{22}-pf_{22}-qg_{22}$. Again, we have

$$f_{12} = \frac{\partial^2 f}{\partial U_m \partial V_n} U_{m+1}V_{n+1} + \frac{\partial F_2}{\partial U_m} U_{m+1} + \frac{\partial F_1}{\partial V_n} V_{n+1} + F_{12},$$

where F_{12} involves only quantities that occur in x, y, z. There are similar expressions for g_{12} and h_{12}; hence, writing

$$k_0 = \frac{\partial^2 h}{\partial U_m \partial V_n} - p\frac{\partial^2 f}{\partial U_m \partial V_n} - q\frac{\partial^2 g}{\partial U_m \partial V_n},$$

$$l_0 = \frac{\partial H_2}{\partial U_m} - p\frac{\partial F_2}{\partial U_m} - q\frac{\partial G_2}{\partial U_m},$$

$$m_0 = \frac{\partial H_1}{\partial V_n} - p\frac{\partial F_1}{\partial V_n} - q\frac{\partial G_1}{\partial V_n},$$

$$n_0 = H_{12} - pF_{12} - qG_{12},$$

we have an expression for $h_{12}-pf_{12}-qg_{12}$. The three expressions thus obtained are

$$\left.\begin{aligned} h_{11}-pf_{11}-qg_{11} &= AU_{m+2} + a_0 U_{m+1}^2 + 2b_0 U_{m+1} + c_0 \\ h_{12}-pf_{12}-qg_{12} &= k_0 U_{m+1}V_{n+1} + l_0 U_{m+1} + m_0 V_{n+1} + n_0 \\ h_{22}-pf_{22}-qg_{22} &= BV_{n+2} + a_0' V_{n+1}^2 + 2b_0' V_{n+1} + c_0' \end{aligned}\right\}.$$

Thus the quantity U_{m+2} occurs only (if at all) in the combination $h_{11}-pf_{11}-qg_{11}$; and V_{n+2} occurs only (if at all) in the combination $h_{22}-pf_{22}-qg_{22}$.

The final differential equation, in some form

$$F(x, y, z, p, q, r, s, t) = 0,$$

is to be the result of eliminating u, v, U, and V and their derivatives among (i.) these three equations when the values of the left-hand sides in terms of r, s, t in § 5 are substituted, (ii.) the two equations giving p and q, and (ii.) the three original equations expressing x, y, z.

The various cases, depending (as in § 2) upon the forms of f, g, h, will be considered in turn.

CASE I.

7. In this case, all the three functions f, g, h involve u and v. Arbitrary functions U and V, as well as their derivatives, can occur in f, g, h.

There are four sub-cases, according to the non-evanescence or the evanescence of A and of B separately: we shall take

sub-case (1), when $A \neq 0,\ B \neq 0$;
„ (2), „ $A \neq 0,\ B = 0$;
„ (3), „ $A = 0,\ B \neq 0$;
„ (4), „ $A = 0,\ B = 0$.

These will be discussed in succession.

First Sub-case.

8. I., (1).—The equations from which the parametric quantities and the arbitrary functions are to be eliminated are

$$x = f, \qquad y = g, \qquad z = h;$$

$$\left.\begin{aligned} AU_{m+1}+H_1-pF_1-qG_1 &= 0 \\ BV_{m+1}+H_2-pF_2-qG_2 &= 0 \end{aligned}\right\};$$

$$AU_{m+2}+a_0U_{m+1}^2+2b_0U_{m+1}+c_0 = rf_1^2+2sf_1g_1+tg_1^2,$$

$$BV_{n+2}+a_0'V_{n+1}^2+2b_0'V_{n+1}+c_0' = rf_2^2+2sf_2g_2+tg_2^2,$$

$$k_0U_{m+1}V_{n+1}+l_0U_{m+1}+m_0V_{n+1}+n_0 = rf_1f_2+s(f_1g_2+f_2g_1)+tg_1g_2.$$

The last equation but two is the only equation which involves U_{m+2}; it can be ignored simultaneously with U_{m+2}. Similarly, the last equation but one can be ignored simultaneously with V_{n+2}. To eliminate U_{m+1} and V_{n+1}, which occur explicitly on the left-hand side and implicitly on the right-hand side, we use the values of U_{m+1} and V_{n+1} in the forms

$$U_{m+1} = -\frac{1}{A}(H_1-pF_1-qG_1),$$

$$V_{n+1} = -\frac{1}{B}(H_2-pF_2-qG_2);$$

and the values of f_1, f, g_1, g_2 are then

$$f_1 = U_{m+1}\frac{\partial f}{\partial U_m}+F_1 = \frac{1}{A}\left\{F_1\frac{\partial h}{\partial U_m}-H_1\frac{\partial f}{\partial U_m}+q\left(G_1\frac{\partial f}{\partial U_m}-F_1\frac{\partial g}{\partial U_m}\right)\right\},$$

$$g_1 = U_{m+1}\frac{\partial g}{\partial U_m}+G_1 = \frac{1}{A}\left\{G_1\frac{\partial h}{\partial U_m}-H_1\frac{\partial g}{\partial U_m}+p\left(F_1\frac{\partial g}{\partial U_m}-G_1\frac{\partial f}{\partial U_m}\right)\right\},$$

$$f_2 = V_{n+1}\frac{\partial f}{\partial V_n}+F_2 = \frac{1}{B}\left\{F_2\frac{\partial h}{\partial V_n}-H_2\frac{\partial f}{\partial V_n}+q\left(G_2\frac{\partial f}{\partial V_n}-F_2\frac{\partial g}{\partial V_n}\right)\right\},$$

$$g_2 = V_{n+1}\frac{\partial g}{\partial V_n}+G_2 = \frac{1}{B}\left\{G_2\frac{\partial h}{\partial V_n}-H_2\frac{\partial g}{\partial V_n}+p\left(F_2\frac{\partial g}{\partial V_n}-G_2\frac{\partial f}{\partial V_n}\right)\right\}.$$

When all these values are substituted in the equation

$$k_0U_{m+1}V_{n+1}+l_0U_{m+1}+m_0V_{n+1}+n_0 = rf_1f_2+s(f_1g_2+f_2g_1)+tg_1g_2,$$

and, when terms are collected, the new equation can (after some simple reductions) be expressed in the form

$$W = N(q^2r-2pqs+p^2t)+2M(qr-ps)+2L(pt-qs)+rD_1-2sD_2+tD_3,$$

where, if

$$\left.\begin{aligned}\alpha &= H_1\frac{\partial g}{\partial U_m}-G_1\frac{\partial h}{\partial U_m}\\ \beta &= F_1\frac{\partial h}{\partial U_m}-H_1\frac{\partial f}{\partial U_m}\\ \gamma &= G_1\frac{\partial f}{\partial U_m}-F_1\frac{\partial g}{\partial U_m}\end{aligned}\right\},\qquad \left.\begin{aligned}\alpha' &= H_2\frac{\partial g}{\partial V_n}-G_2\frac{\partial h}{\partial V_n}\\ \beta' &= F_2\frac{\partial h}{\partial V_n}-H_2\frac{\partial f}{\partial V_n}\\ \gamma' &= G_2\frac{\partial f}{\partial V_n}-F_2\frac{\partial g}{\partial V_n}\end{aligned}\right\},$$

the values of the coefficients in the equation are

$$D_3 = \alpha\alpha',\qquad 2M = \gamma\beta'+\beta\gamma',$$
$$D_1 = \beta\beta',\qquad 2L = \alpha\gamma'+\gamma\alpha',$$
$$N = \gamma\gamma',\qquad 2D_2 = \beta\alpha'+\alpha\beta',$$

and

$$W = k_0(H_1-pF_1-qG_1)(H_2-pF_2-qG_2)$$
$$-l_0B(H_1-pF_1-qG_1)-m_0A(H_2-pF_2-qG_2)+n_0AB.$$

Now $$4(M^2-ND_1) = (\gamma\beta'-\beta\gamma')^2 = 4P^2,\ \text{say},$$
$$4(L^2-ND_3) = (\alpha\gamma'-\gamma\alpha')^2 = 4Q^2,\ \text{say}:$$

then $$4(ND_2-LM) = (\gamma\beta'-\beta\gamma')(\alpha\gamma'-\gamma\alpha') = 4PQ.$$

With these values, the equation has the form

$$\{(Nq+M)^2-P^2\}r-2\{(Np+L)(Nq+M)+PQ\}s+\{(Np+L)^2-Q^2\}t$$
$$= NW.$$

Note 1.—It is an immediate corollary that, if the equation

$$(Nq^2+2Mq+D_1)r-2(Npq+Mp+Lq+D_2)s+(Np^2+2Lp+D_3)t = W$$

is to belong to Ampère's first class and has an integral possessing the preceding character with the latent limitations, a necessary condition is

$$\begin{vmatrix} N, & M, & L \\ L, & D_2, & D_3 \\ M, & D_1, & D_2 \end{vmatrix} = 0,$$

provided N is not zero: while, when N is zero, a necessary condition is

$$L^2D_1-2LMD_2+M^2D_3 = 0.$$

But the equation $\quad (1+q^2)r-2pqs+(1+p^2)t = 0$

does not obey the test: the latent limitations are not satisfied.

Note 2.—The preceding result implicitly assumes that F_1, G_1, H_1 do not vanish simultaneously: likewise as to F_2, G_2, H_2. But, if

$$F_1 = 0, \qquad G_1 = 0, \qquad H_1 = 0;$$

$$F_2 = 0, \qquad G_2 = 0, \qquad H_2 = 0,$$

then

$$\left.\begin{aligned} A &= \frac{\partial h}{\partial U_m} - p\frac{\partial f}{\partial U_m} - q\frac{\partial g}{\partial U_m} = 0 \\ B &= \frac{\partial h}{\partial V_n} - p\frac{\partial f}{\partial V_n} - q\frac{\partial g}{\partial V_n} = 0 \end{aligned}\right\};$$

and the assumption as to elimination is not justified.

Note 3.—The quantities α, β, γ; α', β', γ'; k_0, l_0, m_0, n_0 (except in so far as the last four involve p and q) contain only quantities which occur in the values of x, y, z: hence, if a differential equation is to emerge as the result of the elimination, the ratios of the quantities L, M, N, P, Q, and the coefficients of combinations of p and q in W, are functions of x, y, z alone.

9. Comparing the coefficients in the differential equation with their values in terms of α, β, γ; α', β', γ', we have

$$\begin{aligned}(\alpha\beta'-\alpha'\beta)^2 &= 4(D_2^2-D_1D_3) \\ &= \frac{4}{N^2}\{(LM+PQ)^2-(M^2-P^2)(L^2-Q^2)\} \\ &= \frac{4}{N^2}(LP+MQ)^2,\end{aligned}$$

so that

$$\alpha\beta'-\alpha'\beta = \frac{2}{N}(LP+MQ).$$

Hence $$\alpha\beta' = D_2 + \frac{LP+MQ}{N} = \frac{(M+P)(L+Q)}{N},$$

$$\alpha'\beta = D_2 - \frac{LP+MQ}{N} = \frac{(M-P)(L-Q)}{N},$$

$$\gamma\beta' = M+P, \quad \beta\gamma' = M-P, \quad \alpha\gamma' = L+Q, \quad \alpha'\gamma = L-Q,$$

$$\gamma\gamma' = N, \quad \beta\beta' = D_1, \quad \alpha\alpha' = D_3;$$

so that $$\left.\begin{aligned} \alpha : \beta : \gamma &= L+Q : M-P : N \\ \alpha' : \beta' : \gamma' &= L-Q : M+P : N \end{aligned}\right\}.$$

Also $$\begin{aligned} \alpha U_{m+1} &= H_1(g_1-G_1)-G_1(h_1-H_1) \\ &= H_1 g_1 - h_1 G_1, \\ \beta U_{m+1} &= F_1 h_1 - f_1 H_1, \\ \gamma U_{m+1} &= G_1 f_1 - g_1 F_1; \end{aligned}$$

so that $$\alpha f_1 + \beta g_1 + \gamma h_1 = 0,$$

that is, $$(L+Q)\frac{dx}{du} + (M-P)\frac{dy}{du} + N\frac{dh}{du} = 0.$$

Similarly, $$(L-Q)\frac{dx}{dv} + (M+P)\frac{dy}{dv} + N\frac{dh}{dv} = 0.$$

Again, having regard to the initial and the final form of the differential equation, we have

$$\begin{aligned} \theta f_1 f_2 &= (Nq+M)^2 - P^2, \\ \theta(f_1 g_2 + f_2 g_1) &= -2\{Np+L)(Nq+M)+PQ\}, \\ \theta g_1 g_2 &= (Np+L)^2 - Q^2, \\ \theta(h_{12} - p f_{12} - q g_{12}) &= NW. \end{aligned}$$

From the first three, we have

$$\theta^2 (f_1 g_2 - f_2 g_1)^2 = 4\{P(Np+L)+Q(Nq+M)\}^2,$$

that is, $$\theta(f_1 g_2 - f_2 g_1) = +2\{LP+MQ+N(Pp+Qq)\}.$$

Hence

$$\begin{aligned} \theta f_1 g_2 &= -\{(Np+L)(Nq+M)+PQ\} + \{(Np+L)P+(Nq+M)Q\} \\ &= -(Np+L-Q)(Nq+M-P). \end{aligned}$$

Consequently,
$$\frac{f_2}{g_2} = \frac{\theta f_1 f_2}{\theta f_1 g_2} = -\frac{Nq+M+P}{Np+L-Q},$$
$$\frac{f_1}{g_1} = \frac{\theta f_1 g_2}{\theta g_1 g_2} = -\frac{Nq+M-P}{Np+L+Q};$$
and therefore
$$\frac{f_1}{Nq+M-P} = \frac{g_1}{-(Np+L+Q)} = \frac{h_1(=pf_1+qg_1)}{(M-P)p-(L+Q)q},$$
$$\frac{f_2}{Nq+M+P} = \frac{g_2}{-(Np+L-Q)} = \frac{h_2(=pf_2+qg_2)}{(M+P)p-(L-Q)q}.$$
Again, we have
$$\frac{h_{12}-pf_{12}-qg_{12}}{f_1g_2-f_2g_1} = \frac{\frac{1}{2}NW}{(Np+L)P+(Nq+M)Q},$$
which, on writing
$$W = [1, p, q]_3,$$
because W is a non-homogeneous cubic in p and q, gives
$$\begin{vmatrix} h_{12}, & f_{12}, & g_{12}, \\ h_1, & f_1, & g_1, \\ h_2, & f_2, & g_2, \end{vmatrix} = \tfrac{1}{2}N \frac{[f_1g_2-f_2g_1, \quad \ldots, \quad]_3}{\begin{vmatrix} LP+MQ, & NP, & MQ \\ h_1, & f_1, & g_1 \\ h_2, & f_2, & g_2 \end{vmatrix}}.$$

We thus have equations* for the determination of f, g, h: but their integration, even if possible, is complicated.

But we know that the equation, if it has an integral of the required form, is integrable by Darboux's method. For a compatible equation of any order (*e.g.*, if there is an intermediate integral), there are two systems of equations. If there are two integrals of each system, then
$$\left.\begin{aligned} \theta(x, y, z, p, q) &= u \\ \theta'(x, y, z, p, q) &= U \end{aligned}\right\}, \qquad \left.\begin{aligned} \phi(x, y, z, p, q) &= v \\ \phi'(x, y, z, p, q) &= V \end{aligned}\right\}.$$
Expressing x, y, p, q in terms of z, u, v, substituting in
$$dz = (pf_1+qg_1)\,du+(pf_2+qg_2)\,dv,$$
and integrating, we find
$$z = h.$$
And this consideration, as regards the practicability of the integration, is enough to justify the ignoration of the source of the differential equation after it has been constructed.

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* Several of these equations can be deduced from those of the characteristics of the differential equation. It may be added that
$$(L\pm Q)\,dx+(M\mp P)\,dy+N\,dz, \qquad (Nq+M\pm P)\,dp-(Np+L\mp Q)\,dq+\frac{NW\,dy}{Np+L\pm Q}$$
are multiples of the exact differentials that occur in Monge's method of integration of the equation.

Second Sub-case.

10. I., (2).—In this case, each of the three functions f, g, h involves u and v: but the coefficient of V_{n+2} in Φ vanishes; that is,

$$B = \begin{vmatrix} \frac{\partial f}{\partial V_n}, & \frac{\partial g}{\partial V_n}, & \frac{\partial h}{\partial V_n} \\ f_1, & g_1, & h_1 \\ f_2, & g_2, & h_2 \end{vmatrix} = 0.$$

Taking account of the values of f_1 and f_2 in the form

$$f_1 = \frac{\partial f}{\partial U_m} U_{m+1} + F_1, \qquad f_2 = \frac{\partial f}{\partial V_n} V_{n+1} + F_2,$$

with similar expressions for G_1, G_2, H_1, H_2, the relation is equivalent to the relation

$$\begin{vmatrix} \frac{\partial f}{\partial V_n}, & \frac{\partial g}{\partial V_n}, & \frac{\partial h}{\partial V_n} \\ \frac{\partial f}{\partial U_m} U_{m+1} + F_1, & \frac{\partial g}{\partial U_m} U_{m+1} + G_1, & \frac{\partial h}{\partial U_m} U_{m+1} + H_1 \\ F_2, & G_2, & H_2 \end{vmatrix} = 0.$$

This relation is not satisfied in virtue of relations $x = f$, $y = g$, $z = h$; it must therefore be satisfied identically; hence, noting that U_{m+1} occurs only through terms in the second row and even then only as a linear factor, we have

$$\begin{vmatrix} \frac{\partial f}{\partial V_n}, & \frac{\partial g}{\partial V_n}, & \frac{\partial h}{\partial V_n} \\ \frac{\partial f}{\partial U_m}, & \frac{\partial g}{\partial U_m}, & \frac{\partial h}{\partial U_m} \\ F_2, & G_2, & H_2 \end{vmatrix} = 0, \qquad \begin{vmatrix} \frac{\partial f}{\partial V_n}, & \frac{\partial g}{\partial V_n}, & \frac{\partial h}{\partial V_n} \\ F_1, & G_1, & H_1 \\ F_2, & G_2, & H_2 \end{vmatrix} = 0.$$

Further, remembering that the quantities

$$f_1 g_2 - f_2 g_1, \quad g_1 h_2 - g_2 h_1, \quad h_1 f_2 - h_2 g_1$$

do not vanish and that they are proportional to 1, $-p$, $-q$, the condition $B = 0$ can be written

$$\frac{\partial h}{\partial V_n} - p \frac{\partial f}{\partial V_n} - q \frac{\partial g}{\partial V_n} = 0.$$

Again, when we take the second expression for $B = 0$ in the form

$$\begin{vmatrix} \frac{\partial f}{\partial V_n}, & \frac{\partial g}{\partial V_n}, & \frac{\partial h}{\partial V_n} \\ f_1, & g_1, & h_1 \\ F_2, & G_2, & H_2 \end{vmatrix} = 0,$$

and when we use the relation

$$h_1 - pf_1 - qg_1 = 0,$$

which is general, and the particular relation

$$\frac{\partial h}{\partial V_n} - p\frac{\partial f}{\partial V_n} - q\frac{\partial g}{\partial V_n} = 0,$$

we have
$$H_2 - pF_2 - qG_2 = 0.$$

Consequently, the equation

$$BV_{n+1} + H_2 - pF_2 - qG_2 = 0$$

becomes evanescent.

It appears that there are two relations

$$\frac{\partial h}{\partial V_n} - p\frac{\partial f}{\partial V_n} - q\frac{\partial g}{\partial V_n} = 0,$$

$$H_2 - pF_2 - qG_2 = 0.$$

The first of these is not evanescent: the quantity V_n occurs certainly in one of the three equations $x = f$, $y = g$, $z = h$; and the relation shews that, save for trivial and negligible cases, it must occur in two of the three equations. The second of the relations might be evanescent, through the identical vanishing of F_2, G_2, H_2: reserving this result for consideration, and now assuming that it is not the fact, we have another non-evanescent relation. In the latter event, two alternatives may occur: the two equations may be one and the same equation, not determining p and q; or they may be distinct equations, so that they do determine p and q.

With the former alternative, we have

$$F_2 = \phi\frac{\partial f}{\partial V_n}, \qquad G_2 = \phi\frac{\partial g}{\partial V_n}, \qquad H_2 = \phi\frac{\partial h}{\partial V_n};$$

and the reserved case, when the second equation is evanescent, is included in this alternative by taking $\phi = 0$.

With the latter alternative, the values of p and q are given by the two independent equations

$$\frac{\partial h}{\partial V_n} - p\frac{\partial f}{\partial V_n} - q\frac{\partial g}{\partial V_n} = 0,$$

$$H_2 - pF_2 - qG_2 = 0;$$

obviously the values of p and q involve only those quantities which occur in f, g, h. But these values also must satisfy

$$A\,U_{m+1} + H_1 - pF_1 - qG_1 = 0;$$

and then neither A nor $H_1 - pF_1 - qG_1$, after substitution takes place for p and q, can involve quantities which do not occur in f, g, h. The derivative U_{m+1} of the arbitrary function U cannot be expressible in terms of these magnitudes; we therefore must have

$$A = \frac{\partial h}{\partial U_m} - p\frac{\partial f}{\partial U_m} - q\frac{\partial g}{\partial U_m} = 0,$$

$$H_1 - pF_1 - qG_1 = 0,$$

in this case.

The alternatives must be considered in succession.

11. I., (2), (i.) We are to have

$$F_2 = \phi\frac{\partial f}{\partial V_n}, \qquad G_2 = \phi\frac{\partial g}{\partial V_n}, \qquad H_2 = \phi\frac{\partial h}{\partial V_n};$$

and the special case, in which the relation $H_2 - pF_2 - qG_2 = 0$ becomes evanescent, is covered by the value $\phi = 0$.

Now $$f_2 = V_{n+1}\frac{\partial f}{\partial V_n} + \delta' f = V_{n+1}\frac{\partial f}{\partial V_n} + F_2,$$

where $$\delta' f = V_n\frac{\partial f}{\partial V_{n-1}} + V_{n-1}\frac{\partial f}{\partial V_{n-2}} + \ldots + \frac{\partial f}{\partial v}.$$

Hence, in the present case,

$$f_2 = (V_{n+1} + \phi)\frac{\partial f}{\partial V_n};$$

and therefore $$f_{22} = \left(V_{n+2} + \frac{d\phi}{dv}\right)\frac{\partial f}{\partial V_n} + (V_{n+1} + \phi)\frac{d}{dv}\left(\frac{\partial f}{\partial V_n}\right).$$

There are corresponding expressions for g_{22} and h_{22}; consequently

$$\begin{aligned} a_0' V_{n+1}^2 + 2b_0' V_{n+1} + c_0' &= h_{22} - pf_{22} - qg_{22} \\ &= (V_{n+1}+\phi) \left\{ \frac{d}{dv}\left(\frac{\partial h}{\partial V_n}\right) - p\frac{d}{dv}\left(\frac{\partial f}{\partial V_n}\right) - q\frac{d}{dv}\left(\frac{\partial g}{\partial V_n}\right) \right\}. \end{aligned}$$

It follows that $$a_0'\phi^2 - 2b_0'\phi + c_0' = 0;$$

and therefore also

$$\frac{a_0'\phi - b_0'}{1} = \frac{b_0'\phi - c_0'}{\phi} = (b_0'^2 - a_0' c_0')^{\frac{1}{2}} = \Delta_0',$$

say; and the quantity Δ_0' is easily seen not to vanish.

For, by § 6, we have

$$a_0' = \frac{\partial^2 h}{\partial V_n^2} - p\frac{\partial^2 f}{\partial V_n^2} - q\frac{\partial^2 g}{\partial V_n^2},$$

$$2b_0' = \frac{\partial h}{\partial V_{n-1}} - p\frac{\partial f}{\partial V_{n-1}} - q\frac{\partial g}{\partial V_{n-1}} + 2\left\{\delta'\left(\frac{\partial h}{\partial V_n}\right) - p\delta'\left(\frac{\partial f}{\partial V_n}\right) - q\delta'\left(\frac{\partial g}{\partial V_n}\right)\right\}.$$

Also, $$\delta' f = \phi\frac{\partial f}{\partial V_n},$$

so that $$\frac{\partial}{\partial V_n}(\delta' f) = \frac{\partial \phi}{\partial V_n}\frac{\partial f}{\partial V_n} + \phi\frac{\partial^2 f}{\partial V_n^2},$$

that is, $$\frac{\partial f}{\partial V_{n-1}} + \delta'\left(\frac{\partial f}{\partial V_n}\right) = \frac{\partial \phi}{\partial V_n}\frac{\partial f}{\partial V_n} + \phi\frac{\partial^2 f}{\partial V_n^2},$$

with corresponding relations in g and h; hence

$$\begin{aligned} a_0'\phi &= \phi\left(\frac{\partial^2 h}{\partial V_n^2} - p\frac{\partial^2 f}{\partial V_n^2} - q\frac{\partial^2 g}{\partial V_n^2}\right) \\ &= \frac{\partial h}{\partial V_{n-1}} - p\frac{\partial f}{\partial V_{n-1}} - q\frac{\partial g}{\partial V_{n-1}} + \delta'\left(\frac{\partial h}{\partial V_n}\right) - p\delta'\left(\frac{\partial f}{\partial V_n}\right) - q\delta'\left(\frac{\partial g}{\partial V_n}\right) \\ &= b_0' + \tfrac{1}{2}\left\{\frac{\partial h}{\partial V_{n-1}} - p\frac{\partial f}{\partial V_{n-1}} - q\frac{\partial g}{\partial V_{n-1}}\right\}. \end{aligned}$$

Also, from the relation $$\delta' f = \phi\frac{\partial f}{\partial V_n},$$

we have $$\delta'(\delta' f) = \delta'\phi\frac{\partial f}{\partial V_n} + \phi\delta'\left(\frac{\partial f}{\partial V_n}\right),$$

with similar relations for g and h; so that

$$c'_0 = \delta'(\delta' h) - p\delta'(\delta' f) - q\delta'(\delta' g)$$

$$= \phi\left\{\delta'\left(\frac{\partial h}{\partial V_n}\right) - p\delta'\left(\frac{\partial f}{\partial V_n}\right) - q\delta'\left(\frac{\partial g}{\partial V_n}\right)\right\}$$

$$= \phi\left\{b'_0 - \tfrac{1}{2}\left(\frac{\partial h}{\partial V_{n-1}} - p\frac{\partial f}{\partial V_{n-1}} - q\frac{\partial g}{\partial V_{n-1}}\right)\right\};$$

hence we have
$$\frac{a'_0\phi - b'_0}{1} = \frac{b'_0\phi - c'_0}{\phi}$$

$$= \tfrac{1}{2}\left(\frac{\partial h}{\partial V_{n-1}} - p\frac{\partial f}{\partial V_{n-1}} - q\frac{\partial g}{\partial V_{n-1}}\right).$$

Consequently the value of Δ'_0 is given by

$$2\Delta'_0 = \frac{\partial h}{\partial V_{n-1}} - p\frac{\partial f}{\partial V_{n-1}} - q\frac{\partial g}{\partial V_{n-1}};$$

and there is no general reason in the form of the equations why Δ'_0 should vanish.

Again, if we write
$$f_1 = \frac{\partial f}{\partial U_m} U_{m+1} + \delta f,$$

where
$$\delta f = U_m\frac{\partial f}{\partial U_{m-1}} + \ldots + U_1\frac{\partial f}{\partial U} + \frac{\partial f}{\partial u},$$

the operators δ and δ' are permutable; that is,

$$\delta\delta' = \delta'\delta.$$

Now, in the present case,
$$\delta' f = \phi\frac{\partial f}{\partial V_n},$$

so that
$$\frac{\partial}{\partial U_m}(\delta' f) = \phi\frac{\partial^2 f}{\partial U_m \partial V_n} + \frac{\partial \phi}{\partial U_m}\frac{\partial f}{\partial V_n},$$

with similar equations for g and h; hence

$$l_0 = \frac{\partial}{\partial U_m}(\delta' h) - p\frac{\partial}{\partial U_m}(\delta' f) - q\frac{\partial}{\partial U_m}(\delta' g)$$

$$= \phi\left\{\frac{\partial^2 h}{\partial U_m \partial V_n} - p\frac{\partial^2 f}{\partial U_m \partial V_n} - q\frac{\partial^2 g}{\partial U_m \partial V_n}\right\}$$

$$= \phi k_0.$$

Similarly,
$$\begin{aligned}\delta . \delta' f &= \delta\phi \frac{\partial f}{\partial V_n} + \phi . \delta \left(\frac{\partial f}{\partial V_n}\right) \\ &= \delta\phi \frac{\partial f}{\partial V_n} + \phi \frac{\partial}{\partial V_n} (\delta f) \\ &= \delta\phi \frac{\partial f}{\partial V_n} + \phi \frac{\partial F_1}{\partial V_n},\end{aligned}$$

with corresponding equations for g and h; thus
$$\begin{aligned} n_0 &= \delta\delta' h - p\,\delta\delta' f - q\,\delta\delta' g \\ &= \phi \left\{ \frac{\partial H_1}{\partial V_n} - p \frac{\partial F_1}{\partial V_n} - q \frac{\partial G_1}{\partial V_n} \right\} \\ &= \phi m_0. \end{aligned}$$

Proceeding now to construct the differential equation, we have to perform some eliminations. As A is not zero, we have
$$\begin{aligned} U_{m+1} &= -\frac{1}{A} (H_1 - pF_1 - qG_1), \\ f_1 &= \frac{1}{A} (\beta + \gamma q), \\ g_1 &= -\frac{1}{A} (\alpha + \gamma p), \end{aligned}$$

with the notation of § 8. Also, as A does not vanish, the quantity U_{m+2} occurs only in the expression for $h_{11} - pf_{11} - qg_{11}$; consequently, the combination $rf_1^2 + 2sf_1 g_1 + tg^2$ does not intervene in the elimination. Next, we have
$$\begin{aligned} h_{12} - pf_{12} - qg_{12} &= k_0 U_{m+1} V_{n+1} + l_0 U_{m+1} + m_0 V_{n+1} + n_0 \\ &= (k_0 U_{m+1} + m_0)(V_{n+1} + \phi) \\ &= \left\{ m_0 - \frac{k_0}{A} (H_1 - pF_1 - qG_1) \right\} (V_{n+1} + \phi). \end{aligned}$$

Further, $$rf_1 f_2 + s(f_1 g_2 + f_2 g_1) + tg_1 g_2 = (V_{n+1} + \phi)\, \rho,$$

where $$\rho = rf_1 \frac{\partial f}{\partial V_n} + s_1 \left(f_1 \frac{\partial g}{\partial V_n} + g_1 \frac{\partial f}{\partial V_n}\right) + tg_1 \frac{\partial g}{\partial V_n};$$

and therefore one of the equations to be used in the elimination becomes
$$\left\{ m_0 - \frac{k_0}{A} (H_1 - pF_1 - qG_1) - \rho \right\} (V_{n+1} + \phi) = 0.$$

Again, with the notation of § 6, and now using the condition $B=0$, we have

$$h_{22}-pf_{22}-qg_{22}=a_0' V_{n+1}^2+2b_0' V_{n+1}+c_0';$$

and

$$rf_2^2+2sf_2g_2+tg_2^2=(V_{n+1}+\phi)^2\sigma,$$

where

$$\sigma=r\left(\frac{\partial f}{\partial V_n}\right)^2+2s\frac{\partial f}{\partial V_n}\frac{\partial g}{\partial V_n}+t\left(\frac{\partial g}{\partial V_n}\right)^2.$$

Hence the equation becomes

$$a_0' V_{n+1}^2+2b_0' V_{n+1}+c_0'=(V_{n+1}+\phi)^2\sigma.$$

The part $V_{n+1}+\phi=0$ of the former equation reduces this equation to an identity, because

$$a_0'\phi^2-2b_0'\phi+c_0'=0.$$

Moreover, as ϕ does not involve r, s, t, or V_{n+1}, the relation $V_{n+1}+\phi=0$ cannot hold. Hence the eliminant is merely the other part of the former equation: it is

$$\rho=m_0-\frac{k_0}{A}(H_1-pF_1-qG_1),$$

which has the form

$$(\beta+\gamma q)\left(r\frac{\partial f}{\partial V_n}+s\frac{\partial g}{\partial V_n}\right)-(\alpha+\gamma p)\left(s\frac{\partial f}{\partial V_n}+t\frac{\partial g}{\partial V_n}\right)=\gamma''+\alpha''p+\beta''q,$$

where the ratios of the coefficients of the various combinations of p, q, r, s, t are functions of x, y, and z only.

12. I., (2), (ii.). With the second alternative of § 10, we have*

$$A=0, \qquad B=0;$$

the values of p and q satisfy (and are determined by) the four equations

$$\left.\begin{aligned} H_2-pF_2-qG_2&=0\\ \frac{\partial h}{\partial V_n}-p\frac{\partial f}{\partial V_n}-q\frac{\partial g}{\partial V_n}&=0\end{aligned}\right\}$$

and

$$\left.\begin{aligned} H_1-pF_1-qG_1&=0\\ \frac{\partial h}{\partial U_m}-p\frac{\partial f}{\partial U_m}-q\frac{\partial g}{\partial U_m}&=0\end{aligned}\right\}.$$

The first two of these equations are independent of one another. In the

* This is, in effect, part of the fourth sub-case of (I.), to be dealt with in § 15: as it arises now, the necessary analysis will be developed here.

second pair, the second equation is not evanescent; the first can be evanescent (in which case F_1, G_1, H_1 vanish); or, it can be effectively the same as the second (in which case

$$\frac{\partial F_1}{\partial U_m} = \theta F_1, \quad \frac{\partial G_1}{\partial U_m} = \theta G_1, \quad \frac{\partial H_1}{\partial U_m} = \theta H_1,$$

where θ involves only the quantities occurring in x, y, z); or it can be an independent equation. The last is the most comprehensive possibility among the three; it will be seen to include the other two.

To obtain the partial differential equation, we have first to eliminate U_{m+1} and V_{n+1} between the three equations

$$kU_{m+1}V_{n+1}+lU_{m+1}+mV_{n+1}+n = 0,$$

$$aU_{m+1}^2+2bU_{m+1}+c = 0,$$

$$a'V_{n+1}^2+2b'V_{n+1}+c' = 0,$$

which (by § 3) can be expressed in a form

$$\Theta^2 = 4(kn-lm)^2(b^2-ac)(b'^2-a'c'):$$

and then the coefficients of the various combinations of r, s, t in that eliminant are functions of u, U, U_1, ..., U_m, v, V, V_1, ..., V_n: in the present case all these coefficients must be expressible in terms of x, y, z, p, q.

The expression for Θ in terms of r, s, t can be obtained by direct substitution; but the calculations are very long. They can be considerably abbreviated by the use of variables with umbral coefficients. For this purpose, we write

$$x_1,\ x_2 = \frac{\partial f}{\partial U_m},\ \frac{\partial g}{\partial U_m} = \phi_1,\ \psi_1;$$

$$\xi_1,\ \xi_2 = F_1,\ G_1;$$

$$y_1,\ y_2 = \frac{\partial f}{\partial V_n},\ \frac{\partial g}{\partial V_n} = \phi_2,\ \psi_2;$$

$$\eta_1,\ \eta_2 = F_2,\ G_2;$$

and

$$r = a_1^2 = b_1^2 = c_1^2 = d_1^2 = \ldots,$$

$$s = a_1a_2 = b_1b_2 = c_1c_2 = d_1d_2 = \ldots,$$

$$t = a_2^2 = b_2^2 = c_2^2 = d_2^2 = \ldots,$$

the last being the umbral coefficients. Then

$$\left.\begin{aligned} a &= -a_0+a_x^2 \\ b &= -b_0+a_x a_\xi \\ c &= -c_0+a_\xi^2 \end{aligned}\right\}, \qquad \left.\begin{aligned} a' &= -a_0'+a_y^2 \\ b' &= -b_0'+a_y a_\eta \\ c' &= -c_0'+a_\eta^2 \end{aligned}\right\},$$

$$\left.\begin{aligned} k &= -k_0+a_x a_y \\ l &= -l_0+a_x a_\eta \\ m &= -m_0+a_y a_\xi \\ n &= -n_0+a_\xi a_\eta \end{aligned}\right\}.$$

As regards the expression for Θ, we note that the interchanges

$$\left.\begin{matrix} k \\ m \end{matrix}\right\}, \quad \left.\begin{matrix} n \\ l \end{matrix}\right\}, \quad \left.\begin{matrix} c \\ a \end{matrix}\right\}, \quad \left.\begin{matrix} \phi_1 \\ F_1 \end{matrix}\right\}, \quad \left.\begin{matrix} \psi_1 \\ G_1 \end{matrix}\right\}$$

can be made simultaneously without affecting its value. Similarly, the interchanges

$$\left.\begin{matrix} k \\ l \end{matrix}\right\}, \quad \left.\begin{matrix} m \\ n \end{matrix}\right\}, \quad \left.\begin{matrix} c' \\ a' \end{matrix}\right\}, \quad \left.\begin{matrix} \phi_2 \\ F_2 \end{matrix}\right\}, \quad \left.\begin{matrix} \psi_2 \\ G_2 \end{matrix}\right\}$$

can be made simultaneously without affecting the value of Θ. Similarly, the interchanges

$$\left.\begin{matrix} a \\ a' \end{matrix}\right\}, \quad \left.\begin{matrix} b \\ b' \end{matrix}\right\}, \quad \left.\begin{matrix} c \\ c' \end{matrix}\right\}, \quad \left.\begin{matrix} l \\ m \end{matrix}\right\}, \quad \left.\begin{matrix} \phi_1 \\ \phi_2 \end{matrix}\right\}, \quad \left.\begin{matrix} \psi_1 \\ \psi_2 \end{matrix}\right\}, \quad \left.\begin{matrix} F_1 \\ F_2 \end{matrix}\right\}, \quad \left.\begin{matrix} G_1 \\ G_2 \end{matrix}\right\}$$

can also be made without affecting its value. When these properties are noted, it is possible to dispense with many of the calculations; for then many of the aggregates of terms of a particular type can be set down, when a single aggregate of that type has been actually calculated. The possibility will be utilised.

13. Then, when we adopt the customary process in the umbral calculations connected with binary forms, the terms in Θ, which are independent of a_0, b_0, c_0, a_0', b_0', c_0', are

$$\begin{aligned} &= k^2c_\xi^2d_\eta^2-2klc_\xi^2d_y d_\eta-2kmc_x c_\xi d_\eta^2+2knc_x c_\xi d_y d_\eta+2lmc_x c_\xi d_y d_\eta \\ &\qquad +l^2c_\xi^2d_y^2-2lnc_x c_\xi d_y^2+m^2c_x^2d_\eta^2-2mnc_x^2d_y d_\eta+n^2c_x^2d_y^2 \\ &= (kc_\xi d_\eta-lc_\xi d_y-mc_x d_\eta+nc_x d_y)^2 \\ &= \vartheta^2, \end{aligned}$$

say. The part of ϑ which is independent of k_0, l_0, m_0, n_0 is

$$\begin{aligned} &= a_x a_y c_\xi d_\eta - a_x a_\eta c_\xi d_y - a_y a_\xi c_x d_\eta + a_\xi a_\eta c_x d_y \\ &= (a_x c_\xi - a_\xi c_x)(a_y d_\eta - a_\eta d_y) \\ &= (x\xi)(y\eta)(ac)(ad); \end{aligned}$$

and therefore

$$\vartheta^2 = \{(x\xi)(y\eta)(ac)(ad) - (k_0 c_\xi d_\eta - l_0 c_\xi d_y - m_0 c_x d_\eta + n_0 c_x d_y)\}^2.$$

The coefficient of $(x\xi)^2(y\eta)^2$ in the part of ϑ^2, which does not involve k_0, l_0, m_0, n_0, is

$$= (ac)(ad)(bc)(bd);$$

or, as

$$\begin{aligned} (ac)(ad) &= (a_1 c_2 - a_2 c_1)(a_1 d_2 - a_2 d_1) \\ &= rc_2 d_2 - s(c_2 d_1 + c_1 d_2) + tc_1 d_1, \end{aligned}$$

the said coefficient is

$$\begin{aligned} &= \{rc_2 d_2 - s(c_2 d_1 + c_1 d_2) + tc_1 d_1\}^2 \\ &= 2(rt - s^2)^2, \end{aligned}$$

on reduction: consequently, this part of ϑ^2 in Θ is

$$2(rt-s^2)^2(x\xi)^2(y\eta)^2.$$

Again, the coefficient of $-2k_0$ in ϑ^2 is

$$= (x\xi)(y\eta)(ac)(ad)\, c_\xi d_\eta;$$

or, since $(ac)c_\xi = (a_1 c_2 - a_2 c_1)(c_1\xi_1 + c_2\xi_2) = a_1(s\xi_1 + t\xi_2) - a_2(r\xi_1 + s\xi_2)$, this coefficient is

$$\begin{aligned} (ac)c_\xi(ad)d_\eta &= \{a_1(s\xi_1+t\xi_2) - a_2(r\xi_1+s\xi_2)\}\{a_1(s\eta_1+t\eta_2) - a_2(r\eta_1+s\eta_2)\} \\ &= \quad r(s\xi_1+t\xi_2)(s\eta_1+t\eta_2) + t(r\xi_1+s\xi_2)(r\eta_1+s\eta_2) \\ &\quad -s\{(s\xi_1+t\xi_2)(r\eta_1+s\eta_2) + (r\xi_1+s\xi_2)(s\eta_1+t\eta_2)\} \\ &= (rt-s^2)\{rF_1F_2 + s(F_1G_2+F_2G_1) + tG_1G_2\}; \end{aligned}$$

hence, taking account of the interchanges which leave Θ unaltered, we have the aggregate of terms, which are linear in k_0, l_0, m_0, n_0 and do not involve a_0, b_0, c_0, a_0', b_0', c_0', in the form

$$\begin{aligned} -2(rt-s^2)(x\xi)(y\eta)[\quad & k_0\{rF_1F_2 + s(F_1G_2+F_2G_1) + tG_1G_2\} \\ & -l_0\{rF_1\phi_2 + s(F_1\psi_2 + G_1\phi_2) + tG_1\psi_2\} \\ & -m_0\{r\phi_1F_2 + s(\phi_1G_2 + \psi_1F_2) + t\psi_1G_2\} \\ & +n_0\{r\phi_1\phi_2 + s(\phi_1\psi_2 + \psi_1\phi_2) + t\phi_1\phi_2\}] \\ = -2(rt-s^2)(\phi_1G_1 - \psi_1F_1)(\phi_2G_2 - \psi_2F_2)\{ & k_0 a_\xi a_\eta - l_0 a_\xi a_y - m_0 a_x a_\eta \\ & + n_0 a_x a_y\}. \end{aligned}$$

And the part of ϑ^2, which is of the second degree in k_0, l_0, m_0, n_0, while independent of a_0, b_0, c_0, a'_0, b'_0, c'_0, is

$$(k_0 c_\xi d_\eta - l_0 c_\xi d_y - m_0 c_x d_\eta + n_0 c_x d_y)^2,$$

which can be expressed immediately in non-umbral forms. Hence

$$\begin{aligned}\vartheta^2 = \;& 2(rt-s^2)^2 (\phi_1 G_1 - \psi_1 F_1)^2 (\phi_2 G_2 - \psi_2 F_2)^2 \\ & -2(rt-s^2)(\phi_1 G_1 - \psi_1 F_1)(\phi_2 G_2 - \psi_2 F_2)(k_0 a_\xi a_\eta - l_0 a_\xi a_y \\ & \qquad\qquad - m_0 a_x a_\eta + n_0 a_x a_y) \\ & + (k_0 c_\xi d_\eta - l_0 c_\xi d_y - m_0 c_x d_\eta + n_0 c_x d_y)^2.\end{aligned}$$

We proceed similarly with the other terms in Θ; denoting their aggregate by ϖ, we find

$$\begin{aligned}\varpi = & -(rt-s^2)\left[(y\eta)^2 \{a_0 d_\xi^2 - 2b_0 d_x d_\xi + c_0 d_x^2\}\right. \\ & \qquad \left. + (x\xi)^2 \{a'_0 d_\eta^2 - 2b'_0 d_y d_\eta + c'_0 d_y^2\}\right] \\ & + 2(rt-s^2)\{(\phi_2 G_2 - \psi_2 F_2) I + (\phi_1 G_1 - \psi_1 F_1) J\} \\ & - \{a_0 (n_0 a_y - m_0 a_\eta)^2 - 2(n_0 a_y - m_0 a_\eta)(l_0 a_y - k_0 a_\eta) b_0 + (l_0 a_y - k_0 a_\eta)^2 c_0\} \\ & - \{a'_0 (n_0 a_x - l_0 a_\xi)^2 - 2(n_0 a_x - l_0 a_\xi)(m_0 a_x - k_0 a_\xi) b'_0 + (m_0 a_x - k_0 a_\xi)^2 c'_0\} \\ & - 2\left[k_0 (c_0 a_x - b_0 a_\xi)(c'_0 a_y - b'_0 a_\eta) - l_0 (c_0 a_x - b_0 a_\xi)(b'_0 a_y - a'_0 a_\eta)\right. \\ & \qquad \left. - m_0 (c'_0 a_y - b'_0 a_\eta)(b_0 a_x - a_0 a_\xi) + n_0 (b_0 a_x - a_0 a_\xi)(b'_0 a_y - a'_0 a_\eta)\right] \\ & + \{c_0 a_x b_x - b_0 (a_x b_\xi + b_x a_\xi) + a_0 a_\xi b_\xi\}\{c'_0 a_y b_y - b'_0 (a_y b_\eta + b_y a_\eta) + a'_0 a_\eta b_\eta\} \\ & + \Theta_0,\end{aligned}$$

where

$$\begin{aligned}I = \;& a_0 \{n_0 (\phi_2 G_1 - \psi_2 F_1) + m_0 (G_2 F_1 - F_2 G_1)\} \\ & - b_0 \{k_0 (G_2 F_1 - F_2 G_1) + l_0 (\phi_2 G_1 - \psi_2 F_1) + m_0 (G_2 \phi_1 - F_2 \psi_1) \\ & \qquad + n_0 (\phi_2 \psi_1 - \phi_1 \psi_2)\} \\ & + c_0 \{l_0 (\phi_2 \psi_1 - \psi_2 \phi_1) + k_0 (G_2 \phi_1 - F_2 \psi_1)\}, \\ J = \;& a'_0 \{n_0 (\phi_1 G_2 - \psi_1 F_2) + l_0 (G_1 F_2 - G_2 F_1)\} \\ & - b'_0 \{k_0 (G_1 F_2 - F_1 G_2) + m_0 (\phi_1 G_2 - \psi_1 F_2) + l_0 (G_1 \phi_2 - F_1 \psi_2) \\ & \qquad + n_0 (\phi_2 \psi_1 - \phi_1 \psi_2)\} \\ & + c'_0 \{m_0 (\phi_1 \psi_2 - \phi_2 \psi_1) + k_0 (G_1 \phi_2 - F_1 \psi_2)\}, \\ \Theta_0 = \;& k_0^2 c_0 c'_0 - 2k_0 l_0 c_0 b'_0 - 2k_0 m_0 b_0 c'_0 + 2k_0 n_0 b_0 b'_0 + 2l_0 m_0 b_0 b'_0 + l_0^2 c_0 a'_0 \\ & - 2l_0 n_0 b_0 a'_0 + m_0^2 a_0 c'_0 - 2m_0 n_0 a_0 b'_0 + n_0^2 a_0 a'_0.\end{aligned}$$

Thus $$\Theta = \vartheta^2 + \varpi.$$

Also

$$kn - lm = k_0 n_0 - l_0 m_0 - (k_0 a_\xi a_\eta - l_0 a_\xi a_y - m_0 a_x a_\eta + n_0 a_x a_y) + (rt - s^2)(\phi_1 G_1 - \psi_1 F_1)(\phi_2 G_2 - \psi_2 F_2),$$

$$b^2 - ac = b_0^2 - a_0 c_0 + (a_0 a_\xi^2 - 2b_0 a_x a_\xi - c_0 a_x^2) - (rt - s^2)(\phi_1 G_1 - \psi_1 F_1)^2,$$

$$b'^2 - a'c' = b_0'^2 - a_0' c_0' + (a_0' a_\eta^2 - 2b_0' a_y a_\eta + c_0' a_y^2) - (rt - s^2)(\phi_2 G_2 - \psi_2 F_2)^2.$$

Finally, the eliminant is

$$(\vartheta^2 + \varpi)^2 = 4\,(kn - lm)^2\,(b^2 - ac)(b_0'^2 - a_0' c_0').$$

When expansion takes place, various sets of terms cancel: and the terms of highest and of lowest degrees in r, s, t are indicated in the equation

$$8(rt - s^2)^3 (\phi_1 G_1 - \psi_1 F_1)^3 (\phi_2 G_2 - \psi_2 F_2)^3 \{(\phi_2 G_2 - \psi_2 F_2) I + (\phi_1 G_1 - \psi_1 F_1) J\} + \ldots + \Theta_0^2 - 4(k_0 n_0 - l_0 m_0)(b_0^2 - a_0 c_0)(b_0'^2 - a_0' c_0') = 0,$$

which accordingly is the form of the partial differential equation.

The temporarily omitted alternatives are the special cases of the preceding equation,

(α) when $$F_1 = G_1 = H_1 = 0;$$

(β) when $$F_1 = \theta \frac{\partial F}{\partial U_m}, \quad G_1 = \theta \frac{\partial G}{\partial U_m}, \quad H_1 = \theta \frac{\partial H}{\partial U_m}.$$

Some of the special cases, when the integral system is of the form

$$x, y, z = \text{function of } u + \text{function of } v,$$

so that $h_{12} = 0$, $f_{12} = 0$, $g_{12} = 0$, and therefore $k_0 = l_0 = m_0 = n_0 = 0$, will be considered later (§ 19).

Note.—An example of the general case just considered is provided by the integral system

$$x = f = \frac{u}{v} - V',$$

$$y = g = ve^{-U'},$$

$$z = h = u + U - uU' + V - vV'.$$

The quantities A and B are given by

$$A = \begin{vmatrix} 0, & -ve^{-U'}, & -u \\ \frac{1}{v}, & -ve^{-U'}U'', & 1 - uU'' \\ -\frac{u}{v^2} - V'', & e^{-U'}, & -vV'' \end{vmatrix} = 0,$$

$$B = \begin{vmatrix} -1, & 0, & -v \\ \frac{1}{v}, & -ve^{-U'}U'', & 1-uU'' \\ -\frac{u}{v^2}-V'', & e^{-U'}, & -vV'' \end{vmatrix} = 0;$$

the values of p and q are

$$p = v, \qquad q = \frac{u}{v} - e^{U'};$$

and the partial differential equation of the second order, which is satisfied by the integral system, is

$$y(rt-s^2)+qr-ps = 0.$$

Third Sub-case.

14. I., (3).—In this case, we have $A = 0$ while B is not initially given as a vanishing quantity: thus

$$A = \begin{vmatrix} \frac{\partial h}{\partial U_m}, & \frac{\partial f}{\partial U_m}, & \frac{\partial g}{\partial U_m} \\ h_1, & f_1, & g_1 \\ h_2, & f_2, & g_2 \end{vmatrix} = 0.$$

The analysis follows a line of development exactly the same as in the last case; as the detailed results can be obtained by interchanging the variables u and v, they will be stated without proof.

There are three alternatives. In the first, quantities F_1, G_1, H_1 vanish. In the second, the equations

$$\frac{\partial h}{\partial U_m} - p\frac{\partial F}{\partial U_m} - q\frac{\partial G}{\partial U_m} = 0,$$

$$H_1 - pF_1 - qG_1 = 0$$

are effectively one and the same equation: then

$$F_1 = \theta\frac{\partial F}{\partial U_m}, \quad G_1 = \theta\frac{\partial G}{\partial U_m}, \quad H_1 = \theta\frac{\partial G}{\partial U_m},$$

and the first of the alternatives is derived from the second by making θ vanish. Clearly a non-vanishing θ involves only the same quantities as occur in x, y, z. Also

$$a_0\theta^2 - 2b_0\theta + c_0 = 0,$$

so that

$$\frac{a_0\theta - b_0}{1} = \frac{b_0\theta - c_0}{\theta} = (b_0^2 - a_0c_0)^{\frac{1}{2}}$$

$$= \Delta_0$$

$$= \tfrac{1}{2}\left(\frac{\partial h}{\partial U_{m-1}} - p\frac{\partial f}{\partial U_{m-1}} - q\frac{\partial g}{\partial U_{m-1}}\right).$$

Further, $$m_0 = \theta k_0, \qquad n_0 = \theta l_0;$$

and
$$V_{n-1} = -\frac{1}{B}(H_2 - pF_2 - qG_2),$$
$$f_2 = \frac{1}{B}(\beta' + \gamma' q),$$
$$g_2 = -\frac{1}{B}(\alpha' + \gamma' p),$$

with the notation of § 8. And the final differential equation, which results from the elimination, is

$$(\beta' + \gamma' q)\left(r\frac{\partial f}{\partial U_m} + s\frac{\partial g}{\partial U_m}\right) - (\alpha' + \gamma' p)\left(s\frac{\partial f}{\partial U_m} + t\frac{\partial g}{\partial U_m}\right) = \gamma''' + \alpha''' p + \beta''' q,$$

where the ratios of the coefficients of the various combinations of p, q, r, s, t are functions of x, y, and z only.

In the third alternative, the two equations

$$\frac{\partial h}{\partial U_m} - p\frac{\partial F}{\partial U_m} - q\frac{\partial G}{\partial U_m} = 0, \qquad H_1 - pF_1 - qG_1 = 0$$

are independent of one another: by an argument similar to the earlier argument, we have

$$\frac{\partial h}{\partial V_n} - p\frac{\partial F}{\partial V_n} - q\frac{\partial G}{\partial V_n} = 0, \qquad H_2 - pF_2 - qG_2 = 0.$$

The circumstances now are precisely the same as in the third alternative of the second case: the results are the same as before, in § 13, and need not be restated, either in general, or for the special cases, when the two new equations are one and the same, and when the second of the new equations is evanescent.

Fourth Sub-case.

15. I., (4).—In this case, we have

$$A = 0, \qquad B = 0,$$

that is,

$$\left.\begin{aligned}\frac{\partial h}{\partial U_m} - p\frac{\partial f}{\partial U_m} - q\frac{\partial g}{\partial U_m} &= 0\\ H_1 - pF_1 - qG_1 &= 0\end{aligned}\right\}, \qquad \left.\begin{aligned}\frac{\partial h}{\partial V_n} - p\frac{\partial f}{\partial V_n} - q\frac{\partial g}{\partial V_n} &= 0\\ H_2 - pF_2 - qG_2 &= 0\end{aligned}\right\}.$$

With each pair of equations, we have the three same alternatives as before.

The two first alternatives are that the respective second equations become evanescent.

The two second alternatives are that the second equation in each pair is effectively the same as the first equation in its own pair. In the respective cases, we have

$$F_1 = \theta \frac{\partial f}{\partial U_m}, \qquad G_1 = \theta \frac{\partial g}{\partial U_m}, \qquad H_1 = \theta \frac{\partial h}{\partial U_m};$$

$$F_2 = \phi \frac{\partial f}{\partial V_n}, \qquad G_2 = \phi \frac{\partial g}{\partial V_n}, \qquad H_2 = \phi \frac{\partial h}{\partial V_n}.$$

The vanishing of F_1, G_1, H_1 is given by a vanishing θ; and the vanishing of F_2, G_2, H_2 by a vanishing ϕ. When the two second alternatives coexist, while θ and ϕ do not vanish, we have

$$a_0\theta^2 - 2b_0\theta + c_0 = 0, \qquad a_0'\phi^2 - 2b_0'\phi + c_0' = 0;$$

also $$n_0 = \phi m_0 = \theta l_0 = \theta\phi k_0.$$

Again, writing

$$\rho = r\left(\frac{\partial f}{\partial U_m}\right)^2 + 2s\frac{\partial f}{\partial U_m}\frac{\partial g}{\partial U_m} + t\left(\frac{\partial g}{\partial U_m}\right)^2,$$

$$\sigma = r\frac{\partial f}{\partial U_m}\frac{\partial f}{\partial V_n} + s\left(\frac{\partial f}{\partial U_m}\frac{\partial g}{\partial V_n} + \frac{\partial f}{\partial V_n}\frac{\partial g}{\partial U_m}\right) + t\frac{\partial g}{\partial U_m}\frac{\partial g}{\partial V_n},$$

$$\tau = r\left(\frac{\partial f}{\partial V_n}\right)^2 + 2s\frac{\partial f}{\partial V_n}\frac{\partial g}{\partial V_n} + t\left(\frac{\partial g}{\partial V_n}\right)^2,$$

and $$a = -a_0 + \rho, \qquad b = -b_0 + \rho\theta, \qquad c = -c_0 + \rho\theta^2;$$

$$a' = -a_0' + \tau, \qquad b' = -b_0' + \tau\phi \qquad c' = -c_0' + \tau\phi^2;$$

$$k = -k_0 + \sigma, \qquad l = -l_0 + \phi\sigma, \qquad m = -m_0 + \theta\sigma, \qquad n = -n_0 + \theta\phi\sigma,$$

the three equations, which lead to the partial differential equation, are

$$aU_{m+1}^2 + 2bU_{m+1} + c = 0,$$

$$a'V_{n+1}^2 + 2b'V_{n+1} + c' = 0,$$

$$kU_{m+1}V_{n+1} + lU_{m+1} + mV_{n+1} + n = 0.$$

When the values of a, b, c; a', b', c'; k, l, m, n are substituted, and when the equations are re-arranged, they take the form

$$(U_{m+1}+\theta)^2(\rho - a_0) + 2\Delta_0(U_{m+1}+\theta) = 0,$$

$$(V_{n+1}+\phi)^2(\tau - a_0') + 2\Delta_0'(V_{n+1}+\phi) = 0,$$

$$(\sigma - k_0)(U_{m+1}+\theta)(V_{n+1}+\phi) = 0,$$

where Δ_0 and Δ_0' denote the respective quantities

$$(b_0^2 - a_0c_0)^{\frac{1}{2}} \quad \text{and} \quad (b_0'^2 - a_0'c_0')^{\frac{1}{2}}.$$

Now, as θ contains only the quantities which occur in x, y, z, the relation $U_{m+1}+\theta = 0$ cannot be satisfied; similarly, $V_{n+1}+\phi$ cannot vanish.

Hence the first two equations express U_{m+1} and V_{n+1} respectively in terms of x, y, z, p, q, r, s, t; and the partial differential equation, which survives from the third equation, is

$$\sigma - k_0 = 0,$$

that is,

$$r\frac{\partial f}{\partial U_m}\frac{\partial g}{\partial V_n} + s\left(\frac{\partial f}{\partial U_m}\frac{\partial g}{\partial V_n} + \frac{\partial f}{\partial V_n}\frac{\partial g}{\partial U_m}\right) + t\frac{\partial g}{\partial U_m}\frac{\partial g}{\partial V_n} = k_0.$$

The two third alternatives are that the two equations, in each pair of relations arising from $A = 0$ and $B = 0$, are independent of one another. When these two third alternatives co-exist, we once again have the set of circumstances considered in §§ 12, 13; the results need not be restated. When only one of these third alternatives is valid, together with the first or the second of the alternatives for the other pair of equations, once again we have particular cases of the general result just indicated.

16. In the preceding four cases which have just been considered, the discrimination has been effected by the vanishing or the non-vanishing of the quantities A and B, which are the coefficients of U_{m+2} and of V_{n+2} in $h_{11} - pf_{11} - qg_{11}$ and in $h_{22} - pf_{22} - qg_{22}$ respectively. It might seem as if a new set of cases would arise, discriminated by the vanishing or the non-vanishing of the quantity

$$\begin{vmatrix} \dfrac{\partial^2 h}{\partial U_m \partial V_n}, & \dfrac{\partial^2 f}{\partial U_m \partial V_n}, & \dfrac{\partial^2 g}{\partial U_m \partial V_n} \\ f_1, & g_1, & h_1 \\ f_2, & g_2, & h_2 \end{vmatrix},$$

which effectively is the coefficient of $U_{m+1} V_{n+1}$ in $h_{12} - pf_{12} - qg_{12}$; but this possibility is not actually a fact. The essential difference lies in the property that, whereas U_{m+2} occurs in the expression for $h_{11} - pf_{11} - qg_{11}$ only (so that, if $A = 0$, the form of the expression is substantially changed), and similarly for V_{n+2} and $h_{22} - pf_{22} - qg_{22}$, the quantities U_{m+1} and V_{n+1} do occur elsewhere than in a first term in $h_{12} - pf_{12} - qg_{12}$. The sole effect of the non-occurrence of the term in the last quantity, which involves $U_{m+1} V_{n+1}$, is that $k_0 = 0$; some simplification thereby arises in the preceding formulæ; but simplification, not essential change, is the sole effect of the condition.

The most obvious case arises when

$$x, y, z = \text{function of } u + \text{function of } v;$$

to its fuller consideration, not in this reference alone, we shall proceed (§ 19).

17. In several of the preceding discussions, it has been pointed out that the vanishing of F_1, G_1, H_1 modifies the form of the final equation obtained, and that this final form can be obtained as a limiting case of a more general final form: and similarly for the vanishing of F_2, G_2, H_2. Such conditions, however, admit of a simple discussion of the problem from the beginning.

When $F_1 = 0$, $G_1 = 0$, $H_1 = 0$, then the equation

$$h_1 = pf_1 + qg_1$$

becomes, on the removal of a common factor,

$$\frac{\partial h}{\partial U_m} = p\frac{\partial f}{\partial U_m} + q\frac{\partial g}{\partial U_m}.$$

In this case, A vanishes because its value is

$$A = \begin{vmatrix} \frac{\partial h}{\partial U_m}, & \frac{\partial f}{\partial U_m}, & \frac{\partial g}{\partial U_m} \\ h_1, & f_1, & g_1 \\ h_2, & f_2, & g_2 \end{vmatrix}.$$

Thus U_{m+1} is not expressible in terms of p and q: but it is given by

$$\left\{a_0 - r\left(\frac{\partial f}{\partial U_m}\right)^2 - 2s\frac{\partial f}{\partial U_m}\frac{\partial g}{\partial U_m} - t\left(\frac{\partial g}{\partial U_m}\right)^2\right\} U_{m+1}^2 + 2b_0 U_{m+1} + c_0 = 0.$$

On the supposition that B does not vanish, the value of V_{n+1} is given by the equation

$$V_{n+1} = -\frac{1}{B}(H_2 - pF_2 - qG_2).$$

Moreover, as F_1, G_1, H_1 all vanish, we have $m_0 = 0$, $n_0 = 0$; hence

$$\begin{aligned} k_0 U_{m+1} V_{n+1} &+ l_0 U_{m+1} \\ &= rf_1 f_2 + s(f_1 g_2 + f_2 g_1) + tg_1 g_2 \\ &= U_{m+1}\left[r\frac{\partial f}{\partial U_m}\left(\frac{\partial f}{\partial V_n} V_{n+1} + F_2\right) + t\frac{\partial g}{\partial U_m}\left(\frac{\partial g}{\partial V_n} V_{n+1} + G_2\right)\right. \\ &\qquad \left. + s\left\{\frac{\partial f}{\partial U_m}\left(\frac{\partial g}{\partial V_n} V_{n+1} + G_2\right) + \frac{\partial g}{\partial U_m}\frac{\partial f}{\partial V_n}(V_{n+1} + F_2)\right\}\right], \end{aligned}$$

so that

$$\begin{aligned} \left\{k_0 - r\frac{\partial f}{\partial U_m}\frac{\partial f}{\partial V_n} - s\left(\frac{\partial f}{\partial U_m}\frac{\partial g}{\partial V_n} + \frac{\partial g}{\partial U_m}\frac{\partial f}{\partial V_n}\right) - t\frac{\partial g}{\partial U_m}\frac{\partial g}{\partial V_n}\right\} V_{n+1} \\ = -l_0 + rF_2\frac{\partial f}{\partial U_m} + s\left(G_2\frac{\partial f}{\partial U_m} + F_2\frac{\partial g}{\partial U_m}\right) + t\frac{\partial g}{\partial U_m} G_2. \end{aligned}$$

Substitution of the preceding value of V_{n+1} leads to a relation of the form

$$\alpha r+\beta s+\gamma t = \delta,$$

where α, β, γ, δ are linear functions of p and q, having functions of x, y, z for their coefficients.

When $F_2 = G_2 = H_2 = 0$, while A is not zero, there is a similar form of final equation.

When $F_1 = 0 = G_1 = H_1$, $F_2 = 0 = G_2 = H_2$, the equation

$$h_{12}-pf_{12}-qg_{12} = rf_1f_2+s(f_1g_2+f_2g_1)+tg_1g_2$$

leads to the equation

$$\frac{\partial^2 h}{\partial U_m \partial V_n}-p\frac{\partial^2 f}{\partial U_m \partial V_n}-q\frac{\partial^2 g}{\partial U_m \partial V_n}$$
$$= r\frac{\partial f}{\partial U_m}\frac{\partial f}{\partial V_n}+s\left(\frac{\partial f}{\partial U_m}\frac{\partial g}{\partial V_n}+\frac{\partial f}{\partial V_n}\frac{\partial g}{\partial U_m}\right)+t\frac{\partial g}{\partial U_m}\frac{\partial g}{\partial V_n},$$

which is the final form of the equation in the supposed circumstances; the coefficients of r, s, t, and the term independent of r, s, t, are functions of x, y, z, p, q.

Note.—A simple and well known example of this last case is provided by the equations

$$x = (1-u^2)U''+2uU'-2U+(1-v^2)V''+2vV'-2V,$$
$$y = i[-(1+u^2)U''+2uU'-2U+(1+v^2)V''-2vV'+2V],$$
$$z = 2uU''-2U'+2vV''-2V.$$

Here we have $$p = \frac{u+v}{1-uv}, \qquad q = i\frac{u+v}{1-uv},$$

on reduction; also

$$\frac{\partial^2 h}{\partial U_m \partial V_n} = 0 = \frac{\partial^2 f}{\partial U_m \partial V_n} = \frac{\partial^2 g}{\partial U_m \partial V_n};$$

and the final differential equation is

$$(1-u^2)(1-v^2)r-2i(u^2-v^2)s+(1+u^2)(1+v^2)t = 0,$$

that is, $$(1+q^2)r-2pqs+(1+p^2)t = 0,$$

the equation of minimal surfaces.

It will be noticed that the integral equations are of the special form

$$x, y, z = \text{function of } u + \text{function of } v,$$

so that $$h_{12} = 0, \quad f_{12} = 0, \quad g_{12} = 0.$$

18. We know that, by means of Legendre's transformation,

$$Z = px+qy-z,$$

$$X = p, \quad Y = q, \quad P = x, \quad Q = y,$$

an equation
$$f(x, y, z, p, q, r, s, t) = 0$$
is transformed into the equation

$$f\left(P, Q, PX+QY-Z, X, Y, \frac{T}{RT-S^2}, \frac{-S}{RT-S^2}, \frac{R}{RT-S^2}\right) = 0;$$

hence, when the integral of the former is known, the integral of the latter can be derived.

Thus, for example, the primitive of the equation

$$(1+x^2)\,r+2xys+(1+y^2)\,t = 0$$

can be derived from the primitive of the equation

$$(1+q^2)\,r-2pqs+(1+p^2)\,t = 0;$$

it is easily found to be expressible by the three equations

$$x = \frac{u+v}{1-uv},$$

$$y = i\,\frac{u-v}{1-uv},$$

$$z = \frac{1}{1-uv}\,\{(6uv-2)(U'+V')-4vU-4uV\}.$$

We notice that the form of the integral equations in the primitive of the new equation is no longer

$$x, y, z = \text{function of } u + \text{function of } v,$$

which was the form of the primitive of the original equations.

Equations under Case I., having Integrals of Special Form.

19. In the preceding investigations, the principal aim has been the construction of the form of partial equations of the second order as determined by an assumed set of three equations which constitute the primitive. We now proceed (as promised at the end of § 13 and § 16) to discuss, in greater detail, some of those equations of the second order the primitive of which is constituted by a set of the form

$$x, y, z = \text{function of } u + \text{function of } v.$$

The question will be considered by dealing with the differential equations, which result from using the relation

$$h_{12}-pf_{12}-qg_{12} = rf_1f_2+s(f_1g_2+f_2g_1)+tg_1g_2,$$

without using either of the relations which respectively give the values of $h_{11}-pf_{11}-qg_{11}$, $h_{22}-pf_{22}-qg_{22}$. In the present case,

$$f_{12} = 0, \qquad g_{12} = 0, \qquad h_{12} = 0;$$

so that the equation becomes

$$rf_1f_2+s(f_1g_2+f_2g_1)+tg_1g_2 = 0.$$

As A is not zero, and also B is not zero, the ratios of the coefficients are functions of x, y, z, p, q; hence the equation may be written

$$ar+2bs+ct = 0,$$

where a, b, c are functions of x, y, z, p, q.

Comparing the two forms of the equation, we have a quantity λ, such that

$$f_1f_2 = \lambda a, \qquad f_1g_2+f_2g_1 = 2\lambda b, \qquad g_1g_2 = \lambda c;$$

hence, writing

$$\delta = (b^2-ac)^{\frac{1}{2}},$$

we have

$$f_1g_2 = \lambda(b+\delta), \qquad f_2g_1 = \lambda(b-\delta).$$

Consequently,

$$\frac{b+\delta}{a} = \frac{g_2}{f_2} = \text{function of } v \text{ only},$$

$$\frac{b-\delta}{a} = \frac{g_1}{f_1} = \quad ,, \quad u \text{ only};$$

and therefore

$$f_1\frac{d}{dx}\left(\frac{b+\delta}{a}\right)+g_1\frac{d}{dy}\left(\frac{b+\delta}{a}\right) = 0,$$

$$f_2\frac{d}{dx}\left(\frac{b-\delta}{a}\right)+g_2\frac{d}{dy}\left(\frac{b-\delta}{a}\right) = 0,$$

where

$$\left.\begin{aligned}\frac{d}{dx} &= \frac{\partial'}{\partial' x}+r\frac{\partial}{\partial p}+s\frac{\partial}{\partial q}\\ \frac{d}{dy} &= \frac{\partial'}{\partial' y}+s\frac{\partial}{\partial p}+t\frac{\partial}{\partial q}\end{aligned}\right\}, \qquad \left.\begin{aligned}\frac{\partial'}{\partial' x} &= \frac{\partial}{\partial x}+p\frac{\partial}{\partial z}\\ \frac{\partial'}{\partial' y} &= \frac{\partial}{\partial y}+q\frac{\partial}{\partial z}\end{aligned}\right\};$$

that is,

$$a\frac{d}{dx}\left(\frac{b+\delta}{a}\right)+(b-\delta)\frac{d}{dy}\left(\frac{b+\delta}{a}\right) = 0,$$

$$a\frac{d}{dx}\left(\frac{b-\delta}{a}\right)+(b+\delta)\frac{d}{dy}\left(\frac{b-\delta}{a}\right) = 0.$$

Adding, we have

$$a\frac{d}{dx}\left(\frac{b}{a}\right)+b\frac{d}{dy}\left(\frac{b}{a}\right)-\delta\frac{d}{dy}\left(\frac{\delta}{a}\right)=0;$$

and, subtracting, we have

$$a\frac{d}{dx}\left(\frac{\delta}{a}\right)+b\frac{d}{dy}\left(\frac{\delta}{a}\right)-\delta\frac{d}{dy}\left(\frac{b}{a}\right)=0.$$

The former gives

$$\frac{d}{dx}\left(\frac{b}{a}\right)+\frac{b}{a}\frac{d}{dy}\left(\frac{b}{a}\right)=\frac{\delta}{a}\frac{d}{dy}\left(\frac{\delta}{a}\right)$$

$$=\frac{b}{a}\frac{d}{dy}\left(\frac{b}{a}\right)-\tfrac{1}{2}\frac{d}{dy}\left(\frac{c}{a}\right),$$

so that

$$\frac{d}{dx}\left(\frac{b}{a}\right)+\tfrac{1}{2}\frac{d}{dy}\left(\frac{c}{a}\right)=0;$$

and similarly, after slight reduction, the second gives

$$\frac{d}{dy}\left(\frac{b}{c}\right)+\tfrac{1}{2}\frac{d}{dx}\left(\frac{a}{c}\right)=0.$$

Having regard to the significance of the operations d/dx, d/dy, we notice that these conditions are to be satisfied concurrently with the original differential equation.

20. When b vanishes, so that the equation is $ar+ct=0$, the two conditions give

$$\frac{c}{a}=\text{constant};$$

the equation is at once transformable into the form

$$r+t=0,$$

characteristic of two-dimensional potential.

When b does not vanish (and this now will be assumed), we can divide the partial equation throughout by $2b$ (or, what is the same thing, we can take $2b=1$): the equation then is

$$ar+s+ct=0,$$

and the two conditions become

$$\left.\begin{aligned}\frac{d}{dx}\left(\frac{1}{a}\right)+\frac{d}{dy}\left(\frac{c}{a}\right)&=0\\ \frac{d}{dy}\left(\frac{1}{c}\right)+\frac{d}{dx}\left(\frac{a}{c}\right)&=0\end{aligned}\right\}$$

which are to be satisfied concurrently with the partial differential equation. The former of these conditions is

$$a\frac{dc}{dy}=\frac{da}{dx}+c\frac{da}{dy},$$

that is,

$$a\left\{\frac{\partial' c}{\partial' y}-(ar+ct)\frac{\partial c}{\partial p}+t\frac{\partial c}{\partial q}\right\}$$

$$=\frac{\partial' a}{\partial' x}+r\frac{\partial a}{\partial p}-(ar+ct)\frac{\partial a}{\partial q}+c\left\{\frac{\partial' a}{\partial' y}-(ar+ct)\frac{\partial a}{\partial p}+t\frac{\partial a}{\partial q}\right\}$$

concurrently with the equation ; and therefore we have

$$\left.\begin{aligned} a\frac{\partial' c}{\partial' y}&=\frac{\partial' a}{\partial' x}+c\frac{\partial' a}{\partial' y}\\ a^2\frac{\partial c}{\partial p}&=(ac-1)\frac{\partial a}{\partial p}+a\frac{\partial a}{\partial q}\\ c^2\frac{\partial a}{\partial p}&=ac\frac{\partial c}{\partial p}-a\frac{\partial c}{\partial q}\end{aligned}\right\},$$

apparently three relations. Similarly, the other condition leads to the three relations

$$\left.\begin{aligned} c\frac{\partial' a}{\partial' x}&=\frac{\partial' c}{\partial' y}+a\frac{\partial' c}{\partial' x}\\ c^2\frac{\partial a}{\partial q}&=(ac-1)\frac{\partial c}{\partial q}+c\frac{\partial c}{\partial p}\\ a^2\frac{\partial c}{\partial q}&=ac\frac{\partial a}{\partial q}-c\frac{\partial a}{\partial p}\end{aligned}\right\}.$$

It is easy to verify that the six relations thus obtained are completely satisfied in virtue of the four relations

$$\left.\begin{aligned} a\frac{\partial' c}{\partial' y}&=\frac{\partial' a}{\partial' x}+c\frac{\partial' a}{\partial' y}\\ c\frac{\partial' a}{\partial' x}&=\frac{\partial' c}{\partial' y}+a\frac{\partial' c}{\partial' y}\\ c^2\frac{\partial a}{\partial p}&=ac\frac{\partial c}{\partial p}-a\frac{\partial c}{\partial q}\\ a^2\frac{\partial c}{\partial q}&=ac\frac{\partial a}{\partial q}-c\frac{\partial a}{\partial p}\end{aligned}\right\},$$

which therefore are conditions to be satisfied by a and c in order that the equation may have a primitive of the specified type.

21. Consider, first, the case (if any) in which a and c are functions of x, y, and z only. The third and the fourth of the relations are then satisfied identically: and the other two, which are the effective conditions, can be taken

$$\frac{\partial'}{\partial' x}\left(\frac{1}{a}\right)+\frac{\partial'}{\partial' y}\left(\frac{c}{a}\right)=0,$$

$$\frac{\partial'}{\partial' y}\left(\frac{1}{c}\right)+\frac{\partial'}{\partial' x}\left(\frac{a}{c}\right)=0.$$

As a and c do not involve p or q, these equations give

$$\frac{\partial}{\partial z}\left(\frac{1}{a}\right)=0,\quad \frac{\partial}{\partial z}\left(\frac{c}{a}\right)=0,\quad \frac{\partial}{\partial z}\left(\frac{1}{c}\right)=0,\quad \frac{\partial}{\partial z}\left(\frac{c}{a}\right)=0;$$

hence a and c are functions of x and y only. The first of the relations then shews that some function θ, of x and y, exists such that

$$\frac{c}{a}=\frac{\partial\theta}{\partial x},\qquad -\frac{1}{a}=\frac{\partial\theta}{\partial y};$$

when these values of a and c are substituted in the second relation, it takes the form

$$\frac{\partial^2\theta}{\partial x^2}-\frac{\partial\theta}{\partial y}\frac{\partial^2\theta}{\partial x\partial y}+\frac{\partial\theta}{\partial x}\frac{\partial^2\theta}{\partial y^2}=0.$$

The primitive of this relation can be expressed* in the form

$$-x=\rho''+\sigma'',$$

$$-y=\lambda\rho''-\rho'+\mu\sigma''-\sigma',$$

$$\theta=\lambda^2\rho''-2\lambda\rho'+2\rho+\mu^2\sigma''-2\mu\sigma'+2\sigma,$$

where ρ is any function of the parameter λ, and σ is any function of the parameter μ. We easily find

$$\frac{\partial\theta}{\partial x}=\lambda\mu,\qquad \frac{\partial\theta}{\partial y}=-(\lambda+\mu),$$

so that

$$\frac{c}{a}=\lambda\mu,\qquad \frac{1}{a}=\lambda+\mu.$$

* See my *Theory of Differential Equations*, Vol. VI., p. 343.

Consequently, *the differential equation is*

$$r+(\lambda+\mu)\,s+\lambda\mu t = 0,$$

where (changing the signs of ρ and σ) λ and μ are functions of x and y, given by the equations

$$x = \rho''+\sigma'',$$

$$y = \lambda\rho''-\rho'+\mu\sigma''-\sigma';$$

and, for our purposes, ρ and σ are any functions of λ and μ.

This equation of the second order possesses two intermediate integrals, in the forms

$$p+\lambda q = \Phi'(\rho''), \qquad p+\mu q = \Psi'(\sigma''),$$

where Φ and Ψ are arbitrary functions: and then

$$\begin{aligned} dz &= p\,dx+q\,dy \\ &= (p+\lambda q)\,\rho'''\,d\lambda+(p+\mu q)\,\sigma'''\,d\mu, \end{aligned}$$

so that *the primitive is given by*

$$\left.\begin{aligned} z &= \Phi(\rho'')+\Psi(\sigma'') \\ x &= \rho''+\sigma'' \\ y &= \lambda\rho''-\rho'+\mu\sigma''-\sigma' \end{aligned}\right\},$$

where Φ and Ψ are arbitrary functions of their argument, and ρ and σ are specific functions of λ and μ.

As the equation possesses two intermediate integrals, it is known* to be transformable to $s = 0$ by contact-transformations.

22. Consider, next, the case in which a and c do not explicitly involve x, y, or z: we have

$$ac\frac{\partial c}{\partial p}-c^2\frac{\partial a}{\partial p} = a\frac{\partial c}{\partial q},$$

$$-a^2\frac{\partial c}{\partial q}+ac\frac{\partial a}{\partial q} = c\frac{\partial a}{\partial p},$$

when a and c are functions of p and q only, as the full equations for the determination of a and c.

The first gives $$\frac{\partial}{\partial p}\left(\log\frac{c}{a}\right) = -\frac{\partial}{\partial q}\left(\frac{1}{c}\right),$$

* *Theory of Differential Equations*, Vol. VI., p. 295.

and the second gives $\dfrac{\partial}{\partial q}\left(\log\dfrac{a}{c}\right) = -\dfrac{\partial}{\partial p}\left(\dfrac{1}{a}\right).$

Take a quantity θ such that

$$\frac{1}{c} = -\frac{\partial^2\theta}{\partial p^2},$$

so that, from the modified first relation,

$$\log\frac{c}{a} = \frac{\partial^2\theta}{\partial p\,\partial q}.$$

Hence $$\log\frac{a}{c} = -\frac{\partial^2\theta}{\partial p\,\partial q};$$

and therefore, from the modified second relation,

$$\frac{1}{a} = \frac{\partial^2\theta}{\partial q^2}.$$

If now we write temporarily

$$p,\ q,\ \theta = x,\ y,\ z,$$

the equation for $\theta\ (= z)$, in terms of p and q as the new independent variables, is

$$e^s = \frac{c}{a} = -\frac{t}{r},$$

that is, $$r + te^{-s} = 0.$$

23. This equation can be integrated by Darboux's method. Let λ and μ be the roots of the equation

$$\rho^2 + \rho\,te^{-s} + e^{-s} = 0,$$

so that $$\lambda + \mu = -te^{-s}, \qquad \lambda\mu = e^{-s}.$$

For either of the quantities λ and μ, we have

$$(2\rho + te^{-s})\frac{\partial\rho}{\partial t} + \rho e^{-s} = 0,$$

$$(2\rho + te^{-s})\frac{\partial\rho}{\partial s} - \rho\,te^{-s} - e^{-s} = 0.$$

Hence
$$\begin{aligned}(2\mu + te^{-s})\left(\mu\frac{\partial\mu}{\partial s} - \frac{\partial\mu}{\partial t}\right) &= \mu^2 te^{-s} + \mu e^{-s} + \mu e^{-s} \\ &= -\mu^2(\lambda+\mu) + 2\mu\,\lambda\mu \\ &= \mu^2(\lambda - \mu),\end{aligned}$$

that is,
$$\mu \frac{\partial \mu}{\partial s} - \frac{\partial \mu}{\partial t} = -\mu^2.$$

And
$$(2\mu + te^{-s})\left(\lambda \frac{\partial \mu}{\partial s} - \frac{\partial \mu}{\partial t}\right) = \lambda(\mu t+1)\, e^{-s} + \mu e^{-s}$$
$$= \lambda\mu\, te^{-s} + (\lambda+\mu)\, e^{-s}$$
$$= 0,$$

so that
$$\lambda \frac{\partial \mu}{\partial s} - \frac{\partial \mu}{\partial t} = 0.$$

Similarly,
$$\lambda \frac{\partial \lambda}{\partial s} - \frac{\partial \lambda}{\partial t} = -\lambda^2,$$
$$\mu \frac{\partial \lambda}{\partial s} - \frac{\partial \lambda}{\partial t} = 0.$$

Let
$$u(x, y, z, p, q, r, s, t) = 0$$
be an equation compatible with
$$r + te^{-s} = 0.$$
The equations to be satisfied* by the form of u are
$$\rho^2 \frac{\partial u}{\partial r} - \rho \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = 0,$$
and (as the original equation does not involve x, y, z, p, q)
$$\frac{du}{dx} + \rho' \frac{du}{dy} = 0,$$
where ρ and ρ' are λ and μ in either of the two arrangements.

Now r can be removed from u by means of the original equation: hence we can take the equations for u in a form
$$\lambda \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} = 0,$$
$$\frac{du}{dx} + \mu \frac{du}{dy} = 0.$$
The latter is
$$\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} + r \frac{\partial u}{\partial p} + s \frac{\partial u}{\partial q} + \mu \left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} + s \frac{\partial u}{\partial p} + t \frac{\partial u}{\partial q}\right) = 0;$$

* *Theory of Differential Equations*, Vol. VI., § 261; the method is Darboux's.

and λ, μ are functions of r, s, t only. The condition that the two should co-exist is

$$\left(\lambda \frac{\partial \mu}{\partial s} - \frac{\partial u}{\partial t}\right) \frac{du}{dy} + \lambda \left(\frac{\partial u}{\partial q} + \mu \frac{\partial u}{\partial p}\right) - \mu \frac{\partial u}{\partial q} = 0,$$

that is,
$$(\lambda - \mu) \frac{\partial u}{\partial q} + \lambda\mu \frac{\partial u}{\partial p} = 0,$$

or
$$(\lambda - \mu) \frac{\partial u}{\partial q} + e^{-s} \frac{\partial u}{\partial p} = 0.$$

That this may co-exist with the first, we must have

$$\left(\lambda \frac{\partial \lambda}{\partial s} - \frac{\partial \lambda}{\partial t} - \lambda \frac{\partial \mu}{\partial s} + \frac{\partial \mu}{\partial t}\right) \frac{\partial u}{\partial q} - \lambda e^{-s} \frac{\partial u}{\partial p} = 0,$$

that is,
$$-\lambda^2 \frac{\partial u}{\partial q} - \lambda e^{-s} \frac{\partial u}{\partial p} = 0.$$

Hence
$$\frac{\partial u}{\partial q} = 0, \qquad \frac{\partial u}{\partial p} = 0.$$

That these may co-exist with the second, we must have

$$\frac{\partial u}{\partial z} = 0.$$

Hence the equations for u are

$$\frac{\partial u}{\partial p} = 0, \qquad \frac{\partial u}{\partial q} = 0, \qquad \frac{\partial u}{\partial z} = 0, \qquad \frac{\partial u}{\partial r} = 0,$$

$$\lambda \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} = 0,$$

$$\frac{\partial u}{\partial x} + \mu \frac{\partial u}{\partial y} = 0.$$

Two independent integrals are μ, $y - \mu x$: hence we take

$$y - \mu x = \text{function of } \mu = \mu^2 F''(\mu),$$

as an equation compatible with the original differential equation.

Similarly, from the other system, we have a compatible equation

$$y - \lambda x = \lambda^2 G''(\lambda).$$

Here F and G are arbitrary functions of their arguments: and

$$\left.\begin{aligned} \lambda+\mu &= -te^{-s} = r \\ \lambda\mu &= e^{-s} \\ \frac{1}{\lambda}+\frac{1}{\mu} &= -t \end{aligned}\right\}.$$

Now $$d(rx+sy-p) = x\,dr+y\,ds$$
$$= x(d\lambda+d\mu)-y\left(\frac{d\lambda}{\lambda}+\frac{d\mu}{\mu}\right)$$
$$= -\frac{d\lambda}{\lambda}(y-x\lambda)-\frac{d\mu}{\mu}(y-x\mu)$$
$$= -\lambda G''d\lambda-\mu F''d\mu,$$

so that $$rx+sy-p = -\lambda G'+G-\mu F'+F.$$

Again, $$d(sx+ty-q) = x\,ds+y\,dt$$
$$= -x\left(\frac{d\lambda}{\lambda}+\frac{d\mu}{\mu}\right)+y\left(\frac{d\lambda}{\lambda^2}+\frac{d\mu}{\mu^2}\right)$$
$$= \frac{d\lambda}{\lambda^2}(y-x\lambda)+\frac{d\mu}{\mu^2}(y-x\mu)$$
$$= G''\,d\lambda+F''\,d\mu,$$

so that $$sx+ty-q = G'+F'.$$

Further,

$$d(rx^2+2sxy+ty^2)$$
$$= x^2\,dr+2xy\,ds+y^2\,dt+2(rx+sy)\,dx+2(sx+ty)\,dy,$$

so that

$$d(rx^2+2sxy+ty^2-2z)$$
$$= x^2\,dr+2xy\,ds+y^2\,dt+2(rx+sy-p)\,dx+2(sx+ty-q)\,dy.$$

But $$x^2dr+2xy\,ds+y^2dt = x^2(d\lambda+d\mu)-2xy\left(\frac{d\lambda}{\lambda}+\frac{d\mu}{\mu}\right)+y^2\left(\frac{d\lambda}{\lambda^2}+\frac{d\mu}{\mu^2}\right)$$
$$= (y-\lambda x)^2\frac{d\lambda}{\lambda^2}+(y-\mu x)^2\frac{d\mu}{\mu^2}$$
$$= \lambda^2G''^2\,d\lambda+\mu^2F''^2\,d\mu.$$

And

$$(rx+sy-p)\,dx+(sx+ty-q)\,dy$$
$$= (G+F)\,dx-(\lambda G'+\mu F')\,dx+(G'+F')\,dy$$
$$= (G+F)\,dx+G'(dy-\lambda\,dx)+F'(dy-\mu\,dx)$$
$$= (G+F)\,dx+(G'\,d\lambda+F'\,d\mu)\,x$$
$$+G'(\lambda^2G'''+2\lambda G'')\,d\lambda+F'(\mu^2F'''+2\mu F'')\,d\mu$$
$$=d\{(G+F)x\}+G'(\lambda^2G'''+2\lambda G'')\,d\lambda+F'(\mu^2F'''+2\mu F'')\,d\mu.$$

Consequently,

$$d(rx^2+2sxy+ty^2-2z) = \lambda^2G''^2\,d\lambda+\mu^2F''^2\,d\mu+2d\,\{(G+F)\,x\}$$
$$+2G'(\lambda^2G'''+2\lambda G'')\,d\lambda+2F'(\mu^2F'''+2\mu F'')\,d\mu\,;$$

hence

$$rx^2+2sxy+ty^2-2z-2\,(G+F)\,x$$
$$= 2\lambda^2G'G''+2\mu^2F'F''-\int\lambda^2G''^2\,d\lambda-\int\mu^2F''^2\,d\mu.$$

When r, s, t are eliminated from this equation by the relations

$$\lambda+\mu = r, \qquad \lambda\mu = e^{-s}, \qquad \frac{1}{\lambda}+\frac{1}{\mu} = -t,$$

the resulting value of z is θ: and λ, μ are expressed in terms of x and y by the equations

$$y-\lambda x = \lambda^2G'', \qquad y-\mu x = \mu^2F''.$$

For our purpose, we want $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial y^2}$,

in this notation: where x, y stand for original p and q. Thus

$$\frac{1}{c} = -\frac{\partial^2 z}{\partial x^2} = -\lambda-\mu,$$

$$\frac{1}{a} = \frac{\partial^2 z}{\partial y^2} = -\frac{\lambda+\mu}{\lambda\mu},$$

where $\quad q-\lambda p = \lambda^2G'', \qquad q-\mu p = \mu^2F''.$

Thus the equation, being $\quad ar+s+ct = 0,$

becomes $\quad \lambda\mu r-(\lambda+\mu)\,s+t = 0,$

where λ and μ are expressible* in terms of p and q by relations

$$q-\lambda p = \lambda^2G'', \qquad q-\mu p = \mu^2F''.$$

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* The equation of minimal surfaces is given by

$$\lambda^2G'' = i\,(1+\lambda^2)^{\frac{1}{2}}, \qquad \mu^2F'' = i\,(1+\mu^2)^{\frac{1}{2}}:$$

other equations are given by taking other specific forms for F and G.

24. The primitive of this equation can be obtained in the same way, due to Ampère, as the primitive of the equation of minimal surfaces.* Substituting

$$s' = q' - ty', \qquad r = p' - q'y' + ty'^2,$$

in the equation, and making it evanescent in t, we have

$$\lambda\mu\,(p' - q'y') - (\lambda+\mu)\,q' = 0,$$

$$\lambda\mu y'^2 + (\lambda+\mu)\,y' + 1 = 0.$$

From the second equation, we have

$$\lambda y' + 1 = 0, \qquad \mu y' + 1 = 0.$$

Taking $\lambda y' + 1 = 0$, we have the first equation in the form

$$-q' + \mu p' = 0.$$

But
$$q - \mu p = \mu^2 F''',$$

so that
$$\mu' = 0;$$

thus μ is constant for the set of equations.

Similarly, the other system possesses $\lambda =$ constant as an integral. We therefore take μ and λ as new independent variables.

Now
$$y' = -\frac{1}{\lambda},$$

when μ is constant: that is,

$$\frac{\partial y}{\partial \lambda} = -\frac{1}{\lambda}\,\frac{\partial x}{\partial \lambda}.$$

Similarly,
$$\frac{\partial y}{\partial \mu} = -\frac{1}{\mu}\,\frac{\partial x}{\partial \mu}.$$

Consequently,
$$\frac{\partial}{\partial \mu}\left(\frac{1}{\lambda}\,\frac{\partial x}{\partial \lambda}\right) = \frac{\partial}{\partial \lambda}\left(\frac{1}{\mu}\,\frac{\partial x}{\partial \mu}\right),$$

that is,
$$\frac{\partial^2 x}{\partial \lambda\,\partial \mu} = 0:$$

hence
$$\begin{aligned} x &= \text{function of } \lambda + \text{function of } \mu \\ &= \lambda^2 \phi'(\lambda) + \mu^2 \psi'(\mu) \\ &= \lambda^2 \phi' + \mu^2 \psi', \end{aligned}$$

say. Then
$$\begin{aligned} dy &= -\frac{1}{\lambda}\,\frac{\partial x}{\partial \lambda}\,d\lambda - \frac{1}{\mu}\,\frac{\partial x}{\partial \mu}\,d\mu \\ &= -(\lambda\phi'' + 2\phi')\,d\lambda - (\mu\psi'' + 2\psi')\,d\mu, \end{aligned}$$

* See my *Theory of Differential Equations*, Vol. VI., p. 277.

so that
$$y = -\lambda\phi' - \phi - \mu\psi' - \psi.$$

Finally, we have

$$\begin{aligned} dz &= p\,dx + q\,dy \\ &= \{p(\lambda^2\phi'' + 2\lambda\phi') - q(\lambda\phi'' + 2\phi')\}\,d\lambda \\ &\qquad + \{p(\mu^2\psi'' + 2\mu\psi') - q(\mu\psi'' + 2\psi')\}\,d\mu \\ &= -(\lambda\phi'' + 2\phi')\lambda^2 G''\,d\lambda - (\mu\psi'' + 2\psi')\mu^2 F''\,d\mu. \end{aligned}$$

Modifying the specific functions F and G, we can state the result as follows:—

The primitive of the differential equation

$$\lambda\mu r - (\lambda + \mu)s + t = 0,$$

where λ and μ are functions of p and q determined by the equations

$$q - \lambda p = f(\lambda), \qquad q - \mu p = g(\mu),$$

f and g denoting specific functions, is given by the three equations

$$\begin{aligned} x &= \lambda^2\phi' + \mu^2\psi', \\ -y &= \lambda\phi' + \phi + \mu\psi' + \psi, \\ -z &= \int (\lambda\phi'' + 2\phi')f(\lambda)\,d\lambda + \int (\mu\psi'' + 2\psi')\,g(\mu)\,d\mu, \end{aligned}$$

in which ϕ and ψ denote arbitrary functions of λ and of μ respectively.

Note.—It is easy to verify that the equation

$$\{(Nq+M)^2 - P^2\}\,r - 2\{(Np+L)(Nq+M) + PQ\}\,s + \{(Np+L)^2 - Q^2\}\,t = 0,$$

where L, M, N, P, Q are constants, is a special example of the preceding equation; the functions $f(\lambda)$ and $g(\mu)$ are linear functions of λ and of μ respectively.

25. Consider, next, the case (if it can arise) in which the coefficients a and c in the equation

$$ar + s + ct = 0$$

can be functions of x, y, p, q, subject to the condition that the integral equivalent of the equation is of the specified type.

The relations to be satisfied by a and c are

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(\frac{1}{a}\right) + \frac{\partial}{\partial y}\left(\frac{c}{a}\right) = 0 \\ \frac{\partial}{\partial y}\left(\frac{1}{c}\right) + \frac{\partial}{\partial x}\left(\frac{a}{c}\right) = 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} \frac{\partial}{\partial p}\left(\frac{1}{a}\right) + \frac{\partial}{\partial q}\left(\log\frac{a}{c}\right) = 0 \\ \frac{\partial}{\partial q}\left(\frac{1}{c}\right) + \frac{\partial}{\partial p}\left(\log\frac{c}{a}\right) = 0 \end{aligned}\right\}.$$

In the first pair of equations, the quantities p and q are parametric. We know that the equations are satisfied by taking

$$\frac{1}{a} = u+v, \qquad \frac{1}{c} = \frac{1}{u}+\frac{1}{v},$$

where $\quad x = f'(u)+g'(v), \qquad y = uf'(u)-f(u)+vg'(v)-g(v),$

the functions f and g being any whatever: hence, taking account of the possibility that a and c (and therefore u and v) can involve p and q, as well as x and y, and writing

$$f = f(u, p, q), \qquad g = g(v, p, q),$$

then the first pair of equations are satisfied by

$$x = \frac{\partial f}{\partial u}+\frac{\partial g}{\partial v}, \qquad y = u\frac{\partial f}{\partial u}-f+v\frac{\partial g}{\partial v}-g,$$

$$\frac{1}{a} = u+v, \qquad \frac{1}{c} = \frac{1}{u}+\frac{1}{v}, \qquad \frac{c}{a} = uv;$$

and, so far as the first pair of equations are concerned, the new functions f and g can be any whatever.

The functions f and g are to be determined so as to satisfy the second pair of equations. When the values of $1/a$, $1/c$, c/a are substituted in them, we have

$$\frac{\partial u}{\partial p}+\frac{\partial v}{\partial p}-\frac{1}{u}\frac{\partial u}{\partial q}-\frac{1}{v}\frac{\partial v}{\partial q} = 0,$$

$$-\frac{1}{u^2}\frac{\partial u}{\partial q}-\frac{1}{v^2}\frac{\partial v}{\partial q}+\frac{1}{u}\frac{\partial u}{\partial p}+\frac{1}{v}\frac{\partial v}{\partial p} = 0;$$

that is, the relations in the second pair are satisfied, if

$$u\frac{\partial u}{\partial p} = \frac{\partial u}{\partial q}, \qquad v\frac{\partial v}{\partial p} = \frac{\partial v}{\partial q}.$$

Now u and v are given as functions of x, y, p, q by the preceding two equations which express x and y in terms of f and g; hence f and g, as involving p and q, must be such that the two equations in u and v just obtained may be satisfied. But

$$0 = \frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial p}+\frac{\partial^2 g}{\partial v^2}\frac{\partial v}{\partial p}+\frac{\partial^2 f}{\partial u\,\partial p}+\frac{\partial^2 g}{\partial v\,\partial p},$$

$$0 = u\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial p}+v\frac{\partial^2 g}{\partial v^2}\frac{\partial v}{\partial p}+u\frac{\partial^2 f}{\partial u\,\partial p}+v\frac{\partial^2 g}{\partial v\,\partial p}-\frac{\partial f}{\partial p}-\frac{\partial g}{\partial p},$$

$$0 = \frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial q}+\frac{\partial^2 g}{\partial v^2}\frac{\partial v}{\partial q}+\frac{\partial^2 f}{\partial u\,\partial q}+\frac{\partial^2 g}{\partial v\,\partial q},$$

$$0 = u\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial q}+v\frac{\partial^2 g}{\partial v^2}\frac{\partial v}{\partial q}+u\frac{\partial^2 f}{\partial u\,\partial q}+v\frac{\partial^2 g}{\partial v\,\partial q}-\frac{\partial f}{\partial q}-\frac{\partial g}{\partial q}.$$

Thus there are six equations in which the four first derivatives of u and v occur linearly; when these are eliminated, we should have two equations satisfied by f and g as functions of p, q, u, v. These are easily found to be

$$u\frac{\partial^2 f}{\partial u\,\partial p}-\frac{\partial^2 f}{\partial u\,\partial q}-\frac{\partial f}{\partial p}+v\frac{\partial^2 g}{\partial v\,\partial p}-\frac{\partial^2 g}{\partial v\,\partial q}-\frac{\partial g}{\partial p}=0,$$

$$(u-v)\left(v\frac{\partial^2 g}{\partial v\,\partial p}-\frac{\partial^2 g}{\partial v\,\partial q}\right)+v\frac{\partial f}{\partial p}-\frac{\partial f}{\partial q}+v\frac{\partial g}{\partial p}-\frac{\partial g}{\partial q}=0,$$

which accordingly are two equations for the determination of f and g.

Remembering that f does not involve v, and that g does not involve u, we have, from the first of these two equations,

$$u\frac{\partial^2 f}{\partial u\,\partial p}-\frac{\partial^2 f}{\partial u\,\partial q}-\frac{\partial f}{\partial p}=\theta,$$

$$v\frac{\partial^2 g}{\partial v\,\partial p}-\frac{\partial^2 g}{\partial v\,\partial q}-\frac{\partial g}{\partial p}=-\theta,$$

where θ is a function of p and q alone; and then the second of the two equations becomes

$$-(u-v)\,\theta+v\frac{\partial f}{\partial p}-\frac{\partial f}{\partial q}+u\frac{\partial g}{\partial p}-\frac{\partial g}{\partial q}=0.$$

Differentiating the last with regard to u and to v separately, we have

$$-\theta+v\frac{\partial^2 f}{\partial u\,\partial p}-\frac{\partial^2 f}{\partial u\,\partial q}+\frac{\partial g}{\partial p}=0,$$

$$\theta+u\frac{\partial^2 g}{\partial v\,\partial p}-\frac{\partial^2 g}{\partial v\,\partial q}+\frac{\partial f}{\partial p}=0;$$

and then, differentiating the former of these with respect to v or the latter with respect to u, we have

$$\frac{\partial^2 f}{\partial u\,\partial p}+\frac{\partial^2 g}{\partial v\,\partial p}=0,$$

that is, we may take
$$\frac{\partial^2 f}{\partial u\,\partial p}=\phi=-\frac{\partial^2 g}{\partial v\,\partial p},$$

where ϕ is a function of p and q only, so that the latest deduced equations are

$$\frac{\partial g}{\partial p}-\theta+v\phi-\frac{\partial^2 f}{\partial u\,\partial q}=0,$$

$$\frac{\partial f}{\partial p}+\theta-u\phi-\frac{\partial^2 g}{\partial v\,\partial q}=0.$$

The earlier equations give

$$\frac{\partial f}{\partial p}+\theta-u\phi+\frac{\partial^2 f}{\partial u\,\partial q}=0,$$

$$\frac{\partial g}{\partial p}-\theta+v\phi+\frac{\partial^2 g}{\partial v\,\partial q}=0;$$

hence

$$\frac{\partial^2 f}{\partial u\,\partial q}=-\frac{\partial^2 g}{\partial v\,\partial q}=\psi,$$

where ψ is a function of p and q only; and so we have

$$\frac{\partial f}{\partial p}=u\phi-(\psi+\theta),$$

$$\frac{\partial g}{\partial p}=-v\phi+(\psi+\theta),$$

from the preceding relations. These satisfy

$$\frac{\partial^2 f}{\partial u\,\partial p}=-\frac{\partial^2 g}{\partial v\,\partial p}=\phi.$$

Proceeding from

$$\frac{\partial^2 f}{\partial u\,\partial q}=\psi=-\frac{\partial^2 g}{\partial v\,\partial q},$$

we have

$$\frac{\partial f}{\partial q}=u\psi+\chi,\qquad \frac{\partial g}{\partial q}=-v\psi+\omega,$$

where χ and ω are independent of u and v. When we substitute in the relation

$$-(u-v)\theta+v\frac{\partial f}{\partial p}-\frac{\partial f}{\partial q}+u\frac{\partial g}{\partial p}-\frac{\partial g}{\partial q}=0,$$

we find $\chi+\omega=0$; hence we have

$$\left.\begin{aligned}\frac{\partial f}{\partial p}&=u\phi-(\psi+\theta)\\ \frac{\partial g}{\partial p}&=-v\phi+(\psi+\theta)\end{aligned}\right\},\qquad \left.\begin{aligned}\frac{\partial f}{\partial q}&=u\psi+\chi\\ \frac{\partial g}{\partial p}&=-v\psi-\chi\end{aligned}\right\}.$$

In order that these may co-exist, we must have

$$\frac{\partial\phi}{\partial q}=\quad\frac{\partial\psi}{\partial p}=\frac{\partial^2\rho}{\partial p\,\partial q},$$

$$\frac{\partial(\psi+\theta)}{\partial q}=-\frac{\partial\chi}{\partial p}=\frac{\partial^2\sigma}{\partial p\,\partial q},$$

where ρ and σ are functions of p and q only. Hence

$$f = u\rho - \sigma + F(u),$$
$$g = -v\rho + \sigma + G(v),$$

where F and G are arbitrary functions of u and of v respectively. Now

$$x = \frac{\partial f}{\partial u} + \frac{\partial g}{\partial v}$$
$$= F'(u) + G'(v),$$
$$y = u\frac{\partial f}{\partial u} - f + v\frac{\partial g}{\partial v} - g$$
$$= uF'(u) - F(u) + vG'(v) - v\,;$$

in other words, u and v are functions of x and y only, that is, if u and v involve x and y, they do not involve p and q.

Consequently, *we have two classes of equations*

$$ar + s + ct = 0$$

having an integral system of the required type, such that

$$x,\ y,\ z = \text{function of } u + \text{function of } v\,;$$

in one class, a and c are appropriate functions of x and y alone: in the other class, a and c are appropriate functions of p and q alone.

It is easy to see that the two equations, when subjected to Legendre's transformation, lead to equations of the same form; but, when this transformation is applied to either integral, it leads to new integral systems of a different type. An example has already (§ 18) been given.

26. We may *summarise the results* of the preceding investigations as regards an equation

$$ar + bs + ct = 0,$$

where a, b, c involve no derivatives of order higher than the first, the equation being supposed to have an integral system represented by

$$x,\ y,\ z = \text{function of } u + \text{function of } v\,;$$

as follows :—

(A.) When $b = 0$, then a and c are constant: the equation can be transformed to

$$r + t = 0.$$

(B.) When b is not zero, then we can (by division) replace it by unity.

Then a and c can be functions of x and y alone; and they can be functions of p and q alone; but they cannot be functions of x, y, p, q simultaneously.

When a and c are functions of x and y alone, then, taking two parameters λ and μ, and any two functions ρ and σ of those parameters respectively, we express λ and μ in terms of x and y by the relations

$$x = \rho''+\sigma'', \qquad y = \lambda\rho''-\rho'+\mu\sigma''-\sigma';$$

the partial differential equation is

$$r+(\lambda+\mu)s+\lambda\mu t = 0,$$

and its primitive is

$$z = \Phi(\lambda)+\Psi(\mu),$$

where Φ and Ψ are arbitrary functions.

When a and c are functions of p and q alone, we take two parameters λ and μ, and any two functions f and g of those parameters, and we determine λ and μ in terms of p and q, by the relations

$$q-\lambda p = f(\lambda). \qquad q-\mu p = g(\mu);$$

the differential equation is

$$\lambda\mu r-(\lambda+\mu)s+t = 0;$$

and its primitive is

$$x = \lambda^2\phi'+\mu^2\psi',$$
$$y = \lambda\phi'+\phi+\mu\psi'+\psi,$$
$$-z = \int(\lambda\phi''+2\phi')f(\lambda)\,d\lambda+\int(\mu\psi''+2\psi')g(\mu)\,d\mu,$$

where ϕ and ψ are arbitrary functions.

CASE II.: *with Sub-cases.*

27. In this case, the functions f and h involve both u and v, together (possibly) with the arbitrary functions U and V; while g involves only v and the arbitrary function V, with (possibly) the derivatives of the latter.

There are, as before, sub-cases according to the non-evanescence or the evanescence of the quantities A and B. Most of the initial forms of the results for the present case can be deduced as special forms of the results in the preceding sections; we shall therefore not deal with the sub-cases in the same detail as before. For all the sub-cases, we have

$$g_1 = 0, \quad g_{12} = 0, \quad g_{11} = 0;$$

also

$$p = \frac{h_1}{f_1}, \qquad q = \frac{h_2}{g_2}-\frac{f_2h_1}{f_1g_2}.$$

II., (1).—Proceeding as in the least restricted of the former sub-cases (§ 8), in which the values of A and B do not vanish, being

$$A = \frac{\partial h}{\partial U_m}-p\frac{\partial f}{\partial U_m}, \qquad B = \frac{\partial h}{\partial V_n}-p\frac{\partial f}{\partial V_n}-q\frac{\partial g}{\partial V_n},$$

we have $U_{m+1} = -\frac{1}{A}(H_1 - pF_1), \qquad f_1 = \frac{1}{A}\left(F_1 \frac{\partial h}{\partial U_m} - H_1 \frac{\partial f}{\partial U_m}\right),$

and V_{n+1}, f_2 have the same formal values as before. We proceed from the equation
$$h_{12} - pf_{12} - qg_{12} = rf_1 f_2 + s(f_1 g_2 + f_2 g_1) + tg_1 g_2,$$
which now is
$$h_{12} - pf_{12} = rf_1 f_2 + sf_1 g_2;$$
and we find, on substitution, the equation
$$(q+b)r - (p+a)s = w,$$
where
$$\left.\begin{aligned} a\left(F_2 \frac{\partial g}{\partial V_n} - G_2 \frac{\partial f}{\partial V_n}\right) &= G_2 \frac{\partial h}{\partial V_n} - H_2 \frac{\partial g}{\partial V_n} \\ b\left(F_2 \frac{\partial g}{\partial V_n} - G_2 \frac{\partial f}{\partial V_n}\right) &= H_2 \frac{\partial f}{\partial V_n} - F_2 \frac{\partial h}{\partial V_n} \end{aligned}\right\},$$
and w is a polynomial in p and q, of degree two in p and degree one in q: and where it is assumed that $\frac{1}{F_2}\frac{\partial f}{\partial V_n} - \frac{1}{G_2}\frac{\partial g}{\partial V_n}$ is not zero. The quantities a and b, as well as the coefficients in w, involve only the magnitudes which occur in x, y, z: hence, if an equation of the second order is to be the equivalent of the integral system, these quantities must be functions of x, y, z alone.

We evidently have
$$a \frac{\partial f}{\partial V_n} - b \frac{\partial g}{\partial V_n} = -\frac{\partial h}{\partial V_n},$$
$$aF_2 + bG_2 = -H_2;$$
so that
$$\frac{dz}{dv} + a\frac{dx}{dv} + b\frac{dy}{dv} = 0.$$

28. The application of Ampère's method to the equation
$$(q+b)r - (p+a)s = w$$
requires the substitution of
$$r = p' - q'y' + ty'^2, \qquad s = q' - ty';$$
and the modified equation is made evanescent in t. Thus
$$(q+b)y'^2 + (p+a)y' = 0,$$
$$(q+b)(p' - q'y') - (p+a)q' = w.$$
The former equation gives either
$$y' = 0,$$
or
$$(q+b)y' + (p+a) = 0.$$

Hence there are two subsidiary systems of equations, viz.,

$$y' = 0, \qquad (q+b)\,p' - (p+a)\,q' = w, \qquad z' = p\,;$$

and $$(q+b)\,y' + (p+a) = 0, \qquad (q+b)\,p' = w, \qquad z' = p+qy'.$$

Consider, in particular, the special case when w is zero, so that the equation is

$$(q+b)\,r - (p+a)\,s = 0\,;$$

and we set aside, as trivial, the instances (i.) when a and b are constants; (ii.) when a is a constant, and when b is a function of y only. The first group of subsidiary equations then gives

$$y = \text{constant},$$

when β is variable: that is, we take

$$y = \alpha,$$

and then
$$(q+b)\frac{\partial p}{\partial \beta} - (p+a)\frac{\partial q}{\partial \beta} = 0,$$
$$\frac{\partial z}{\partial \beta} = p\,\frac{\partial x}{\partial \beta}.$$

Similarly, from the second set of equations, we have

$$p = \text{constant},$$

when α is variable: that is, we take

$$p = \beta,$$

and then
$$(q+b)\frac{\partial y}{\partial \alpha} + (p+a)\frac{\partial x}{\partial \alpha} = 0,$$
$$\frac{\partial z}{\partial \alpha} = p\,\frac{\partial x}{\partial \alpha} + q\,\frac{\partial y}{\partial \alpha}.$$

Accordingly, as connected with the equation

$$(q+b)\,r - (p+a)\,s = 0,$$

we can take y and p as the independent variables: and the aggregate of equations then is

$$\left.\begin{aligned} q+b-(p+a)\frac{\partial q}{\partial p} &= 0 \\ q+b+(p+a)\frac{\partial x}{\partial y} &= 0 \\ \frac{\partial z}{\partial p} &= p\,\frac{\partial x}{\partial p} \\ \frac{\partial z}{\partial y} &= p\,\frac{\partial x}{\partial y} + q \end{aligned}\right\}.$$

From the first and the second of these, we have

$$\frac{\partial x}{\partial y}+\frac{\partial q}{\partial p}=0,$$

which is the condition of coexistence of the third and the fourth. Consequently, a function (say ϕ) of y and p exists such that

$$x=\frac{\partial \phi}{\partial p}, \qquad q=-\frac{\partial \phi}{\partial y};$$

and then, from the third and the fourth of the equations, it follows that

$$z=p\frac{\partial \phi}{\partial p}-\phi.$$

The function ϕ then satisfies the equation

$$\frac{\partial^2 \phi}{\partial y\,\partial p}+\frac{q+b}{p+a}=0,$$

when for q, and for x and z in a and b, we substitute their values. From the nature of the case, ϕ must be an explicit function, expressible in a form that is free from partial quadratures; and, consequently, the equation must be integrable by Darboux's method. Now, when we change the notation for the variables, the equation can be written

$$s'+f(x', y', z', p', q')=0;$$

and the cases in which this equation is integrable by Darboux's method have been discussed by Goursat:* we need not therefore consider the equation further, for the case in which $w=0$. The general case still remains.

29. One special form of the case discussed arises, when the integral system is of the form

$$x, z = \text{function of } u+\text{function of } v,$$

$$y = \text{function of } v.$$

We then have $\qquad f_{12}=0, \qquad g_{12}=0, \qquad h_{12}=0,$

so that $w=0$; and the partial equation is

$$f_1 f_2 r+f_1 g_2 s=0,$$

that is, $$f_2 r+g_2 s=0.$$

* See chapter viii. of his treatise quoted at the beginning of this paper; also *Annales de Toulouse*, 2e Sér., t. I. (1899), pp. 31–78, 439–463.

Now $$\frac{g_2}{f_2} = \text{function of } v \text{ only}$$
$$= \quad ,, \quad y \text{ only}$$
$$= 1/Y',$$

say; so that, writing $$Y'\,dy = dy',$$
the equation becomes reduced to

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y'} = 0,$$

and its primitive is easily obtainable in the form

$$z - F(y') = G(x - y'),$$

where F and G are arbitrary functions.

30. II., (2).—In this sub-case, we have $B = 0$; and there are two alternatives. In the first of them, the two equations

$$\frac{\partial h}{\partial V_n} - p\frac{\partial f}{\partial V_n} - q\frac{\partial g}{\partial V_n} = 0,$$
$$H_2 - pF_2 - qG_2 = 0,$$

are effectively one and the same, so that

$$F_2 = \phi\frac{\partial f}{\partial V_n}, \qquad G_2 = \phi\frac{\partial g}{\partial V_n}, \qquad H_2 = \phi\frac{\partial h}{\partial V_n};$$

the circumstances, when F_2, G_2, H_2 vanish, are given by $\phi = 0$. The quantities α and γ of § 8 vanish, owing to the form of g, which is a quantity independent of u; and thus the degenerate expression of the equation of § 11 becomes*

$$r + as = c',$$

where a and c' are functions of x, y, z, p, q. But this form assumes that $\partial f/\partial V_n$ is not zero; if it should vanish, the equation is

$$s = c'.$$

In the other of the two alternatives, we have $A = 0$ as well as $B = 0$; we shall proceed to this almost immediately as the last of the sub-cases to be considered.

* An instance of this possibility is (taking account of the interchange of variables covered by the explanations in § 2) provided by

$$x = \phi''(u), \quad y = (u+v)^2\,\phi''(u) - 2(u+v)\,\phi'(u) + 2\phi(u) + \psi(v), \quad z = u + v;$$

the equation is $$s + tz^2 + 2q^2 z = 0,$$
and its primitive is easily obtained by quadratures.

31. II., (3).—In this sub-case, we have $A = 0$: again, there are two alternatives. The second of the alternatives is, for the moment, deferred until the discussion of the similar alternative for the last sub-case.

In the first of them, the two equations

$$\frac{\partial h}{\partial U_m} - p\frac{\partial f}{\partial U_m} = 0, \qquad H_1 - pF_1 = 0,$$

can be only one and the same equation; and it gives the value of p. As B is not zero for this alternative, the partial equation is

$$(\beta' + \gamma' q)\, r \frac{\partial f}{\partial U_m} - (a' + \gamma' p)\, s \frac{\partial f}{\partial U_m} = \gamma''' + a''' p + \beta''' q,$$

that is,

$$(q + b)\, r - (p + a)\, s = w,$$

where now w is linear in p and in q; but otherwise the final equation is the same as in § 27.

32. II., (4).—In this sub-case, we have $A = 0$, $B = 0$; it thus includes the two alternatives respectively deferred from the consideration of the last two cases.

We must have

$$H_1 = \theta \frac{\partial h}{\partial U_m}, \qquad F_1 = \theta \frac{\partial g}{\partial U_m};$$

if it should happen that

$$H_2 = \phi \frac{\partial h}{\partial V_n}, \qquad F_2 = \phi \frac{\partial f}{\partial V_n}, \qquad G_2 = \phi \frac{\partial g}{\partial V_n},$$

then, by § 11, we easily see that the final partial equation is

$$r + s = k,$$

where k is a function of x, y, z, p, q at the utmost: the equation is transformable to

$$s + f(x, y, z, p, q) = 0,$$

and therefore (having regard to the integral system, which is derivable by Darboux's method) it belongs to the class already considered by Goursat (p. 170, foot-note).

If it should happen that the relations

$$H_2 = \phi \frac{\partial h}{\partial V_n}, \qquad F_2 = \phi \frac{\partial f}{\partial V_n}, \qquad G_2 = \phi \frac{\partial g}{\partial V_n}$$

are not satisfied, either for a non-zero value or for a zero value of ϕ, then we have the alternative considered in § 13; and the result for the present case can be deduced from the result there obtained by putting

$$G_1 = 0, \qquad \psi_1 = 0:$$

the result is of the form

$$\{(rt-s^2)(ar+\beta s+\gamma t+\delta)+U\}^2 = a'^2\beta'\{a''-(rt-s^2)\beta''\},$$

where a, β, γ, δ, β'' are functions of x, y, z, p, q; U is a polynomial in r, s, t, 1 of the second degree, and a', β', a'' are polynomials in r, s, t, 1 of the first degree only.

If, in particular, while g is a function of v only, and the assumptions as to the conditions under which $A = 0$ and $B = 0$ are satisfied still hold, it should happen that

$$x,\ z = f,\ h = \text{function of } u + \text{function of } v,$$

so that

$$k_0 = 0 = l_0 = m_0 = n_0,$$

it is easy to deduce (from earlier results) that the final equation is of the form

$$rt-s^2 = 0.$$

This conclusion can be verified from the fact that the two independent equations

$$H_2-pF_2-qG_2 = 0, \qquad \frac{\partial h}{\partial V_n}-p\frac{\partial f}{\partial V_n}-q\frac{\partial g}{\partial V_n} = 0$$

determine p and q as functions of v alone, so that we have

$$\chi(p, q) = 0;$$

and therefore

$$rt-s^2 = 0.$$

CASE III.: *with Sub-cases.*

33. In this case, f is a function of u only, g is a function of v only, while h is a function of u and v, the arbitrary function U and its derivatives occurring with u, and likewise for V and v.

We thus have

$$f_2 = 0, \qquad f_{12} = 0, \qquad f_{22} = 0;$$

$$g_1 = 0, \qquad g_{12} = 0, \qquad g_{11} = 0.$$

If the quantity A, which now is $\dfrac{\partial h}{\partial U_m}-p\dfrac{\partial f}{\partial U_m}$, does not vanish, and. if similarly B, which now is $\dfrac{\partial h}{\partial V_n}-q\dfrac{\partial g}{\partial V_n}$, which does not vanish, then

$$U_{m+1} = -\frac{1}{A}(H_1-pF_1), \qquad V_{n+1} = -\frac{1}{B}(H_2-qG_2),$$

$$f_1 = \frac{1}{A}\left(F_1\frac{\partial h}{\partial U_m}-H_1\frac{\partial f}{\partial U_m}\right), \qquad g_2 = \frac{1}{B}\left(G_2\frac{\partial h}{\partial V_n}-H_2\frac{\partial g}{\partial V_n}\right).$$

where p and q are given by the relations

$$h_1 = pf_1, \qquad h_2 = qg_2.$$

The equation $h_{12} - pf_{12} - qg_{12} = f_1 f_2 r + (f_1 g_2 + f_2 g_1) s + g_1 g_2 t$

now becomes $$h_{12} = f_1 g_2 s;$$

and, on substituting the values of U_{m+1} and V_{n+1} in h_{12} and reducing, we find the equation

$$s = \alpha + \beta p + \gamma q + \zeta pq,$$

where $\alpha, \beta, \gamma, \zeta$ are functions of x, y, z only; and the value of ζ is given by

$$\left(F_1 \frac{\partial h}{\partial U_m} - H_1 \frac{\partial f}{\partial U_m}\right)\left(G_2 \frac{\partial h}{\partial V_n} - H_2 \frac{\partial g}{\partial V_n}\right)\zeta$$

$$= F_1 G_2 \frac{\partial^2 h}{\partial U_m \partial V_n} - F_1 \frac{\partial g}{\partial V_n}\frac{\partial(\delta' h)}{\partial U_m} - G_2 \frac{\partial f}{\partial U_m}\frac{\partial(\delta h)}{\partial V_n} + \frac{\partial f}{\partial U_m}\frac{\partial g}{\partial V_n}(\delta\delta' h),$$

in the notation of § 11.

This differential equation is to have an integral equivalent of the form

$$x = \text{function of } u, \qquad y = \text{function of } v,$$

$$z = \text{function of } u \text{ and } v.$$

The problem of the determination of the different forms of the equation with their respective integral equivalents is similar to the problem propounded by Moutard, and solved by Lloyd Tanner and Cosserat;* the necessary modifications of the analysis are omitted from the present communication.

When A is zero but not B, or when B is zero but not A, the final differential equation is easily seen to be

$$s = a + bp + cq,$$

where a, b, c are functions of x, y, z alone: it is only a particular case of the preceding form.

34. There remains the sub-case, for which $A = 0$ and $B = 0$, that is,

$$\frac{\partial h}{\partial U_m} - p\frac{\partial f}{\partial U_m} = 0, \qquad H_1 - pF_1 = 0,$$

$$\frac{\partial h}{\partial V_n} - q\frac{\partial g}{\partial V_n} = 0, \qquad H_2 - qG_2 = 0.$$

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* See my *Theory of Differential Equations*, Vol. VI., ch. XV.

The form of the final differential equation might be deduced from the result of § 13, by inserting the appropriate simplifications; it is as easy to proceed directly from the equations

$$h_{11}-pf_{11}=f_1^2 r, \qquad h_{12}=f_1 g_2 s, \qquad h_{22}-qg_{22}=g_2^2 t,$$

which, on writing

$$\left.\begin{aligned} a &= -\phi^2 r+a_0 \\ b &= -\xi\phi r+b_0 \\ c &= -\xi^2 r+c_0 \end{aligned}\right\}, \qquad \left.\begin{aligned} a' &= -\psi^2 t+a_0' \\ b' &= -\eta\psi t+b_0' \\ c' &= -\eta^2 t+c_0' \end{aligned}\right\},$$

$$\left.\begin{aligned} k &= -\phi\psi s+k_0 \\ l &= -\phi\eta s+l_0 \\ m &= -\psi\xi s+m_0 \\ n &= -\eta\xi s+n_0 \end{aligned}\right\},$$

where $\phi=\dfrac{\partial f}{\partial U_m}, \qquad \psi=\dfrac{\partial g}{\partial V_n}, \qquad \xi=F_1, \qquad \eta=G_2,$

and where $a_0, b_0, c_0, a_0', b_0', c_0', k_0, l_0, m_0, n_0$ have their former signification, become

$$aU_{m+1}^2+2bU_{m+1}+c=0,$$

$$a'V_{n+1}^2+2b'V_{n+1}+c'=0,$$

$$kU_{m+1}V_{n+1}+lU_{m+1}+mV_{n+1}+n=0.$$

The final equation is the result of eliminating U_{m+1} and V_{n+1} between these three equations. Let

$$\alpha_1=k_0\xi\eta-l_0\xi\psi-m_0\eta\phi+n_0\phi\psi,$$

$$\beta_1=a_0\xi^2-2b_0\xi\phi+c_0\phi^2,$$

$$\gamma_1=a_0'\eta^2-2b_0'\eta\psi+c_0'\psi^2,$$

$$E=(a_0', -b_0', c_0' \between l_0\xi-n_0\phi,\ k_0\xi-m_0\phi)^2,$$

$$G=(a_0, -b_0, c_0 \between l_0\eta-n_0\psi,\ k_0\eta-m_0\psi)^2,$$

$$\begin{aligned} F = {} & k_0(c_0\phi-b_0\xi)(c_0'\psi-b_0'\eta)-l_0(c_0\phi-b_0\xi)(b_0'\psi-a_0'\eta) \\ & -m_0(b_0\phi-a_0\xi)(c_0'\psi-b_0'\eta)+n_0(b_0\phi-a_0\xi)(b_0'\psi-a_0'\eta), \end{aligned}$$

$$\theta_0=k_0^2 c_0 c_0'-\ldots+n_0^2 a_0 a_0',$$

as before; then the eliminant, being the final equation, is

$$\Theta^2 = 4(a_1 s - k_0 n_0 + l_0 m_0)^2 (b_0^2 - a_0 c_0 + \beta_1 r)(b_0'^2 - a_0' c_0' + \gamma_1 t),$$

where
$$\Theta = a_1^2 rt + \beta_1 \gamma s^2 - Er - 2Fs - Gt + \theta_0.$$

The aggregate of terms of the highest degree in this equation is made up of those of degree 4; it is

$$(a_1^2 rt - \beta_1 \gamma s^2)^2.$$

Simpler forms, however, occur for special forms of the functions. Thus, in particular, if the integral equivalent is

$$x = \text{function of } u, \qquad y = \text{function of } v,$$

$$h = \text{function of } u + \text{function of } v,$$

the equation reduces to
$$s = 0,$$

as (obviously beforehand) should be the final form.