11.

De formatione et proprietatibus Determinantium.

(Auct. *C. G. J. Jacobi* prof. ord. math. Regiom.)

1.

Sunt quidem notissimi Algorithmi, qui aequationum linearium litteralium resolutioni inserviunt. Neque tamen video eorum proprietates praecipuas, ita breviter enarratas atque in conspectum positas esse, quantum optare debemus propter earum in gravissimis quaestionibus Analyticis usum. Scilicet illae proprietates quamvis elementares non omnes ita tritae sunt, ut quas indemonstratas relinquere deceat et valde molestum est earum demonstrationibus altiorum ratiociniorum decursum interrumpere. Cui defectui hic supplere volo quo commodius in aliis commentationibus ad hanc recurrere possim; neutiquam vero mihi propono totam illam materiam absolvere. Adjeci sub finem Propositiones quasdam ad Methodum minimorum Quadratorum pertinentes, quibus explicetur quomodo incognitarum valores eorumque Pondera, Methodo illa determinata, pendeant a diversis valoribus et ponderibus quae obtinentur pro diversis Combinationibus numeri Observationum numero incognitarum aequalis, qui earum determinationi sufficit. Quae ad computum inutilia, facere tamen possunt ad naturam illorum valorum et Ponderum melius cognoscendam.

2.

Proponatur productum conflatum ex omnibus $\frac{n(n+1)}{2}$ differentiis $n+1$ quantitatum a_0, a_1, ... a_n,

$$\begin{aligned} P = (a_1-a_0)(a_2-a_0)(a_3-a_0) & \ldots\ldots (a_n-a_0) \\ (a_2-a_1)(a_3-a_1) & \ldots\ldots (a_n-a_1) \\ (a_3-a_2) & \ldots\ldots (a_n-a_2) \\ & \ldots\ldots\ldots\ldots \\ & (a_n-a_{n-1}); \end{aligned}$$

quod productum omnimodis permutando quantitates a_i valorem absolutum mutare non potest, sed aut valorem eundem servat aut in oppositum abit. Vocemus eas indicum 0, 1, ... n Permutationes, pro quibus P valorem eundem servat, *positivas;* eas, pro quibus P valorem oppositum induit, *negativas,* sive priores dicamus pertinere ad *classem positivam Permutationum,* posteriores ad *classem negativam.* Binis propositis Permutationibus quibuscunque, certa exstabit Permutatio, qua post alteram adhibita altera prodit. *Pertinebunt duae Permutationes propositae ad classem eandem aut ad classes oppositas, prout Permutatio, qua altera ex altera obtinetur, ad classem positivam aut negativam pertinet.* Tribus enim Permutationibus abeat P respective in εP, $\varepsilon' P$, $\varepsilon'' P$, ipsis ε, ε', ε'' denotantibus ± 1; si secunda Permutatio *post* primam adhibetur, abit P successive in εP, $\varepsilon . \varepsilon' P$; unde si secundam Permutationem post primam adhibendo nascitur tertia, fit

$$\varepsilon'' = \varepsilon \varepsilon'.$$

Hinc prout ε' aut $+1$ aut -1, hoc est prout Permutatio qua tertia e prima obtinetur ad classem positivam aut negativam pertinet, Permutationes prima et tertia ad classem eandem aut oppositam pertinent, et vice versa. Sequitur ex antecc., Permutationes ad eandem classem pertinentes, si nova fiat Permutatio, aut cunctas simul in eadem classe manere aut cunctas simul in alteram classem transire. Scilicet fit illud aut hoc, prout Permutatio ad classem positivam aut negativam pertinet. Si plures Permutationes aliae post alias adhibentur, diversae nasci possunt Permutationes pro diverso quo aliae post alias adhibentur *ordine.* Etenim Permutatione aliqua loco 0, 1, 2 etc. ponatur i_0, i_1, i_2 etc. atque alia quadam Permutatione k_0, k_1, k_2 etc.; secunda post primam adhibita, ipsorum 0, 1, 2 etc. locum occupabunt

$$k_{i_0}, \quad k_{i_1}, \quad k_{i_2} \text{ etc.},$$

prima vero post secundum adhibita,

$$i_{k_0}, \quad i_{k_1}, \quad i_{k_2} \text{ etc.}$$

neque necessarium est fieri

$$k_{i_m} = i_{k_m}.$$

At prorsus eadem methodo, qua Propositio praecedens, demonstratur, *Permutationes diversas quae nascantur pro diverso ordine quo Permutationes complures aliae post alias adhibentur ad eandem pertinere classem.*

Designantibus i et i' binos indices quoscunque, productum P sic exhibere licet:

$$P = \pm(a_i - a_{i'}) . \Pi(a_k - a_i)(a_k - a_{i'}) . \Pi(a_k - a_{k'}),$$

siquidem designant

$$\Pi(a_k - a_i)(a_k - a_{i'}), \qquad \Pi(a_k - a_{k'})$$

producta omnium ipsius P factorum $(a_k - a_i)(a_k - a_{i'})$ vel $a_k - a_{k'}$, qui obtinentur tribuendo ipsi k vel utrique k, k' valores ab i et i' diversos. Quae duo producta alterum ipsorum i, i' respectu symmetricum est alterum iis vacat unde permutando indices i et i' non mutantur. Contra ex permutatione factor singularis $a_i - a_{i'}$ valorem oppositum induit; unde *ipsum productum propositum P permutando binos indices valorem oppositum induit.* Duorum igitur indicum commutatio est Permutatio negativa, unde Permutationes positivae si denuo bini indices commutantur cunctae in negativas, negativae cunctae in positivas transeunt.

Reciprocas vocare licet binas Permutationes, quibus altera post alteram adhibitis positio primitiva non mutatur. Statuamus Permutatione aliqua loco 0, 1, 2 etc. poni i_0, i_1, i_2 etc.; erit Permutatio reciproca qua 0, 1, 2 etc. loco i_0, i_1, i_2 etc. ponitur. Binae Permutationes reciprocae ad eandem classem pertinent, cum altera post alteram adhibita ipsum P non mutetur.

3.

Ut cognoscatur an Permutatio proposita sit positiva an negativa, variae assignari possunt regulae. Statuamus indicibus permutatis loco

$$0, \quad 1, \quad 2 \quad \dots\dots \quad n$$

respective positos esse

$$i_0, \quad i_1, \quad i_2 \quad \dots\dots \quad i_n,$$

ac quaeratur an hac permutatione productum P immutatum maneat an signum mutet. Producti P factores singuli ita exhibiti sunt ut elementum minore indice affectum de elemento maiore indice affecto detrahatur. Itaque si r et s bini sunt indicum 0, 1, 2 .... n, atque $r < s$, erit ipsius P factor

$$a_s - a_r$$

qui factor permutatione assignata abit in

$$a_{i_s} - a_{i_r}$$

qui et ipse seu illi oppositus erit inter ipsius P factores prout $i_s > r$ aut $i_s < r$. Itaque si in serie numerorum,

$$i_0, \quad i_1, \quad i_2 \quad \dots\dots \quad i_n,$$

m vicibus evenit ut post numerum aliquem i_r invenietur minor numerus i_s, totidem vicibus producti P factor aliquis signum mutat sive Permutatione indicata mutatur P in

$$(-1)^m P,$$

eritque Permutatio positiva aut negativa prout m par aut impar est. Quam regulam olim *Cel. Cramer* dedit, *ill. Laplace* demonstravit.

Sint

$$i_0, \quad i_1, \quad i_2 \quad \ldots . \quad i_m$$

quicunque indicum $0, 1, 2 \ldots . n-1$, ac consideremus eam Permutationem qua mutatur i_0 in i_1, i_1 in i_2 etc. ac postremo i_m in i_0. Ad eandem Permutationem pervenimus, si primum i_0 cum i_1, deinde i_0 cum i_2 etc. postremo i_0 cum i_m commutamus. Unde una illa Permutatio obtinetur m vicibus commutando duo elementa ideoque est Permutatio positiva aut negativa prout m par aut impar sive prout indicum numerus $m+1$ impar aut par est.

Ponamus Permutatione aliqua proposita quacunque mutari indices i_0 in i_1, i_1 in i_2, i_2 in i_3 ac generaliter i_{k-1} in i_k: pervenitur tandem ad indicem i_m qui in i_0 mutatur, neque antea ad aliquem praecedentium indicum reditur. Ponamus enim in serie indicum i_0, i_1, $i_2 \ldots .$ inveniri indicem i_λ qui in indicem aliquem praecedentem i_k mutetur; cum Permutatione quacunque unus tantum index in datum quendam indicem mutetur, fieri debet $i_\lambda = i_{k-1}$ ideoque etiam $i_{\lambda-1} = i_{k-2}$, $i_{\lambda-2} = i_{k-3}$ et ita porro usque dum habeatur $i_{\lambda-k+1} = i_0$. Unde fit $i_{\lambda-k+1} = i_m$, ideoque indicem i_λ, qui in indicem aliquem praecedentem i_k mutatur, semper antecedit index i_m qui in i_0 mutatur. Si indices i_0, $i_1 \ldots . i_m$ non cunctos effingunt indices $0, 1, 2 \ldots . n$, et Permutatione proposita reliqui indices quoque inter se commutantur: sit eorum aliquis h_0, rursus habetur *cyclus* indicum

$$h_0, \quad h_1, \quad h_2 \quad \ldots . \quad h_l,$$

qui Permutatione proposita quilibet in proxime sequentem, ultimus in primum mutantur. Si ita pergimus usque dum omnes indices exhauriantur, patet pro unaquaque Permutatione indices una quadam et necessaria ratione disponi posse in cyclos, ita ut indices in singulos cyclos dispositi ea Permutatione quilibet in proxime sequentem, ultimus in primum abeat.

Proposita Permutatione aliqua, disponantur secundum antecedentia indices $0, 1, 2 \ldots . n$ in cyclos, quorum numerus sit p singulique cycli respective formentur

$$\alpha_1, \quad \alpha_2, \quad \alpha_3 \quad \ldots . \quad \alpha_p$$

indicibus ita ut sit

$$\alpha_1 + \alpha_2 + \alpha_3 \ldots . + \alpha_p = n+1.$$

Si cyclus aliquis unico indice constat sive antecedentium numerorum α_1 etc. aliquis unitati aequalis est, index ille non in alium neque alius in eum mutatur. Cuilibet cyclo k indicibus constanti vidimus respondere Permutationem quae obtineri potest $k-1$ vicibus duos indices inter se commutando. Unde Permutatio proposita obtineri potest,

$$\alpha_1+\alpha_2+\alpha_3 \ldots\ldots+\alpha_p-p=n+1-p$$

vicibus duo elementa inter se permutando *). Unde Permutatio proposita est positiva aut negativa prout $n+1-p$ par aut impar est sive ***prout detrahendo de numero indicum numerum cyclorum in quos indices Permutatione proposita discedunt, residuum par aut impar fit.*** Hanc pulchram regulam qua Permutatio proposita positiva an negativa sit cognoscatur, dedit ill. ***Cauchy (Éc. Pol. cah. 17. p. 41).***

4.

Propositis $(n+1)^2$ quantitatibus

$$a_k^{(i)},$$

in quibus indices et superiores i et inferiores k valores omnes $0, 1, 2 \ldots\ldots n$ induant, producatur terminus

$$a\, a_1'\, a_2'' \ldots\ldots a_n^{(n)} \text{ **)};$$

ex eoque numerus $1.2.3 \ldots\ldots (n+1)$ terminorum similium formetur indices aut superiores aut inferiores omnimodis inter se permutando. Singulis deinde terminis signum aut positivum aut negativum praefigatur, prout Permutationes quibus e termino $a\, a_1'\, a_2'' \ldots\ldots a_n^{(n)}$ obtinentur positiva aut negativa sunt, omniumque $1.2.3 \ldots\ldots (n+1)$ terminorum suis signis acceptorum fiat Aggregatum, quod designabo per

$$R=\Sigma \pm a\, a_1'\, a_2'' \ldots\ldots a_n^{(n)}.$$

Eiusmodi Aggregatum R praeunte ill. *Gauss* aliisque ***Determinans*** appellabo, ipsas quantitatis $a_k^{(i)}$ Determinantis *elementa* et cum ipsius R terminus quilibet e $n+1$ elementis producatur ipsum R dicam Determinans $n+1^{ti}$ *gradus*.

Quilibet Determinantis R *terminus*

$$a_k\, a'_{k'}\, a''_{k''} \ldots\ldots a^{(n)}_{k^{(n)}}$$

*) Patet simul, paucioribus commutationibus duorum elementorum Permutationem propositam obtineri non posse.

**) Indicem (0) in genere non scribo ita ut $a^{(i)}$, a_k, a loco $a_0^{(i)}$, $a_k^{(0)}$, $a_0^{(0)}$ ponatur vel quantitates $a^{(i)}$; a_k, a respective dicantur inferiore aut superiore aut utroque indice (0) affecti.

e termino $a_0 a'_1 a''_2 \ldots . a^{(n)}_n$ duplici modo obtineri potest, sive loco indicum inferiorum 0, 1, 2 .... n ponendo respective $k, k', k'' \ldots . k^{(n)}$, sive ponendo 0, 1, 2 .... n loco indicum superiorum $k, k', k'' \ldots . k^{(n)}$. Quae Permutationes sunt reciprocae ideoque ad eandem classem pertinent; unde Determinantis termini iisdem signis afficiuntur, regula signorum apposita sive inferiorum sive superiorum indicum permutationibus adhibeatur. Cum nova Permutatione quacunque facta eiusdem classis Permutationes simul omnes in eadem classe maneant sive omnes simul in oppositam classem transeant, sequitur, *quacunque indicum superiorum inferiorumve Permutatione Determinans aut non mutari aut valorem oppositum induere.* Porro cum binorum indicum Permutatione classis Permutationum positiva in negativam, negativa in positivam abeat, sequitur *binos quoscunque sive superiores sive inferiores indices permutando Determinans valorem oppositum induere.* Quae Determinantis proprietas principalis et characteristica est. Unde haec altera fluit propositio fundamentalis, *evanescere Determinans quoties bini indices sive superiores sive inferiores inter se aequales existant,* siquidem breviter indices inter se aequales dicimus ubi quantitates iis affectae aequales sunt. Scilicet si duo indices inter se aequales sunt, eorum permutatione nihil mutatur, qua tamen permutatione cum per proprietatem characteristicam Determinans in valorem oppositum abeat, fieri debet $R = -R$ sive $R = 0$.

5.

Adnotamus casus quosdam speciales quibus Determinantia in simpliciorem formam sive etiam in unicum terminum redeunt. Exhibito Determinante R sequente modo,

$$1. \quad R = \Sigma \pm a a'_1 \ldots . a^{(m)}_m \, a^{(m+1)}_{m+1} \ldots . a^{(n)}_n,$$

ubi $m < n$; ponamus pro omnibus ipsius i valoribus

$$0, \; 1, \; 2, \; \ldots . \; m-1,$$

esse

$$2. \quad a^{(m)}_i = a^{(m+1)}_i \ldots . = a^{(n)}_i = 0.$$

Reiiciendo Determinantis terminos evanescentes, ii tantum remanent termini,

$$\pm a_i a'_{i'} \ldots . a^{(m)}_{i^m} \ldots . a^{(n)}_{i^n},$$

in quibus indices inferiores,

$$i^m, \quad i^{m+1}, \quad \ldots . \quad i^n,$$

conveniunt cum numeris

$$m, \quad m+1, \quad \dots. \quad n;$$

ordinis respectu non habito. Nam si indicum i^m, i^{m+1} etc. vel unus aequaret aliquem numerorum 0, 1, 2, ... $m-1$, terminus ex hypothesi facta evanesceret. Unde sequitur, quia in quolibet Determinantis termino indices elementis subscripti omnes inter se diversi esse debent, reliquos indices inferiores

$$i, \quad i', \quad i'', \quad \dots. \quad i^{m-1}$$

ordinis respectu non habito convenire cum numeris,

$$0, \quad 1, \quad 2, \quad \dots. \quad m-1,$$

neque valores m, $m+1$ etc. induere. Qua de re eruuntur cuncti Determinantis termini ex uno

$$\pm a a'_1 a''_2 \dots. a^{(m-1)}_{m-1} . \pm a^{(m)}_m a^{(m+1)}_{m+1} \dots. a^{(n)}_n,$$

seorsim inter se permutando indices

$$0, \quad 1, \quad 2, \quad \dots. \quad m-1$$

atque indices

$$m, \quad m+1, \quad m+2, \quad \dots. \quad n,$$

signis insuper ancipitibus $\pm$ ita determinatis ut termini qui binorum indicum permutatione alter in alterum abeunt signis oppositis afficiantur. Unde fit

$$3. \quad R = \Sigma \pm a a'_1 \dots. a^{(m-1)}_{m-1} . \Sigma \pm a^{(m)}_m a^{(m+1)}_{m+1} \dots. a^{(n)}_n,$$

sive habetur Propositio:

I. Quoties pro indicis k valoribus 0, 1, 2, ... $m-1$ evanescant elementa $a^{(m)}_k$, $a^{(m+1)}_k$, ... $a^{(n)}_k$, Determinans

$$\Sigma \pm a a'_1 a''_2 \dots. a^{(n)}_n$$

abire in productum a duobus Determinantibus

$$\Sigma \pm a a'_1 \dots. a^{(m-1)}_{m-1} . \Sigma \pm a^{(m)}_m a^{(m+1)}_{m+1} \dots. a^{(n)}_n.$$

Prorsus eadem valet Propositio si pro indicis i valoribus 0, 1, 2, ... $m-1$ elementa $a^{(i)}_m$, $a^{(i)}_{m+1}$, ... $a^{(i)}_n$ evanescunt. Si in Propositione antecedente insuper pro indicis i valoribus 0, 1, ... $l-1$ evanescunt elementa $a^{(l)}_i$, $a^{(l+1)}_i$, ... $a^{(m)}_i$, Determinans R in productum e tribus Determinantibus abit et ita porro.

Est casus simplicissimus Propositionis antecedentis quo elementa certo quodam indice superiore affecta pro indicibus inferioribus praeter unum omnibus evanescunt, quippe quo casu alterum Determinantium e quibus R producitur in simplex elementum abit. Sit enim

$$a^{(n)} = a_1^{(n)} \;\ldots\; = a_{n-1}^{(n)} = 0,$$

fit:

$$4.\qquad \Sigma \pm a a'_1 a''_2 \ldots a_{n-1}^{(n-1)} a_n^{(n)} = a_n^{(n)} \Sigma \pm a a'_1 \ldots a_{n-1}^{(n-1)}.$$

Si insuper fit,

$$a^{(n-1)} = a_1^{(n-1)} \;\ldots\; = a_{n-2}^{(n-1)} = 0,$$

eadem ratione e (4.) sequitur:

$$\Sigma a a'_1 a''_2 \ldots a_n^{(n)} = a_{n-1}^{(n-1)} a_n^{(n)}.\Sigma \pm a a'_1 \ldots a_{n-2}^{(n-2)}.$$

Sic pergendo eruimus Propositionem hanc:

II. Evanescentibus elementis omnibus,

$$a_k^{(m)},\quad a_k^{(m+1)} \;\ldots\; a_k^{(n)},$$

in quibus respective index inferior k indicibus superioribus m, $m+1$, ... n, minor est, fieri

$$\Sigma \pm a a'_1 a''_2 \ldots a_n^{(n)} = a_m^{(m)} a_{m+1}^{(m+1)} \ldots a_n^{(n)} \Sigma \pm a a'_1 \ldots a_{m-1}^{(m-1)}.$$

Unde ponendo $m = 1$ sequitur:

III. Evanescentibus elementis omnibus in quibus index inferior indice superiore minor est, Determinans in unicum terminum abire vel fieri

$$\Sigma \pm a a'_1 a''_2 \;\ldots\; a_n^{(n)} = a a'_1 a''_2 \;\ldots\; a_n^{(n)}.$$

E Propositione II. sequitur hoc Corollarium:

IV. Evanescentibus elementis omnibus,

$$a_k^{(m)},\quad a_k^{(m+1)} \;\ldots\; a_k^{(n)},$$

in quibus indices inferiores superioribus minores sunt, si insuper habetur,

$$a_m^{(m)} = a_{m+1}^{(m+1)} \;\ldots\; = a_n^{(n)} = 1$$

fit

$$\Sigma \pm a a'_1 a''_2 \;\ldots\; a_n^{(n)} = \Sigma \pm a a'_1 \;\ldots\; a_{m-1}^{(m-1)}.$$

E qua Propositione patet quodlibet inferioris gradus Determinans haberi posse pro Determinantis altioris gradus casu speciali.

6.

Designemus per

$$a_g^{(f)} A_g^{(f)}$$

Aggregatum omnium Determinantis R terminorum qui per quantitatem $a_g^{(f)}$ multiplicati sunt. In quovis ipsius R termino

$$\pm a_k a'_{k'} a''_{k''} \;\ldots\; a_{k^{(n)}}^{(n)}$$

elementa a_k, $a_{k'}$ etc. indicibus cum superioribus tum inferioribus omnibus inter se diversis gaudent. Unde terminos Aggregati $A_g^{(f)}$ non ingredi possunt quantitates $a_k^{(i)}$, in quibus index superior valorem f vel inferior valorem g habet. Porro cum in quovis ipsius R termino elementum unum sit nec plura quod datum indicem superiorem i, unum nec plura quod datum indicem inferiorem k habeat, sequitur, singulos Determinantis R terminos per unum elementorum $a^{(i)}$, $a_1^{(i)}$, ... $a_n^{(i)}$ neque vero per plura eorum simul multiplicari nec non per unum elementorum a_k, a'_k, ... $a_k^{(n)}$ neque vero per plura eorum simul multiplicari. Vocabantur autem,

$$a^{(i)}A^{(i)},\quad a_1^{(i)}A_1^{(i)},\quad \dots\dots\quad a_n^{(i)}A_n^{(i)},$$

Aggregata terminorum Determinantis R respective per $a^{(i)}$, $a_1^{(i)}$ .... $a_n^{(i)}$ multiplicatorum, unde fieri debet,

$$1.\quad R = a^{(i)}A^{(i)} + a_1^{(i)}A_1^{(i)} \dots\dots + a_n^{(i)}A_n^{(i)};$$

porro erant,

$$a_kA_k,\quad a'_kA'_k,\quad \dots\dots\quad a_k^{(n)}A_k^{(n)},$$

Aggregata terminorum Determinantis R respective per a_k a'_k .... $a_k^{(n)}$ multiplicatorum, unde fieri debet,

$$2.\quad R = a_kA_k + a'_kA'_k + \dots\dots + a_k^{(n)}A_k^{(n)}.$$

Tribuendo indici i vel k valores 0, 1, 2 .... n, e quaque duarum formularum (1.) et (2.) obtinentur $n+1$ repraesentationes diversae Determinantis R.

Determinans R est singularum quantitatum $a_k^{(i)}$ respectu expressio linearis, atque ipsius $a_k^{(i)}$ Coefficientem, qua in Determinante R afficitur, vocavimus $A_k^{(i)}$; unde adhibita differentialium notatione ipsum $A_k^{(i)}$ exhibere licet per formulam,

$$3.\quad A_k^{(i)} = \frac{\partial R}{\partial a_k^{(i)}}.$$

Hinc si quantitatibus $a_k^{(i)}$ incrementa infinite parva tribuimus,

$$d a_k^{(i)},$$

simulque R incrementum dR capit, fit

$$4.\quad dR = \Sigma A_k^{(i)} d a_k^{(i)},$$

siquidem sub signo summatorio utrique indici i et k valores 0, 1, 2 .... n conferuntur.

Binos indices superiores i et i' commutando cum R in $-R$ abeat, sequitur, Aggregatum terminorum ipsius R per $a_k^{(i)}$ multiplicatorum, $a_k^{(i)}A_k^{(i)}$, ea commutatione abire in Aggregatum terminorum ipsius $-R$ per $a_k^{(i')}$ multiplicatorum, $-a_k^{(i')}A_k^{(i')}$. Unde sequitur, *ponendo i loco i' abire $A_k^{(i)}$ in $-A_k^{(i')}$*; eademque ratione probatur, *ponendo k loco k' abire $A_k^{(i)}$ in $-A_{k'}^{(i)}$*. Unde etiam sequitur, *simul ponendo i loco i', k loco k', siquidem i et i', k et k' inter se diversi sint, abire $A_k^{(i)}$ in $A_{k'}^{(i')}$.*

Obtinetur $a_i^{(i)}A_i^{(i)}$ si in termino

$$\pm\, a a_1' a_2'' \;\ldots\; a_i^{(i)} \;\ldots\; a_n^{(n)}$$

elementum $a_i^{(i)}$ immutatum manet, reliquorum indicibus superioribus vel inferioribus permutatis, unde fit

$$A_i^{(i)} = \Sigma \pm a a_1' \;\ldots\; a_{i-1}^{(i-1)} a_{i+1}^{(i+1)} \;\ldots\; a_n^{(n)},$$

unde prodit $A_k^{(i)}$ loco inferioris indicis k ponendo i et signa mutando sive fit,

$$A_k^{(i)} = -\Sigma \pm a a_1' \;\ldots\; a_{i-1}^{(i-1)} a_{i+1}^{(i+1)} \;\ldots\; a_{k-1}^{(k-1)} a_i^{(k)} a_{k+1}^{(k+1)} \;\ldots\; a_n^{(n)}.$$

Vel etiam si i et k a o diversi, obtinetur $A_k^{(i)}$ ex

$$A = \Sigma \pm a_1' a_2'' \;\ldots\; a_n^{(n)},$$

loco indicis superioris i et inferioris k ponendo o.

Commutando indices inferiores cum superioribus non mutatur Determinans R; simul termini in $a_k^{(i)}$ ducti, $a_k^{(i)}A_k^{(i)}$, abeunt in terminos in $a_i^{(k)}$ ductos, $a_i^{(k)}A_i^{(k)}$; unde *in quantitatibus $a_k^{(i)}$ commutando indices inferiores cum superioribus abeunt quantitates $A_k^{(i)}$ in $A_i^{(k)}$ sive etiam in quantitatibus $A_k^{(i)}$ indices inferiores cum inferioribus commutantur.* Hinc etiam sequitur, *quoties pro omnibus indicibus i et k fiat,*

$$a_k^{(i)} = a_i^{(k)},$$

fieri etiam

$$A_k^{(i)} = A_i^{(k)}.$$

Commutatis enim indicibus superioribus et inferioribus omnium $a_k^{(i)}$, ipsa $A_k^{(i)}$ non mutatur cum eius elementis aequivalentia substituantur; ea autem commutatione vidimus abire $A_k^{(i)}$ in $A_i^{(k)}$, unde utrumque inter se aequale evadere debet.

Statuamus, pro *datis* duobus indicibus i et i' fieri,

$$5. \quad a^{(i)} = a^{(i')}, \quad a_1^{(i)} = a_1^{(i')}, \;\ldots\; a_n^{(i)} = a_n^{(i')},$$

propter proprietatem eius fundamentalem evanescit valor Determinantis R. Hinc repraesentando Determinans R per formulam (1.) ac substituendo (5.) eruimus,

$$6. \quad 0 = a^{(i')}A^{(i)} + a_1^{(i')}A_1^{(i)} \;\ldots\; + a_n^{(i')}A_n^{(i)}.$$

Haec aequatio inventa quidem est supponendo, quantitates $a_k^{(i')}$ ipsis $a_k^{(i)}$ respective aequales esse, sed cum expressionem ad dextram aequationis (6.) quantitates $a_1^{(i)}$, $a_2^{(i)}$ .... $a_n^{(i)}$ omnino non ingrediantur, *aequatio* (6.) *identica esse debet.* Ac perinde invenitur, designante k' indicem quemcunque a k diversum, quoties sit

$$7.\qquad a_k = a_{k'},\quad a'_k = a'_{k'},\quad \ldots\ldots\quad a_k^{(n)} = a_{k'}^{(n)},$$

identice fieri:

$$8.\qquad o = a_{k'}A_k + a'_{k'}A'_k \;\ldots\ldots + a_{k'}^{(n)}A'_k.$$

Substituendo formulas (3.), inventas formulas (1.), (2.), (6.), (8.) sic quoque exhibere licet:

$$9.\qquad R = a^{(i)}\frac{\partial R}{\partial a^{(i)}} + a_1^{(i)}\frac{\partial R}{\partial a_1^{(i)}} \;\ldots\ldots + a_n^{(i)}\frac{\partial R}{\partial a_n^{(i)}}$$

$$= a_k\frac{\partial R}{\partial a_k} + a'_k\frac{\partial R}{\partial a'_k} \;\ldots\ldots + a_k^{(n)}\frac{\partial R}{\partial a_k^{(n)}},$$

$$10.\qquad o = a^{(i')}\frac{\partial R}{\partial a^{(i)}} + a_1^{(i')}\frac{\partial R}{\partial a_1^{(i)}} \;\ldots\ldots + a_n^{(i')}\frac{\partial R}{\partial a_n^{(i)}},$$

$$o = a_{k'}\frac{\partial R}{\partial a_k} + a'_{k'}\frac{\partial R}{\partial a'_k} \;\ldots\ldots + a_k^{(n)}\frac{\partial R}{\partial a_k^{(n)}}.$$

Quae sunt aequationes differentiales partiales quibus Determinans R satisfacit.

7.

Per formulas §. pr. traditas theoria resolutionis algebraicae aequationum linearium facile absolvitur.

Proponantur enim aequationes lineares,

$$1.\qquad \begin{cases} u = at + a_1 t \;\ldots\ldots + a_n t_n \\ u_1 = a't + a'_1 t_1 \;\ldots\ldots + a'_n t_n \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ u_n = a^{(n)}t + a_1^{(n)}t_1 \;\ldots\ldots + a_n^{(n)}t_n; \end{cases}$$

ut eruatur incognita t_k, aequationes propositae respective per A_k, A'_k .... $A_k^{(n)}$ multiplicentur et post multiplicationem factam instituatur summatio: in summa illa evanescunt e (8.) §. pr. Coëfficientes ipsarum t, t_1, ... t_n praeter Coëfficientem ipsius t_k qui e (2.) §. pr. aequalis evadit Determinanti R, sive fit

$$2.\qquad Rt_k = A_k u + A'_k u_1 \;\ldots\ldots + A_k^{(n)}u_n.$$

Qua in formula tribuendo ipsi k valores 0, 1, 2 .... n, eruimus hoc systema aequationum, quod incognitarum valores suppeditat,

$$3.\quad \begin{cases} Rt = A\,u + A'\,u_1 \ldots\ldots + A^{(n)}\,u_n \\ Rt_1 = A_1\,u + A'_n\,u_1 \ldots\ldots + A_1^{(n)}\,u_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Rt_n = A_n u + A'_n\,u_1 \ldots\ldots + A_n^{(n)}\,u_n. \end{cases}$$

Prorsus eadem ratione, propositis aequationibus linearibus,

$$4.\quad \begin{cases} s = a\;r + a'\;r_1 \ldots\ldots + a^{(n)}\,r_n \\ s_1 = a_1 r + a'_1 r_1 \ldots\ldots + a_1^{(n)}\,r_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ s_n = a_n r + a'_n r_1 \ldots\ldots + a_n^{(n)} r_n, \end{cases}$$

eruimus e (6.), (2)., §. pr.,

$$5.\quad \begin{cases} Rr = As + A_1 s_1 \ldots\ldots + A_n s_n \\ Rr_1 = A's + A'_1 s_1 \ldots\ldots + A'_n s_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Rr_n = A^{(n)} s + A_1^{(n)} s_1 \ldots\ldots + A_n^{(n)} s_n. \end{cases}$$

Commutando elementorum $a_n^{(i)}$ indices superiores et inferiores aequationes (1.) et (4.) in se abeunt; et cum simul ipsorum $A_k^{(i)}$ indices superiores et inferiores commutentur, Determinans R immutatum maneat, simul etiam aequationes (3.) in (5.) abire debent. Unde alterum aequationum systema de altero derivari potest. Sed idem obtinetur absque ulla cognitione rationis qua quantitates R et $A_k^{(i)}$ ex elementis $a_k^{(i)}$ componuntur, observando e (1.) et (4.) fieri:

$$6.\quad ur + u_1 r_1 \ldots\ldots + u_n r_n = ts + t_1 s_1 \ldots\ldots + t_n s_n,$$

ac substituendo in hac aequatione ipsarum t, t_1 etc. valores e (3.) petitos. Ipsum Determinans R dicamus ad aequationes (1.) vel (4.) *pertinere* sive *earum aequationum Determinans* esse.

Aequationes (6.) §. pr. docent, propositis n aequationibus,

$$7.\quad \begin{cases} 0 = at + a_1 t_1 \ldots\ldots + a_n t_n \\ 0 = a't + a'_1 t_1 \ldots\ldots + a'_n t_n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 0 = a^{(n)} t + a_1^{(n)} t_1 \ldots\ldots + a_n^{(n)} t_n, \end{cases}$$

in quibus ipsi a non tribuatur index superior i, fieri

$$8.\quad t : t \ldots\ldots : t_n = A^{(i)} : A_1^{(i)} \ldots\ldots : A_n^{(i)},$$

nisi omnes t, t_1, ... t_n simul evanescant. Ut etiam aequatio

$$0 = a^{(i)} t + a_1^{(i)} t_1 \ldots\ldots + a_n^{(i)} t_n$$

locum habeat sive ut in (1.) quantitates u, u' etc. simul omnes evanescere possint, fieri debet e (1.) §. pr.

$$9. \quad R = 0.$$

Eademque ratione patet, evanescere Determinans, si exstent $n+1$ quantitates non simul omnes evanescentes r, r_1, ... r_n tales ut simul locum habeant $n+1$ aequationes,

$$10. \quad \begin{cases} 0 = ar + a'r_1 \ldots\ldots + a^{(n)} r_n \\ 0 = a_1 r + a_1' r_1 \ldots\ldots + a_1^{(n)} r_n \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ 0 = a_n r + a_n' r_1 \ldots\ldots + a_n^{(n)} r_n. \end{cases}$$

Scilicet, multiplicentur aequationes praecedentes per $A^{(i)}$, $A_1^{(i)}$, ... $A_n^{(i)}$, invenitur addendo e (1.), (6.) §.

$$0 = r_i R,$$

qua aequatione, cum e suppositione facta unum certe non evanescat r_i, Determinans R evanescere patet. *Quoties igitur aequationes* (7.) *vel* (10.) *locum habent neque earum Determinans R evanescit, certo incognitae t, t_1, ... t_n vel r, r_1, ... r_n omnes simul evanescere debent.*

Quaecunque proponantur aequationes lineares (1.), ex iis semper sequntur aequationes (3.) neque ullus est exceptionis locus. Eruntque incognitarum valores aequationibus (3.) prorsus determinati iique finiti nisi evanescat Determinans. Evanescente autem Determinante usu venit ut incognitae aut in infinitum abeant aut indeterminatae evadant. Scilicet aequationum (3.) parte dextra simul evanescente atque Determinante, incognitarum valores formam indeterminatam,

$$\frac{0}{0}$$

induunt. Sed haec res variis adhuc quaestionibus ansam praebet. Fieri enim potest ut inter quantitates infinitas vel indeterminatas variae relationes locum habeant, unde evanescente Determinante varii casus evenire possunt et pro singulis criteria propria assignanda erunt. Afferam exemplum Geometricum. Proposita superficie secundi gradus, dantur Coordinatae centri tribus aequationibus linearibus. Quarum aequationum Determinante non evanescente, habentur Ellipsoidae et Hyperboloidae. Sed evanescente Determinante habentur Paraboloidae si Coordinatarum valores evadunt infiniti, ita tamen ut centrum licet infinite remotum in data recta iaceat. Prodit Cylindrus ellipticus aut hyperbolicus aut systema duorum Planorum se intersecantium, si evanescente Determinante Coordinatarum valores indeterminati evadunt, ita tamen ut centrum rursus in data recta sed ubicunque iaceat. Cylindrus fit parabolicus si centrum in infinitum removetur, ita tamen ut in dato plano iaceat. Determinante igitur evanescente inter varias adhuc casus naturae maxime diversae distinguendum est et pro singulis criteria algebraica afferenda erunt. Quod tamen pro numero quocunque aequationum linearium paullo, prolixum videtur negotium.

8.

Adnotavit M. *Laplace,* unumquodque Determinans repraesentari posse ut Aggregatum productorum plurium Determinantium inferiorum graduum. Quae res ita se habet. Discerpatur numerus n in plures alios numeros veluti in *quatuor* ita ut sit,

$$n = i + k + l + m;$$

distribuantur indices 0, 1, 2, ... n in quatuor classes $i+1$, k, l, m indicibus constantes. Ex. gr. constituant indices,

$$\begin{array}{rl} 0,\ 1,\ 2\ \ldots\ i & \text{primam,} \\ i+1,\ i+2\ \ldots\ k & \text{secundam,} \\ k+1,\ k+2\ \ldots\ l & \text{tertiam} \\ l+1,\ l+2\ \ldots\ n & \text{quartam} \end{array}$$

classem. Quae classes omnimodis sibi invicem inserantur ordine numerorum cuiusvis classis non mutato, ita ut in Permutatione proveniente non fiat ut index minorem aliquem eiusdem classis antecedat. Sit eiusmodi Permutatio,

$$\alpha^0,\ \alpha^1,\ \alpha^2\ \ldots\ \alpha^n$$

ac designemus per,

$$S \pm \alpha^0\ \alpha^1\ \alpha^2\ \ldots\ \alpha^n,$$

Aggregatum omnium expressionum quae e data expressione eiusmodi Permutationibus proveniunt, signis + aut —1 praefixis prout Permutatio positiva aut negativa est. His positis in singulis terminis expressionis

$$S \pm a^{0}_{\alpha^0} a^{1}_{\alpha^1}\ \ldots\ a^{(i)}_{\alpha^i} \,.\, a^{(i+1)}_{\alpha^{i+1}} a^{(i+2)}_{\alpha^{i+2}}\ \ldots\ a^{(k)}_{\alpha^k}\ \ldots\ a^{(n)}_{\alpha^n}$$

loco factorum

$$\begin{array}{ll} a^{0}_{\alpha^0} a^{1}_{\alpha^1}\ \ldots\ a^{(i)}_{\alpha^i}, & a^{(i+1)}_{\alpha^{i+1}} a^{(i+2)}_{\alpha^{i+2}}\ \ldots\ a^{(k)}_{\alpha^k}, \\ a^{(k+1)}_{\alpha^{k+1}} a^{(k+2)}_{\alpha^{k+2}}\ \ldots\ a^{(l)}_{\alpha^l}, & a^{(l+1)}_{\alpha^{l+1}} a^{(l+2)}_{\alpha^{l+2}}\ \ldots\ a^{(n)}_{\alpha^n} \end{array}$$

scribantur Determinantia

$$\begin{array}{ll} \Sigma \pm a^{0}_{\alpha^0} a^{1}_{\alpha^1}\ \ldots\ a^{(i)}_{\alpha^i}, & \Sigma \pm a^{(i+1)}_{\alpha^{i+1}} a^{(i+2)}_{\alpha^{i+2}}\ \ldots\ a^{(k)}_{\alpha^n}, \\ \Sigma \pm a^{(k+1)}_{\alpha^{k+1}} a^{(k+2)}_{\alpha^{k+2}}\ \ldots\ a^{(l)}_{\alpha^l}, & \Sigma \pm a^{(l+1)}_{\alpha^{l+1}} a^{(l+2)}_{\alpha^{l+2}}\ \ldots\ a^{(n)}_{\alpha^n}, \end{array}$$

prodit:

$$R = \Sigma \pm a a'_1 a''_2 \ldots\ldots a^{(n)}_n =$$

$$S \pm \left(\Sigma \pm a^{0}_{\alpha_0} a^{1}_{\alpha_0} \ldots\ldots a^{(i)}_{\alpha_i} \,.\, \Sigma \pm a^{(i+1)}_{\alpha_{i+1}} a^{(i+2)}_{\alpha_{i+2}} \ldots\ldots a^{(k)}_{\alpha_k} \,.\, \Sigma \pm a^{(k+1)}_{\alpha_{k+1}} a^{(k+2)}_{\alpha_{k+2}} \ldots\ldots a^{(l)}_{\alpha_l} \,.\right.$$

$$\left. \Sigma \pm a^{(l+1)}_{\alpha_{l+1}} a^{(l+2)}_{\alpha_{l+2}} \ldots\ldots a^{(n)}_{n}\right).$$

Demonstratio inde patet quod *omnes* obtineantur Permutationes primum indices ita permutando ut indices eiusdem classis certum quendam ordinem servent ac deinde rursus eiusmodi classis indices omnimodis permutando. Numerus productorum Determinantium quae Aggregatum S amplectitur est,

$$\frac{1.2.3 \ldots\ldots (n+1)}{1.2.3 \ldots\ldots (i+1)\,.\,1.2.3 \ldots\ldots k\,.\,1.2.3 \ldots\ldots l\,.\,1.2.3 \ldots\ldots m}.$$

Formula proposita expediri potest Determinantis indagatio si Determinantia partialia, quae singulorum productorum factores constituunt, valoribus simplicibus gaudent.

9.

Accuratius examinemus Determinantia $n-1^{\text{ti}}$ gradus e quibus per Determinantia secundi gradus multiplicatis Determinans R componitur. Proposito determinante,

$$R = \Sigma \pm a a'_1 \ldots\ldots a^{(n)}_n,$$

terminorum eius per $a^{(f)}_g a^{(f')}_{g'}$ multiplicatorum vocemus Aggregatum

$$1.\quad a^{(f)}_g a^{(f')}_{g'}\,.\quad A^{f,f'}_{g,f'}.$$

Ipsi f et f' nec non g et g' quilibet esse possunt indices ex ipsis 0, 1, ... n, a se diversi. In terminis Aggregati

$$2.\quad A^{f,f'}_{g,g'}$$

non inveniuntur elementa indicibus superioribus f et f' neque elementa indicibus inferioribus g et g' affecta, quippe idem Determinatis R terminus binos non habet factores eodem indice superiore vel inferiore affectos. Qua de re indices f et f' vel g et g' inter se permutando ipsum $A^{f,f'}_{g,g'}$ mutationem non subit, ideoque abit expressio (1.) in

$$3.\quad a^{(f')}_g a^{(f)}_{g'}\,.\ A^{f,f'}_{g,g'}.$$

Eadem autem permutatione cum R in $-R$ mutetur, erit (3.) Aggregatum ipsius $-R$ terminorum qui per

$$4.\quad a^{(f')}_g a^{(f)}_{g'}$$

multiplicantur, ideoque erit

$$-a^{(f')}_g a^{(f)}_{g'}\,.\ A^{f,f'}_{g,g'}$$

terminorum ipsius R per $a_g^{(f')} a_{g'}^{(f)}$ multiplicatorum Aggregatum sive

$$5. \quad A^{f,f'}_{g',g} = A^{f',f}_{g,g'} = -A^{f,f'}_{g,g'}.$$

Qua de re continebit R terminos provenientes e producto,

$$6. \quad (a_g^{(f)} a_{g'}^{(f')} - a_{g'}^{(f)} a_g^{(f')}) A^{f,f'}_{g,g'},$$

iisque termini Determinantis R erunt omnes in quibus duo elementa indicibus superioribus f et f' affecta indicibus inferioribus g et g' gaudent. At quivis ipsius R terminus continet duo elementa alterum indice superiore f alterum indice superiore f' affectum nec non duo elementa alterum indice inferiore g alterum indice inferiore g' affectum, quia cuiusvis termini elementa singula singulis indicibus cum superioribus tum inferioribus afficiuntur. Unde obtinetur R summando omnes expressiones (6.) in quibus pro iisdem f et f' sumuntur pro g et g' bini indicum $0, 1, 2, \dots n$ vel etiam in quibus pro iisdem g et g' bini indicum $0, 1, 2, \dots n$ ipsis f et f' substituuntur. Qua de re si pro i, i' vel pro k, k' bini diversi indicum $0, 1, 2, \dots n$ sumuntur, ipsi autem f, f', g, g' dati indices sunt, obtinetur

$$7. \quad R = \Sigma (a_k^{(f)} a_{k'}^{(f')} - a_{k'}^{(f)} a_k^{(f')}) A^{f,f'}_{k,k'}$$
$$= \Sigma (a_g^{(i)} a_{g'}^{(i')} - a_{g'}^{(i)} a_g^{(i')}) A^{i,i'}_{g,g'}.$$

Facile etiam ipsa $A_g^{(f)}$ e quantitatibus $A^{f,f'}_{g,g'}$ componitur. Erat enim $a_g^{(f)} A_g^{(f)}$ Aggregatum terminorum Determinantis R per $a_g^{(f)}$ multiplicatorum; qui termini cum insuper per unum elementorum,

$$a^{(f')}, \quad a_1^{(f')}, \quad a_2^{(f')}, \quad \dots \quad a_n^{(f')},$$

omisso elemento $a_g^{(f')}$, vel etiam per unum elementorum,

$$a_{g'}, \quad a'_{g'}, \quad a''_{g'}, \quad \dots \quad a^{(n)}_{g'},$$

omisso l elemento $a_{g'}^{(f)}$ multiplicati esse debeant, obtinetur:

$$8. \quad A_g^{(f)} = a^{(f')} A^{f,f'}_{g,0} + a_1^{(f')} A^{f,f'}_{g,1} \dots + a_n^{(f')} A^{f,f'}_{g,n}$$

sive

$$9. \quad A_g^{(f)} = a_{g'} A^{f,0}_{g,g'} + a'_{g'} A^{f,1}_{g,g'} \dots + a^{(n)}_{g'} A^{f,n}_{g,g'},$$

ubi respective termini per $a_g^{(f')}$, $a_{g'}^{(f)}$ multiplicati omittendi sunt.

Designemus br. causa per (k, k') expressionem

$$10. \quad A^{f,f'}_{k,k'} = (k, k'),$$

ita ut sit

$$(k, k') = -(k' \; k).$$

Fit e (8.) ipsi g substituendo numeros $0, 1, 2 \dots n$:

$$11. \quad \begin{cases} A^{(f)} = \quad * \quad + a_1^{(f')}(0,1) + a_2^{(f')}(0,2) \ldots\ldots + a_n^{(f')}(0,n) \\ A_1^{(f)} = a^{(f')}(1,0) + \quad * \quad + a_2^{(f')}(1,2) \ldots\ldots + a_n^{(f')}(1,n) \\ A_2^{(f)} = a^{(f')}(2,0) + a_1^{(f')}(2,1) + \quad * \quad \ldots\ldots + a_n^{(f')}(2,n) \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ A_n^{(f)} = a^{(f')}(n,0) + a_1^{(f')}(n,1) + a_2^{(f')}(n,2) \ldots\ldots + \quad * \end{cases}$$

Similes formulae e (9.) derivari possunt. In aequationibus (11.) ipsorum $a^{(f')}$, $a_1^{(f')}$, etc. Coëfficientes in Diagonali positi evanescunt, bini quilibet Coëfficientes Diagonalis respectu symmetrice positi valoribus oppositis gaudent. Quae est species aequationum linearium memorabilis in variis quaestionibus analyticis obveniens.

10.

Quomodum supra differentiando R elementi $a_g^{(f)}$ respectu ipsum $A_g^{(f)}$ obtinuimus, ita ipsum $A_{g,g'}^{f,f'}$ obtinetur bis differentiando R elementorum $a_g^{(f)}$, $a_{g'}^{(f')}$ respectu. Ex ipsa enim Aggregati $A_{g,g'}^{f,f'}$ definitione eruimus formulas,

$$1. \quad A_{g,g'}^{f,f'} = \frac{\partial^2 R}{\partial a_g^{(f)} \partial a_{g'}^{(f')}} = -\frac{\partial^2 R}{\partial a_g^{(f')} \partial a_{g'}^{(f)}}$$
$$= \frac{\partial . A_g^{(f)}}{\partial a_{g'}^{(f')}} = \frac{\partial . A_{g'}^{(f')}}{\partial a_g^{(f)}} = -\frac{\partial . A_g^{(f')}}{\partial a_{g'}^{(f)}} = -\frac{\partial . A_{g'}^{(f)}}{\partial a_g^{(f')}}.$$

In aequationibus (10.) §. 6. ponatur i'', k'' loco i', k', fit:

$$0 = a^{(i'')} \frac{\partial R}{\partial a^{(i)}} + a_1^{(i'')} \frac{\partial R}{\partial a_1^{(i)}} \ldots\ldots + a_n^{(i'')} \frac{\partial R}{\partial a_n^{(i)}}$$
$$0 = a_{k''} \frac{\partial R}{d\, a_k} + a'_{k''} \frac{\partial R}{\partial a'_k} \ldots\ldots + a_{k''}^{(n)} \frac{\partial R}{\partial a_k^{(n)}}.$$

Quae aequationes elementorum $a_k^{(i')}$, $a_{k'}^{(i)}$, respectu differentiamus. Ubi i, i', i'' nec non k, k', k'' a se diversi sunt, obtinemus:

$$2. \quad \begin{cases} 0 = a^{(i'')} A_{0,k}^{i,i'} + a_1^{(i'')} A_{1,k}^{i,i'} \ldots\ldots + a_n^{(i'')} A_{n,k}^{i,i'} \\ 0 = a_{k''} A_{k,k'}^{0,i} + a'_{k''} A_{k,k'}^{1,i} \ldots\ldots + a_{k''}^{(n)} A_{k,k'}^{n,i}. \end{cases}$$

Si i'' ipsi i' vel i aut k'' ipsi k' aut k aequalis est, eruimus:

$$3. \quad \begin{cases} -A_k^{(i)} = a^{(i')} A_{0,k}^{i,i'} + a_1^{(i')} A_{1,k}^{i,i'} \ldots\ldots + a_n^{(i')} A_{n,k}^{i,i'} \\ -A_k^{(i)} = a_{k'} A_{k,k'}^{0,i} + a'_{k'} A_{k,k'}^{1,i} \ldots\ldots + a_{k'}^{(n)} A_{k,k'}^{n,i}. \end{cases}$$

In formulis (2.), (3.) statuendum est:

$$A_{k,k'}^{i,i} = A_{k,k}^{i,i'} = 0,$$

sive omittendi sunt termini in quibus ipsius $A^{i,\,i'}_{k,\,k'}$ indices sive superiores sive inferiores aequales existunt.

Multiplicemus aequationes sequentes,

$$\begin{aligned} 0 &= a\,A_k + a'\,A'_k \;\dots\; + a^{(n)}\,A^{(n)}_k, \\ 0 &= a_1 A_k + a'_1 A'_k \;\dots\; + a^{(n)}_1\,A^{(n)}_k, \\ &\dots\dots\dots\dots \\ R &= a_k A_k + a'_k A'_k \;\dots\; + a^{(n)}_k\,A^{(n)}_k, \\ 0 &= a_n A_k + a'_n A'_k \;\dots\; + a^{(n)}_n\,A^{(n)}_k \end{aligned}$$

per factores

$$A^{i,\,i'}_{0,\,k'},\quad A^{i,\,i'}_{1,\,k'}\;\dots\; A^{i,\,i'}_{k,\,k'}\;\dots\; A^{i,\,i'}_{n,\,k'},$$

additione facta secundum (2.) evanescunt in dextra parte termini omnes per A_k, A'_k etc. multiplicati praeter eos qui per $A^{(i)}_k$, $A^{(i')}_k$ multiplicati sunt et qui secundum (3.) evadunt,

$$A^{(i')}_{k'} . A^{(i)}_k, \quad -A^{(i)}_{k'} . A^{(i')}_k.$$

Unde prodit formula:

$$4. \qquad R . A^{i,\,i'}_{k,\,k'} = A^{(i)}_k A^{(i')}_{k'} - A^{(i)}_{k'} A^{(i')}_k$$

sive

$$5. \qquad R . \frac{\partial^2 R}{\partial a^{(i)}_k\,\partial a^{(i')}_{k'}} = \frac{\partial R}{\partial a^{(i)}_k} . \frac{\partial R}{\partial a^{(i')}_{k'}} - \frac{\partial R}{\partial a^{(i)}_{k'}} . \frac{\partial R}{\partial a^{(i')}_k}.$$

Evolutione productorum facta habetur formula identica,

$$\begin{aligned} &(A^{(i)}_k A^{(i')}_{k'} - A^{(i)}_{k'} A^{(i')}_k)(A^{(i)}_{k''} A^{(i')}_{k'''} - A^{(i)}_{k'''} A^{(i')}_{k''}) \\ +\,&(A^{(i)}_k A^{(i')}_{k''} - A^{(i)}_{k''} A^{(i')}_k)(A^{(i)}_{k'''} A^{(i')}_{k'} - A^{(i)}_{k'} A^{(i')}_{k'''}) \\ +\,&(A^{(i)}_k A^{(i')}_{k'''} - A^{(i)}_{k'''} A^{(i')}_k)(A^{(i)}_{k'} A^{(i')}_{k''} - A^{(i)}_{k''} A^{(i')}_{k'}) = 0. \end{aligned}$$

In qua substituendo (4.) et dividendo per R prodit formula:

$$6. \qquad A^{i,\,i'}_{k,\,k'} A^{i,\,i'}_{k'',\,k'''} + A^{i,\,i'}_{k,\,k''} A^{i,\,i'}_{k''',\,k'} + A^{i,\,i'}_{k,\,k'''} A^{i,\,i'}_{k',\,k''} = 0,$$

ac similiter demonstratur

$$7. \qquad A^{i,\,i'}_{k,\,k'} A^{i'',\,i'''}_{k,\,k'} + A^{i,\,i''}_{k,\,k'} A^{i''',\,i'}_{k,\,k'} + A^{i,\,i'''}_{k,\,k'} A^{i',\,i''}_{k,\,k'} = 0.$$

Per aliam formulam identicam notissimam obtinetur e (4.):

$$8. \qquad \begin{cases} A^{(i)}_k A^{i,\,i'}_{k',\,k''} + A^{(i)}_{k'} A^{i,\,i'}_{k'',\,k} + A^{(i)}_{k''} A^{i,\,i'}_{k,\,k'} = 0 \\ A^{(i)}_k A^{i',\,i''}_{k,\,k'} + A^{(i')}_k A^{i'',\,i}_{k,\,k'} + A^{(i'')}_k A^{i,\,i'}_{k,\,k'} = 0 \end{cases}$$

sive

$$9. \qquad \begin{cases} \dfrac{\partial R}{\partial a^{(i)}_k} . \dfrac{\partial^2 R}{\partial a^{(i)}_{k'}\,\partial a^{(i')}_{k''}} + \dfrac{\partial R}{\partial a^{(i)}_{k'}} . \dfrac{\partial^2 R}{\partial a^{(i)}_{k''}\,\partial a^{(i')}_k} + \dfrac{\partial R}{\partial a^{(i)}_{k''}} . \dfrac{\partial^2 R}{\partial a^{(i)}_k\,\partial a^{(i')}_{k'}} = 0 \\ \dfrac{\partial R}{\partial a^{(i)}_k} . \dfrac{\partial^2 R}{\partial a^{(i')}_k\,\partial a^{(i'')}_{k'}} + \dfrac{\partial R}{\partial a^{(i')}_k} . \dfrac{\partial^2 R}{\partial a^{(i'')}_k\,\partial a^{(i)}_{k'}} + \dfrac{\partial R}{\partial a^{(i'')}_k} . \dfrac{\partial^2 R}{\partial a^{(i)}_k\,\partial a^{(i')}_{k'}} = 0. \end{cases}$$

Advocando formulas (1.) obtines e (8.)

$$14. \quad \left\{ \frac{\partial . \frac{A^{(i)}_{k'}}{A^{(i)}_{k}}}{\partial a^{(i')}_{k''}} = -\frac{A^{(i)}_{k''} A^{i,\,i'}_{k,\,k'}}{A^{(i)}_{k} A^{(i)}_{k}}, \qquad \frac{\partial . \frac{A^{(i)}_{k}}{A^{(i')}_{k}}}{\partial a^{(i'')}_{k'}} = -\frac{A^{(i'')}_{k} A^{i,\,i'}_{k,\,k'}}{A^{(i)}_{k} A^{(i)}_{k}}. \right.$$

Formulae praecedentes Cl°. ***Bézout*** bene in~~str~~notuerunt, earumque in variis quaestionibus usus est.

11.

Formula (4.) §. pr. ad generalius formularum systema pertinet. Vidimus §. 7. ex aequationibus,

$$1. \quad \begin{cases} at + a_1 t_1 \dots + a_n t_n = u \\ a't + a'_1 t_1 \dots + a'_n t_n = u_1 \\ \dots\dots\dots\dots\dots \\ a^{(n)} t + a^{(n)}_1 t_1 \dots + a^{(n)}_n t_n = u_n \end{cases}$$

sequi

$$2. \quad \begin{cases} A\, u + A' u_1 \dots + A^{(n)} u_n = R.t \\ A_1 u + A'_1 u_1 \dots + A^{(n)}_1 u_n = R.t_1 \\ \dots\dots\dots\dots\dots \\ A_n u + A'_n u_1 \dots + A^{(n)}_n u_n = R.t_n, \end{cases}$$

quibus in formulis erat,

$$3. \quad \begin{cases} R = \Sigma \pm a a' \dots a^{(n)}_n, \quad A^{(n)}_n = \Sigma \pm a a'_1 \dots a^{(n-1)}_{n-1}, \\ \qquad\qquad A = \Sigma \pm A A'_1 \dots A^{(n-1)}_{n-1}. \end{cases}$$

Aequationum (1.) tantum $k+1$ primas consideremus,

$$4. \quad \begin{cases} at + a_1 t_1 \dots + a_k t_k + a_{k+1} t_{k+1} \dots + a_n t_n = u, \\ a't + a'_1 t_1 \dots + a'_k t_k + a'_{k+1} t_{k+1} \dots + a'_n t_n = u_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a^{(k)} t + a^{(k)}_1 t_1 \dots + a^{(k)}_k t_k + a^{(k)}_{k+1} t_{k+1} \dots + a^{(k)}_n t_n = u_k, \end{cases}$$

earumque ope determinemus $t,\ t_1 \dots t_k$ per reliquas quantitates t_{k+1}, t_{k+2} etc., atque per $u,\ u_1 \dots u_k$. Prodeat,

$$5. \quad C_k t_k + C_{k+1} t_{k+1} \dots + C_n t_n = Du + D_1 u_1 \dots + D_k u_k,$$

qua in formula erit,

$$6. \quad C_n = \Sigma \pm a a'_1 a''_2 \dots a^{(k)}_k, \qquad D_k = \Sigma \pm a a'_1 a''_2 \dots a^{(k-1)}_{k-1}.$$

Quod patet observando, obtineri aequationes (4.) e (1.) ponendo $n = k$ ac ipsorum u_i loco ponendo,

$$u_i - a^{(i)}_{k+1} t_{k+1} - a^{(i)}_{k+2} t_{k+2} \dots - a^{(i)}_n t_n.$$

Similiter e $n-k+1$ postremis aequationum (2.) determinemus quantitates $u_k, u_{k+1} \dots u_n$ per reliquas $u, u_1 \dots u_{k-1}$ et per quantitates $t_k, t_{k+1} \dots t_n$: quo facto prodeat:

$$7.\quad Eu + E_1 u_1 \dots + E_k u_k = F_k t_k + F_{k+1} t_{k+1} \dots + F_n t_n,$$

erit

$$8.\quad E_k = \Sigma \pm A_k^{(k)} A_{k+1}^{(k+1)} \dots A_n^{(n)}, \quad F_k = R.\Sigma \pm A_{k+1}^{(k+1)} A_{k+2}^{(k+2)} \dots A_n^{(n)}.$$

Aequationes (5.) et (7.) inter se convenire debent, nam per aequationes propositas (1.) unico tantum modo exprimi potest t_k per $t_{k+1}, t_{k+2} \dots t_n$, $u, u_1 \dots u_k$. Unde fieri debet

$$\frac{D_k}{C_k} = \frac{E_k}{F_k}$$

sive

$$9.\quad \frac{\Sigma \pm a a'_1 a''_2 \dots a_{k-1}^{(k-1)}}{\Sigma \pm a a'_1 a''_2 \dots a_k^{(k)}} = \frac{\Sigma \pm A_k^{(k)} A_{k+1}^{(k+1)} \dots A_n^{(n)}}{R.\Sigma \pm A_{k+1}^{(k+1)} A_{k+2}^{(k+2)} \dots A_n^{(n)}}.$$

In hac formula generali ipsi k tribuendo valores

$$n-1, \quad n-2, \quad n-3, \quad \dots \quad 1,$$

prodit:

$$10.\quad \begin{cases} \dfrac{\Sigma \pm a a'_1 \dots a_{n-2}^{(n-2)}}{\Sigma \pm a a'_1 \dots a_{n-1}^{(n-1)}} = \dfrac{\Sigma \pm A_{n-1}^{(n-1)} A_n^{(n)}}{R A_n^{(n)}} \\ \dfrac{\Sigma \pm a a'_1 \dots a_{n-3}^{(n-3)}}{\Sigma \pm a a'_1 \dots a_{n-1}^{(n-2)}} = \dfrac{\Sigma \pm A_{n-2}^{(n-2)} A_{n-1}^{(n-1)} A_n^{(n)}}{R \Sigma \pm A_{n-1}^{(n-1)} A_n^{(n)}} \\ \dots \dots \dots \dots \\ \dfrac{a}{\Sigma \pm a a'_1} = \dfrac{\Sigma \pm A'_1 A''_2 \dots A_n^{(n)}}{R \Sigma \pm A''_2 A'''_3 \dots A_n^{(n)}}. \end{cases}$$

Harum aequationum prima suppeditat,

$$\Sigma \pm A_{n-1}^{(n-1)} A_n^{(n)} = \Sigma \pm a a'_1 \dots a_{n-2}^{(n-2)} = R A_{n-1,\, n}^{n-1,\, n},$$

quae cum formula (4.) §. pr. convenit. Deinde aequationum (10.) duas, tres, quatuor etc. primas inter se multiplicando, prodit formularum systema hoc:

$$11.\quad \begin{cases} \Sigma \pm A_{n-1}^{(n-1)} A_n^{(n)} = R \Sigma \pm a a'_1 \dots a_{n-2}^{(n-2)} \\ \Sigma \pm A_{n-2}^{(n-2)} A_{n-1}^{(n-1)} A_n^{(n)} = R^2 \Sigma \pm a a'_1 \dots a_{n-3}^{(n-3)} \\ \dots \dots \dots \dots \\ \Sigma \pm A'_1 A''_2 \dots A_n^{(n)} = a R^{n-1}. \end{cases}$$

Quas formulas amplectitur formula generalis,

$$12.\quad \Sigma \pm A_{k+1}^{(k+1)} A_{k+2}^{(k+2)} \dots A_n^{(n)} = R^{n-k-1} \Sigma \pm a a'_1 \dots a_k^{(k)}.$$

E qua aliae plurimae profluunt, indices

$$0,\ 1,\ 2\ \dots .\ k,\ k+1,\ k+2\ \dots .\ n$$

omnimodis permutando. Veluti si formularum (11.) postremam sic repraesentamus:

$$\frac{\partial \Sigma \pm A A'_1 A''_2 \dots . A^{(n)}}{\partial A} = a R^{(n-1)},$$

generaliter habebitur

$$\frac{\partial \Sigma \pm A A'_1 A''_2 \dots . A^{(n)}}{\partial A^{(i)}_k} = a^{(i)}_k R^{(n-1)}.$$

Ponendo

$$\Sigma \pm A A'_1 \dots . A^{(n)}_n = r,$$

fit

$$\begin{aligned} r &= A \frac{\partial r}{\partial A} + A_1 \frac{\partial r}{\partial A_1} \dots . + A_n \frac{\partial r}{\partial A_n} \\ &= R^{n-1}\{A a + A_1 a_1 \dots . + A_n a_n\}, \end{aligned}$$

sive e §. 6.:

$$13. \qquad \Sigma \pm A A'_1 \dots . A^{(n)}_n = R^{(n)}.$$

Quae notissima formula est.

12.

Substituendo secundum (3.) §. 6 ipsis $A^{(i)}_k$ expressiones,

$$1. \qquad A^{(i)}_k = \frac{\partial R}{\partial a^{(i)}_k},$$

sequitur e formulis §. 7., *propositis aequationibus linearibus*

$$2. \quad \begin{cases} u = a t + a_1 t_1 \dots . + a_n t_n \\ u_1 = a' t + a'_1 t_1 \dots . + a'_n t_n \\ \dots\dots\dots\dots \\ u_n = a^{(n)} t + a^{(n)}_1 t_1 \dots . + a^{(n)}_n t_n \end{cases}$$

fieri

$$3. \quad \begin{cases} R.t = \frac{\partial R}{\partial a} u + \frac{\partial R}{\partial a'} u_1 \dots . + \frac{\partial R}{\partial a^{(n)}} u_n \\ R.t_1 = \frac{\partial R}{\partial a_1} u + \frac{\partial R}{\partial a'} u_1 \dots . + \frac{\partial R}{\partial a^{(n)}_1} u_n \\ \dots\dots\dots\dots \\ R.t_n = \frac{\partial R}{\partial a_n} u + \frac{\partial R}{\partial a'_n} u_1 \dots . + \frac{\partial R}{\partial a^{(n)}_1} u_n. \end{cases}$$

Proponamus systemata aequationum linearium in quibus omnibus iidem sint incognitarum Coëfficientes et quae solis terminis mere constantibus inter se differunt. Quarum aequationum typus generalis sit,

$$4.\quad \begin{cases} a t + a_1 t_1 \ldots\ldots + a_n t_n = \delta a_k \\ a' t + a'_1 t_1 \ldots\ldots + a'_n t_n = \delta a'_k \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a^{(n)} t + a_1^{(n)} t_1 \ldots\ldots + a_n^{(n)} t_n = \delta a_k^{(n)}, \end{cases}$$

e quibus aequationibus $n+1$ systemata proposita obtineantur ponendo ipsius k loco indices 0, 1, 2 .... n. Valores incognitarum e systemate aequationum (4.) provenientes vocemus

$$t^{(k)},\quad t_1^{(k)},\quad \ldots\ldots\quad t_n^{(k)},$$

erit secundum (3.):

$$5.\quad R\,.\,t_k^{(k)} = \frac{\partial R}{\partial a_k}\delta a_k + \frac{\partial R}{\partial a'_k}\delta a_k \ldots\ldots + \frac{\partial R}{\partial a_k^{(n)}}\delta a_k^{(n)}.$$

Unde tribuendo indici k cunctos valores 0, 1 .... n et summando prodit formula:

$$6.\quad R\{t + t'_1 + t''_2 \ldots\ldots + t_n^{(n)}\} = \delta R$$

sive etiam

$$7.\quad t + t'_1 + t''_2 \ldots\ldots + t_n^{(n)} = \delta \log R.$$

Expressio ad dextram formulae praecedentis est summa e valore primae incognitae e primo aequationum systemate, e valore secundae incognitae e secundo systemate eruto etc. Signum variationis — δ — functioni alicui U elementorum $a_k^{(i)}$ praefigendo intelligo summam,

$$8.\quad \delta U = \Sigma \frac{\partial U}{\partial a_k^{(i)}}\delta a_k^{(i)},$$

designantibus $\delta a_k^{(i)}$ quantitates quascunque et summa ad utriusque indicis i et k cunctos valores 0, 1 .... n sive quod idem est ad cuncta Determinantis elementa extensa.

Supponamus typum aequationum propositarum esse

$$9.\quad \begin{cases} a\, t^{(k)} + a_1\, t_1^{(k)} \ldots\ldots + a_n\, t_n^{(k)} = \delta a_k + (0,\, k) \\ a'\, t^{(k)} + a'_1\, t_1^{(k)} \ldots\ldots + a'_n\, t_n^{(k)} = \delta a'_k + (1,\, k) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a^{(n)} t^{(k)} + a_1^{(n)} t_1^{(k)} \ldots\ldots + a_n^{(n)} t_n^{(k)} = \delta a_k^{(n)} + (n,\, k); \end{cases}$$

mutabitur formula (5.) in hanc,

$$10.\quad R t_k^{(k)} = \frac{\partial R}{\partial a_k}\,\delta a_k + \frac{\partial R}{\partial a'_k}\,\delta a'_k \;\ldots\; + \frac{\partial R}{\partial a_k^{(n)}}\,\delta a_k^{(n)}$$
$$+ \frac{\partial R}{\partial a_k}(0,\,k) + \frac{\partial R}{\partial a'_k}(1,\,k) \;\ldots\; + \frac{\partial R}{\partial a_k^{(n)}}(n,\,k),$$

ideoque mutabitur (11.) in hanc,

$$11.\quad R\{t + t'_1 + t''_2 \;\ldots\; t_n^{(n)}\} = \delta R + \Sigma \frac{\partial R}{\partial a_k^{(i)}}(i,\,k),$$

summa ad indicum i et k cunctos valores $0, 1, 2 \ldots n$ extensa. Ponamus inter Coëfficientes aequationum linearium propositarum locum habere aequationes,

$$a_k^{(i)} = a_i^{(k)},$$

unde quantitates,

$$\frac{\partial R}{\partial a_k^{(i)}} = A_k^{(i)}, \qquad \frac{\partial R}{\partial a_i^{(k)}} = A_i^{(k)},$$

inter se aequales existunt §. 6. Supponamus porro quantitates $(i,\,k)$ indicum i et k permutatione valores induere oppositos, sive fieri

$$12.\quad (k,\,i) = -(i,\,k), \quad (k,\,k) = 0.$$

Quibus positis in summa

$$\Sigma \frac{\partial R}{\partial a_k^{(i)}}(i,\,k),$$

termini

$$\frac{\partial R}{\partial a_k^{(k)}}(k,\,k)$$

evanescunt; porro pro i et k diversis bini termini,

$$\frac{\partial R}{\partial a_k^{(i)}}(i,\,k) + \frac{\partial R}{\partial a_i^{(k)}}(k,\,i),$$

sese mutuo destruunt, unde tota summa

$$\Sigma \frac{\partial R}{\partial a_k^{(i)}}(i,\,k)$$

evanescit. Habemus igitur hanc Propositionem.

Propositio.

„Proponatur aequationum linearium systema,

$$\begin{array}{l}
a t^{(k)} + a_1 t_1^{(k)} \;\ldots\; + a_n t_n^{(k)} = \delta a_k + (0,\,k) \\
a' t^{(k)} + a'_1 t_1^{(k)} \;\ldots\; + a'_n t_n^{(k)} = \delta a'_k + (1,\,k) \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
a^{(n)} t^{(k)} + a_1^{(n)} t_1^{(k)} \;\ldots\; + a_n^{(n)} t_n^{(k)} = \delta a_k^{(n)} + (n,\,k),
\end{array}$$

in quibus

$$a_k^{(i)} = a_i^{(k)}$$

atque (i, k) sunt quantitates quaecunque pro quibus fit,

$$(i, k) = -(k, i), \qquad (k, k) = 0;$$

e systemate aequationum proposito formentur $n+1$ systemata, ponendo pro ipso k indices $0, 1, 2 \ldots n$, atque e primo systemate eruatur valor primae incognitae, e secundo secundae etc.: omnium summa aequatur variationi logarithmi Determinantis aequationum propositarum, sive fit

$$t + t_1' + t_2'' \ldots + t_n^{(n)} = \delta \log \Sigma \pm a a_1' a_2'' \ldots a_n^{(n)}."$$

Hac interdum Propositione solvere licet quaestiones Analyticas gravissimas quae primo intuitu valde complicatae videntur. Cuius rei olim occasione tradam exempla.

13.

Statuamus

$$1. \quad c_k^{(i)} = S\alpha^{(i)} a^{(k)} = \alpha^{(i)} a^{(k)} + \alpha_1^{(i)} a_1^{(k)} \ldots + \alpha_p^{(i)} a_p^{(k)},$$

ac vocemus P Determinans ad elementa $c_k^{(i)}$ pertinens, quod rursus $n+1^{\text{ti}}$ gradus sit, ita ut habeatur,

$$2. \quad P = \Sigma \pm c c_1' c_2'' \ldots c_n^{(n)}.$$

Est productum

$$3. \quad \pm c c_1' c_2'' \ldots c_m^{(n)} = \pm S\alpha a \,.\, S\alpha' a' \,.\, S\alpha'' a'' \ldots S\alpha^{(n)} a^{(n)},$$

quod summarum productum per unam summam repraesentare licet,

$$4. \quad \pm c c_1' c_2'' \ldots c_n^{(n)} = \pm S\alpha_m a_m \,.\, \alpha_{m'}' a_{m'}' \,.\, \alpha_{m''}'' a_{m''}'' \ldots \alpha_{m^{(n)}}^{(n)} a_{m^{(n)}}^{(n)}$$
$$= \pm S\alpha_m \alpha_{m'}' \ldots \alpha_{m^{(n)}}^{(n)} \,.\, a_m a_{m'}' \ldots a_{m^{(n)}}^{(n)},$$

siquidem signum summatorium S ad solos indices inferiores m, m' etc. referimus, quibus singulis cuncti valores tribuendi sunt

$$0, 1, 2 \ldots p.$$

Permutando quantitatum c indices superiores, indices superiores ipsorum α easdem Permutationes subeunt; contra permutando quantitatum c indices inferiores, elementorum a indices superiores easdem Permutationes subeunt. Prodit Determinans P ex aequationis (4.) laeva parte, indices ipsius c superiores $0, 1, 2 \ldots n$ omnibus modis permutando simulque signum positivum aut negativum praefigendo prout eorum indicum permutatio positiva aut negativa est. Qua de re obtinetur P ex expressione,

$$S . \pm \alpha_m \alpha'_{m'} \ldots\ldots \alpha^{(n)}_{m^{(n)}} . a_m a'_{m'} \ldots\ldots a^{(n)}_{m^{(n)}},$$

indices ipsius α superiores omnimodis permutando, signo positivo aut negativo praefixo prout Permutatio positiva aut negativa est, unde fit

$$5. \quad P = S . \left(a_m a'_{m'} \ldots\ldots a^{(n)}_{m^{(n)}} . \Sigma \pm \alpha_m \alpha'_{m'} \ldots\ldots \alpha^{(n)}_{m^{(n)}} \right).$$

At secundum Determinantium proprietatem fundamentalem evanescit Determinans

$$\Sigma \pm \alpha_m \alpha'_{m'} \ldots\ldots \alpha^{(n)}_{m^{(n)}},$$

quoties indicum

$$m, \quad m' \quad \ldots\ldots \quad m^{(n)}$$

duo quicunque inter se aequales existunt. Qua de re sufficit in aequatione (5.) signum S referre ad indicum m, m' etc. *valores a se diversos* quocunque modo e numero indicum 0, 1, 2 .... p petitos. Distinguamus iam inter tres casus quibus $p < n$, $p = n$, $p > n$.

Sit $p < n$; non licet indicibus m, m' .... $m^{(n)}$ quorum numerus est $n+1$ valores inter se diversos e numero $p+1$ indicum 0, 1, 2 .... p tribuere; qua de re semper evanescit Determinans,

$$\Sigma \pm \alpha_m \alpha'_{m'} \ldots\ldots \alpha^{(n)}_{m^{(n)}},$$

ideoque totum Aggregatum quod signum S amplectitur. Qua de re hanc habemus Propositionem.

Propositio I.

„Sit

$$c^{(i)}_k = \alpha^{(i)} a^{(k)} + \alpha^{(i)}_1 a^{(k)}_1 \ldots\ldots + \alpha^{(i)}_p a^{(k)}_p,$$

quoties $p < n$ evanescit Determinans

$$\Sigma \pm c c'_1 c''_2 \ldots\ldots c^{(n)}_n."$$

Iam secundum casum examinemus qui prae ceteris momenti est.

Sit $p = n$; indices inter se diversi m, m' .... $m^{(n)}$ ex indicibus 0, 1, 2 .... n sumi debent ideoque cum utrorumque idem numerus sit, indices m, m' etc. cum indicibus 0, 1, 2 .... n conveniunt, ordinis respectu non habito. Qua de re eruitur P e formula,

$$P = S . a a'_1 \ldots\ldots a^{(n)}_n \Sigma \pm \alpha \alpha'_1 \ldots\ldots \alpha^{(n)}_n,$$

indicibus inferioribus 0, 1 .... omnimodis permutatis ita tamen ut in utroque factore

$$a a'_1 \ldots\ldots a^{(n)}_n, \qquad \Sigma \pm \alpha \alpha'_1 \ldots\ldots \alpha^{(n)}_n$$

eadem adhibeatur Permutatio. At iis Permutationibus Determinans,

$$\Sigma \pm \alpha\alpha'_1 \ldots\ldots \alpha_n^{(n)},$$

aut non mutatur aut tantum signum mutat prout Permutatio positiva aut negativa est. Qua de re eruimus P si in expressione,

$$\pm a a'_1 \ldots\ldots a_n^{(n)} . \Sigma \pm \alpha\alpha'_1 \ldots\ldots \alpha_n^{(n)},$$

indices ipsorum α inferiores omnimodis permutantur signo positivo aut negativo electo prout Permutatio positiva ut nevativa est. Unde si ponimus

$$6. \quad \Sigma \pm a a'_1 \ldots\ldots a_n^{(n)} = R, \qquad \Sigma \pm \alpha\alpha'_1 \ldots\ldots \alpha_n^{(n)} = \mathrm{P},$$

fit

$$7. \quad P = \mathrm{P} R.$$

Qua formula haec continetur Propositio in his quaestionibus fundamentalis

Propositio II.

„Datis binis quibuscunque eiusdem gradus Determinantibus eorum productum exhiberi potest ut eiusdem gradus Determinans cuius elementa sunt expressiones rationales integrae elementorum Determinantium propositorum; videlicet posito

$$c_k^{(i)} = \alpha^{(i)} a^{(k)} + \alpha_1^{(i)} a_1^{(k)} \ldots\ldots + \alpha_n^{(i)} a_n^{(k)}$$

atque

$$R = \Sigma \pm a a'_1 \ldots\ldots a_n^{(n)}, \quad \mathrm{P} = \Sigma \pm \alpha\alpha'_1 \ldots\ldots \alpha_n^{(n)}, \quad P = \Sigma \pm c c'_1 \ldots\ldots c_n^{(n)},$$

fit

$$P = \mathrm{P} R."$$

E Propositione antecedente fluit generalior:

datis quotcunque eiusdem gradus Determinantibus eorum productum ut eiusdem gradus exhiberi posse Determinans cuius elementa expressiones sint rationales integrae elementorum Determinantium propositorum.

Non essentiale est, quod Prop. II. supponitur, utriusque Determinantis eundem gradum esse; vidimus enim §. 5., quodlibet Determinans $m+1^{\text{ti}}$ gradus,

$$\Sigma \pm \alpha\alpha'_1 \ldots\ldots \alpha_m^{(m)},$$

etiam pro altioris gradus Determinante haberi posse. Sit $m > n$ atque supponamus evanescere cuncta elementa,

$$\alpha_k^{(m+1)}, \quad \alpha_k^{(m+2)}, \quad \ldots\ldots \quad \alpha_k^{(n)},$$

in quibus inferior index superiore minor est, porro esse,

$$\alpha_{m+1}^{(m+1)} = \alpha_{m+2}^{(m+2)} \ldots\ldots = \alpha_n^{(n)} = 1;$$

erit secundum §. 5. IV.:

$$\Sigma \pm \alpha \alpha'_1 \alpha''_2 \dots \alpha^{(n)}_n = \Sigma \pm \alpha \alpha'_1 \dots \alpha^{(m)}_m.$$

Eo igitur casu fit:

$$8. \quad \Sigma \pm \alpha \alpha'_1 \alpha''_2 \dots \alpha^{(m)}_m . \Sigma \pm a a'_1 a''_2 \dots a^{(n)}_n = \Sigma \pm c c'_1 c''_2 \dots c^{(n)}_n,$$

sive habetur

Propositio III.

„Sit pro indicis i valoribus 0, 1, 2 .... m,

$$c^{(i)}_k = \alpha^{(i)} a^{(k)} + \alpha^{(i)}_1 a^{(k)}_1 \dots + \alpha^{(i)}_n a^{(k)}_n,$$

pro indicibus i valoribus maioribus quam m,

$$c^{(i)}_k = a^{(k)}_i + \alpha^{(i)}_{i+1} a^{(k)}_{i+1} + \alpha^{(i)}_{i+2} a^{(k)}_{i+2} \dots + \alpha^{(i)}_n a^{(k)}_n:$$

erit

$$\Sigma \pm \alpha \alpha'_1 \dots \alpha^{(m)}_m \, \Sigma \pm a a'_1 \dots a^{(n)}_n = \Sigma \pm c c'_1 c''_2 \dots c^{(n)}_n."$$

In parte laeva aequationis (8.) non inveniuntur elementa α quorum index superior ipso m maior est, unde in Prop. antec. de valoribus eorum ex arbitrio statuere licet. Quos si evanescere ponimus, fit pro ipso $i \leqq m$,

$$c^{(i)}_k = \alpha^{(i)} a^{(k)} + \alpha^{(i)}_1 a^{(k)}_1 \dots + \alpha^{(i)}_m a^{(k)}_m,$$

pro $i > m$,

$$c^{(i)}_k = a^{(k)}_i.$$

14.

Accedamus ad casum quo $p > n$; secundum formulam (5.) §. pr. fit P summa expressionum,

$$a_m a'_{m'} \dots a^{(n)}_{m^{(n)}} . \Sigma \pm \alpha_m \alpha'_{m'} \dots \alpha^{(n)}_{m^{(n)}},$$

indicibus m, m' etc. tributis quibuscunque $n+1$ valoribus a se diversis e numero indicum 0, 1, 2 .... p. Qua de re ex ipsis 0, 1, 2 .... p electis $n+1$ numeris diversis, hi numeri omnimodis inter se permutati pro indicibus inferioribus m, m' .... $m^{(n)}$ sumi debent, omnibusque illis Permutationibus pro quibuscunque $n+1$ numeris factis, singula Aggregata $1.2 \dots n+1$ terminorum sic provenientia summanda sunt. At illis indicum inferiorum m, m' etc. Permutationibus Determinans

$$\Sigma \pm \alpha_m \alpha'_{m'} \dots \alpha^{(n)}_{m^{(n)}}$$

non mutatur aut solum signum mutat prout Permutatio positiva aut negativa est. Qua de re fit,

$$P = S . \Sigma \pm \alpha_m \alpha'_{m'} \dots \alpha^{(n)}_{m^{(n)}} \Sigma \pm a_m a'_{m'} \dots a^{(n)}_{m^{(n)}},$$

sive fit P Aggregatum e

$$\frac{p+1.p.p-1\ldots\ldots p-n+1}{1.2.3\ldots\ldots n+1} = \frac{p+1.p.p-1\ldots\ldots n+2}{1.2.3\ldots\ldots p-n}$$

productis binorum Determinantium,

$$\Sigma \pm \alpha_m \alpha'_{m'} \ldots\ldots \alpha^{(n)}_{m^{(n)}} \,.\, \Sigma \pm a_m a'_{m'} \ldots\ldots a^{(n)}_{m^{(n)}},$$

quae obtinentur quoscunque $n+1$ diversos numeros ex ipsis 0, 1, 2 .... p pro indicibus inferioribus $m, m' \ldots\ldots m^{(n)}$ sumendo. Habemus igitur sequentem Propositionem:

Propositio IV.

„Formentur producta binorum Determinantium,

$$\Sigma \pm \alpha_m \alpha'_{m'} \ldots\ldots \alpha^{(n)}_{m^{(n)}} \,.\, \Sigma \pm a_m a'_{m'} \ldots\ldots a^{(n)}_{m^{(n)}},$$

pro indicibus inferioribus m, m' etc. quoscunque sumendo $n+1$ numeros ex ipsis 0, 1, 2 .... p, ubi $p > n$: cunctorum eiusmodi productorum summa aequatur Determinanti

$$\Sigma \pm c c'_1 \ldots\ldots c^{(n)}_n,$$

cuius elementa dantur per formulam,

$$c^{(i)}_k = \alpha^{(i)} a^{(k)} + \alpha^{(i)}_1 a^{(k)}_1 \ldots\ldots + \alpha^{(i)}_p a^{(k)}_p."$$

Casu particulari quo pro omnibus ipsorum i et k valoribus fit,

$$\alpha^{(i)}_m = a^{(i)}_m,$$

e Propp. antecc. haec fluit:

Propositi V.

„Posito

$$c^{(i)}_k = c^{(k)}_i = a^{(i)} a^{(k)} + a^{(i)}_1 a^{(k)}_1 \ldots\ldots + a^{(i)}_p a^{(k)}_p,$$

sit Determinans

$$\Sigma \pm c c'_1 \ldots\ldots c^{(n)}_n = P;$$

ubi $p < n$ fit

$$P = 0;$$

ubi $p = n$ fit

$$P = \{\Sigma \pm a a'_1 \ldots\ldots a^{(n)}_n\}^2;$$

ubi $p > n$ fit

$$P = S\{\Sigma \pm a_m a'_{m'} \ldots\ldots a^{(n)}_{m^{(n)}}\}^2,$$

siquidem pro indicibus inferioribus m, m' etc. sumuntur quilibet $n+1$ diversi e numeris 0, 1, 2 .... p."

Hinc ut Corollarium sequitur, *quoties quantitates $a^{(i)}_k$ reales sint Determinans*

$$\Sigma \pm c c'_1 \ldots\ldots c^{(n)}_n$$

evanescere non posse nisi Determinantia

$$\Sigma \pm a_m a'_{m'} \ldots\ldots a^{(n)}_{m^{(n)}}$$

singula evanescant.

Propositiones II., IV. ill. *Cauchy* demonstravit loco citato.

15.

Proponantur aequationes lineares

$$1. \quad \begin{cases} c x + c_1 x_1 \ldots\ldots + c_n x_n = \gamma \\ c' x + c'_1 x_1 \ldots\ldots + c'_n x_n = \gamma' \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ c^{(n)} x + c^{(n)}_1 x \ldots\ldots + c^{(n)}_n x_n = \gamma^{(n)}, \end{cases}$$

ubi

$$2. \quad c^{(i)}_k = \alpha^{(i)} a^{(k)} + \alpha^{(i)}_1 a^{(k)}_1 \ldots\ldots + \alpha^{(i)}_p a^{(k)}_p$$

$$3. \quad \gamma^{(i)}_k = \alpha^{(i)} l + \alpha^{(i)}_1 l_1 \ldots\ldots + \alpha^{(i)}_p l_p.$$

Quae proveniunt aequationes si $p+1$ aequationes,

$$4. \quad \begin{cases} a\, x + a' x_1 \ldots\ldots + a^{(n)} x_n = l \\ a_1 x + a'_1 x_1 \ldots\ldots + a^{(n)}_1 x_n = l_1 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ a_p x + a'_p x_1 \ldots\ldots + a^{(n)}_p x_n = l_p \end{cases}$$

per factores

$$\alpha^{(i)}, \quad \alpha^{(i)}_1, \quad \ldots\ldots \quad \alpha^{(i)}_p$$

multiplicatae adduntur. Si $p < n$ aequationum (1.) Determinans secundum Prop. I. §. 13. evanescit, quo casu incognitarum valores aut infiniti aut indeterminati evadunt. Indeterminatos eos evadere inde patet quod aequationibus (1.) satisfit quoties aequationibus (4.) satisfactum est; si vero $p < n$, aequationum (4.) numerus numero incognitarum minor est ideoque aequationibus illis infinitis modis satisfieri potest.

Sit $p \geqq n$, ac statuamus rursus

$$5. \quad P = \Sigma \pm c c'_1 \ldots\ldots c^{(n)}_n;$$

resolvendo (1.) provenit,

$$6. \quad \begin{cases} P.x \;\; = \dfrac{\partial P}{\partial c}\gamma + \dfrac{\partial P}{\partial c'}\gamma' \ldots\ldots + \dfrac{\partial P}{\partial c^{(n)}}\gamma^{(n)} \\ P.x_1 = \dfrac{\partial P}{\partial c_1}\gamma + \dfrac{\partial P}{\partial c'_1}\gamma' \ldots\ldots + \dfrac{\partial P}{\partial c^{(n)}_1}\gamma^{(n)} \\ \cdot\;\cdot \\ P.x_n = \dfrac{\partial P}{\partial c_n}\gamma + \dfrac{\partial P}{\partial c'_n}\gamma' \ldots\ldots + \dfrac{\partial P}{\partial c^{(n)}_n}\gamma^{(n)}. \end{cases}$$

In quibus valoribus ipsius $\gamma^{(i)}$ expressiones (3.) substituamus, quo facto prodeat:

$$7.\quad \left\{\begin{aligned} P.x &= \beta l + \beta_1 l_1 \ldots + \beta_p l_p \\ P.x_1 &= \beta' l + \beta'_1 l'_1 \ldots + \beta'_p l_p \\ &\ldots\ldots\ldots\ldots \\ P.x_n &= \beta^{(n)} l + \beta_1^{(n)} l_1 \ldots + \beta_p^{(n)} l_p; \end{aligned}\right.$$

erit

$$8.\quad \beta_m^{(k)} = \alpha_m \frac{\partial P}{\partial c_k} + \alpha'_m \frac{\partial P}{\partial c'_k} \ldots + \alpha_m^{(n)} \frac{\partial P}{\partial c_k^{(n)}}.$$

Quas expressiones secundum (2.) sic quoque exhibere licet,

$$9.\quad \beta_m^{(k)} = \frac{\partial P}{\partial c_k}.\frac{\partial c_k}{\partial a_m^{(k)}} + \frac{\partial P}{\partial c'_k}.\frac{\partial c'_k}{\partial a_k^{(n)}} \ldots + \frac{\partial P}{\partial c_k^{(n)}}.\frac{\partial c_k^{(n)}}{\partial a_m^{(k)}}.$$

Inter omnes quantitates c solae sunt quantitates c_k, c'_k .... $c_k^{(n)}$ quae elementum $a_m^{(k)}$ implicant; expresso igitur P per quantitates α, a ope formularum (2.), fit

$$10.\quad \beta_m^{(k)} = \frac{\partial P}{\partial a_m^{(k)}}.$$

Unde incognitarum valores (7.) per has formulas exhiberi possunt:

$$11.\quad \left\{\begin{aligned} P.x &= \frac{\partial P}{\partial a} l + \frac{\partial P}{\partial a_1} l_1 \ldots + \frac{\partial P}{\partial a_p} l_p \\ P.x_1 &= \frac{\partial P}{\partial a'} l + \frac{\partial P}{\partial a'_1} l_1 \ldots + \frac{\partial P}{\partial a'_p} l_p \\ &\ldots\ldots\ldots\ldots \\ P.x_n &= \frac{\partial P}{\partial a^{(n)}} l + \frac{\partial P}{\partial a_1^{(n)}} l_1 \ldots + \frac{\partial P}{\partial a_p^{(n)}} l_p. \end{aligned}\right.$$

Ponamus rursus,

$$12.\quad R = \Sigma \pm a_m a'_{m'} \ldots a_{m^{(n)}}^{(n)}, \qquad \mathrm{P} = \Sigma \pm \alpha_m \alpha'_{m'} \ldots \alpha_{m^{(n)}}^{(n)}$$

atque signum summatorium — S — extendamus ad qualibet ipsorum m, m' .... $m^{(n)}$ systemata in quibus m, m' etc. $n+1$ diversis numeris ex ipsis 0, 1, 2 .... p aequantur. Erit secundum Prop. IV. §. pr.,

$$13.\quad P = S.\mathrm{P}R.$$

Qua formula in (11.) substituta fit

$$14.\quad \left\{\begin{aligned} \{S.\mathrm{P}R\}x &= S.\mathrm{P}\left\{\frac{\partial R}{\partial a} l + \frac{\partial R}{\partial a_1} l_1 \ldots + \frac{\partial R}{\partial a_p} l_p\right\} \\ \{S.\mathrm{P}R\}x_1 &= S.\mathrm{P}\left\{\frac{\partial R}{\partial a'} l + \frac{\partial R}{\partial a'_1} l_1 \ldots + \frac{\partial R}{\partial a'_p} l_p\right\} \\ &\ldots\ldots\ldots\ldots \\ \{S.\mathrm{P}R\}x_n &= S.\mathrm{P}\left\{\frac{\partial R}{\partial a^{(n)}} l + \frac{\partial R}{\partial a_1^{(n)}} l_1 \ldots + \frac{\partial R}{\partial a_p^{(n)}} l_p\right\}. \end{aligned}\right.$$

Expressionem R non ingrediuntur omnia elementa,

$$a^{(i)},\quad a_1^{(i)},\quad \ldots\ldots\quad a_p^{(i)},$$

sed tantum elementa,

$$a_m^{(i)},\quad a_{m'}^{(i)},\quad \ldots\ldots\quad a_{m^{(n)}}^{(i)}.$$

Qua de re valores (14.) sic quoque exhibere licet:

$$15.\quad \begin{cases} \{S.\mathrm{P}R\}x \;=\; S.\mathrm{P}\left\{\dfrac{\partial R}{\partial a_m}l_m + \dfrac{\partial R}{\partial a_{m'}}l_{m'}\ldots\ldots + \dfrac{\partial R}{\partial a_{m^{(n)}}}l_{m^{(n)}}\right\} \\ \{S.\mathrm{P}R\}x_1 \;=\; S.\mathrm{P}\left\{\dfrac{\partial R}{\partial a'_m}l_m + \dfrac{\partial R}{\partial a'_{m'}}l_{m'}\ldots\ldots + \dfrac{\partial R}{\partial a'_{m^{(n)}}}l_{m^{(n)}}\right\} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \{S.\mathrm{P}R\}x_n \;=\; S.\mathrm{P}\left\{\dfrac{\partial R}{\partial a_m^{(n)}}l_m + \dfrac{\partial R}{\partial a_{m'}^{(n)}}l_{m'}\ldots\ldots + \dfrac{\partial R}{\partial a_{m^{(n)}}^{(n)}}l_{m^{(n)}}\right\}. \end{cases}$$

Designemus per

$$(x),\quad (x_1),\quad \ldots\ldots\quad (x_n),$$

valores incognitarum $x, x_1 \ldots\ldots x_n$ provenientes e $n+1$ aequationibus e numero aequationum (4.) in quibus termini constantes sint,

$$l_m,\quad l_{m'},\quad \ldots\ldots\quad l_{m^{(n)}};$$

erunt expressiones uncis inclusae quae in dextra parte aequationum (15.) sub signo S inveniuntur per P multiplicatae, aequales ipsis

$$R(x),\quad R(x_1),\quad \ldots\ldots\quad R(x_n).$$

Unde poterunt iam formula (15.) sic exhiberi:

$$16.\quad \begin{cases} \{S.\mathrm{P}R\}x \;=\; S.\mathrm{P}R(x) \\ \{S.\mathrm{P}R\}x_1 = S.\mathrm{P}R(x_1) \\ \ldots\ldots\ldots\ldots \\ \{S.\mathrm{P}R\}x_n = S.\mathrm{P}R(x_n), \end{cases}$$

unde

$$17.\quad x = \frac{S.\mathrm{P}R(x)}{S.\mathrm{P}R},\quad x_1 = \frac{S.\mathrm{P}R(x_1)}{S.\mathrm{P}R},\quad \ldots\ldots\quad x_n = \frac{S.\mathrm{P}R(x_n)}{S.\mathrm{P}R}.$$

Quae sequens est Propositio:

Propositio I.

„Quascunque $n+1$ combinando e $p+1$ aequationibus (4.) veluti $m+1^{\text{tam}}$, $m'+1^{\text{tam}}$, ... $m^{(n)}+1^{\text{tam}}$, eruantur incognitarum $x, x_1 \ldots\ldots x_n$ valores,

$$(x),\quad (x_1),\quad \ldots\ldots\quad (x_n);$$

qui valores omnes per *eandem* quantitatem multiplicentur,

$$\mathrm{P}R = \Sigma \pm \alpha_m \alpha'_{m'} \ldots\ldots \alpha_{m^{(n)}}^{(n)} . \Sigma \pm a_m a'_{m'} \ldots\ldots a_{m^{(n)}}^{(n)};$$

pro omnibus illis combinationibus quarum numerus est,

$$\frac{p+1 \cdot p \ldots\ldots p-n+1}{1.2 \ldots\ldots n+1} = \frac{p+1 \cdot p \ldots\ldots n+2}{1.2 \ldots\ldots p-n},$$

producta singula,

$$\mathbf{P}R.(x), \quad \mathbf{P}R.(x_1), \quad \ldots\ldots \quad \mathbf{P}R.(x_n),$$

summando ac dividendo per summam ipsorum $\mathbf{P}R$ prodeunt incognitarum valores quales per aequationes (1.) determinantur,

$$x = \frac{S.\mathbf{P}R(x)}{S.\mathbf{P}R}, \quad x_1 = \frac{S.\mathbf{P}R(x_1)}{S.\mathbf{P}R}, \quad x_n = \frac{S.\mathbf{P}R(x_n)}{S.\mathbf{P}R}.$$

Si rursus fit,

$$\alpha_m^{(i)} = a_m^{(i)}$$

abeunt aequationes (1.) in sequentes,

$$18. \quad \begin{cases} (a\,a)x + (a\,a')x_1 \ldots\ldots + (a\,a^{(n)})x_n = (a\,l) \\ (a'\,a)x + (a'\,a')x_1 \ldots\ldots + (a'\,a^{(n)})x_n = (a'\,l) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (a^{(n)}\,a)x + (a^{(n)}\,a')x_1 \ldots\ldots + (a^{(n)}\,a^{(n)})x_n = (a^{(n)}\,l), \end{cases}$$

siquidem

$$(a^{(i)}a^{(k)}) = (a^{(k)}a^{(i)}) = a^{(i)}a^{(k)} + a_1^{(i)}a_1^{(k)} \ldots\ldots + (a_p^{(i)}a_p^{(k)})$$
$$(a^{(i)}l) = a^{(i)}l + a_1^{(i)}l_1 \ldots\ldots + a_p^{(i)}l_p.$$

Aequationes (18.) eaedem sunt atque adhibentur ad determinationem incognitarum x, x_1 etc. per Methodum *Minimorum Quadratorum*, si Observationes numerum aequationum (4.) suppeditarunt ipsum incognitarum numerum excedentem. Ponendo enim

$$U = S.(a_m x + a'_m x_1 \ldots\ldots + a_m^{(n)} x_n - l_m)^2,$$

summa S extensa ad ipsius m valores 0, 1 .... p, aequationes (18.) conveniunt cum his

$$\tfrac{1}{2}\frac{\partial U}{\partial x} = 0, \quad \tfrac{1}{2}\frac{\partial U}{\partial x_1} = 0, \quad \tfrac{1}{2}\frac{\partial U}{\partial x_n} = 0,$$

pro quibus U valorem Minimum nanciscitur.

Habemus igitur hanc Propositionem:

Propositio II.

„Proponantur aequationes,

$$\begin{aligned} a x + a' x_1 + a'' x_2 \ldots\ldots + a^{(n)} x_n &= l \\ a_1 x + a'_1 x_1 + a''_1 x_2 \ldots\ldots + a_1^{(n)} x_n &= l_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ a_p x + a'_p x_1 + a''_p x_2 \ldots\ldots + a_p^{(n)} x_n &= l_p, \end{aligned}$$

quarum numerus incognitarum numerum excedat; e quolibet systemate $n+1$ aequationum praecedentium valor incognitae eruatur atque per quadratum Determinantis eius systematis, RR, multiplicetur; quibus factis pro singulis aequationum propositarum combinationibus omnium illorum productorum summa per summam omnium RR dividatur: eruitur incognitae valor idem atque invenitur, si aequationes propositae per Methodum Minimorum Quadratorum tractantur."

Observandum est, valores omnium incognitarum qui ex eadem aequationum propositarum combinatione proveniant secundum Prop. praec. per *eandem* quantitatem RR multiplicari, quam ideo in applicationibus ad *Methodum Minimorum Quadratorum* convenit appellare ***Pondus Combinationis,*** a pondere valoris incognitae bene distinguendum.

16.

Statuamus ex aequationibus (18.) §. pr. sequi,

$$P.x = H(al) + H'(a'l) \dots + H^{(n)}(a^{(n)}l),$$

ubi P sicuti supra designat aequationum illarum Determinans. Consueverunt Astronomi, quantitatem

$$\frac{P}{H} = \mathfrak{P}$$

appellare incognitae x ***Pondus*** seu potius Pondus determinationis incognitae x quae omnibus Observationibus per Methodum Min. Quadr. combinatis eruitur. Restituendo ipsius $(a^{(i)}a^{(k)})$ loco elementum $c_k^{(i)}$ fit,

$$P = \Sigma \pm c\,c'_1 c''_2 \dots c_n^{(n)}, \qquad H = \Sigma \pm c'_1 c''_2 \dots c_n^{(n)}.$$

Unde secundum Prop. V. §. 14.,

$$P = S\left\{\Sigma \pm a_m a'_{m'} a''_{m''} \dots a^{(n)}_{m^{(n)}}\right\}^2$$
$$H = S\left\{\Sigma \pm \quad a'_{m'} a''_{m''} \dots a^{(n)}_{m^{(n)}}\right\}^2,$$

siquidem in altera formula pro ipsis $m, m', m'' \dots m^{(n)}$, in altera pro ipsis $m', m'' \dots m^{(n)}$ omnibus modis quibus fieri potest sumuntur indicum $0, 1, 2 \dots p$ seu $n+1$ seu n diversi. Si tantummodo tot combinamus Observationes quot sunt incognitae, ex. gr. Observationes quantitatibus

$$l_0, \quad l_1 \dots l_n$$

respondontes, fit Pondus ipsius x ea Combinatione determinatae,

$$\frac{\left\{\Sigma \pm a_0 a'_1 a''_2 \dots a_n^{(n)}\right\}^2}{S\left\{\Sigma \pm \quad a'_1 a''_2 \dots a_n^{(n)}\right\}^2} = (\mathfrak{P}),$$

siquidem in denominatore sub signo S pro indicibus inferioribus sumuntur omnibus modis n diversi e $n+1$ indicibus 0, 1, 2 .... n. Si vocamus quantitatem

$$\{\Sigma \pm a_0 a'_1 a''_2 \ \dots \ a^{(n)}_n\}^2 = RR$$

Combinationis Pondus, erit,

$$S\{\Sigma \pm a'_1 a''_2 \ \dots \ a^{(n)}_n\}^2 = \frac{RR}{(\mathfrak{P})},$$

ipsius x per Combinationem illam determinatae Pondus inversum, multiplicatum per Pondus Combinationis RR. Quantitas quae Aggregato praecedente continetur,

$$\{\Sigma \pm a'_0 a''_1 \ \dots \ a^{(n)}_{n-1}\}^2,$$

etiam in aliis Combinationibus obvenit, videlicet in iis quae quantitatibus l_0, l_1 .... l_{n-1} atque uni e reliquis l_n, l_{n+1} .... l_p respondent ideoque in

$$p+1-n$$

Combinationibus. Quam ob rem si pro singulis Combinationibus $p+1$ Observationum ad numerum $n+1$, qui determinandis incognitis sufficit, determinamus ipsius x Pondus inversum, multiplicatum per Combinationis Pondus: omnium eiusmodi productorum summa aequatur quantitati,

$$(p+1-n)\, S\{\Sigma \pm a'_{m'} a''_{m''} \ \dots \ a^{(n)}_{m^{(n)}}\}^2 = (p+1-n)H,$$

sive fit

$$S\frac{RR}{(\mathfrak{P})} = (p+1-n)H = (p+1-n)\cdot\frac{P}{\mathfrak{P}} = (p+1-n)\cdot\frac{S.RR}{\mathfrak{P}},$$

unde

$$\frac{S.\dfrac{RR}{(\mathfrak{P})}}{S.RR} = \frac{p+1-n}{\mathfrak{P}}.$$

Hac formula incognitae per M.M.Q ex omnibus $p+1$ Obss. determinatae pondus $\mathfrak{P}$ determinatur eiusdem quantitatis ponderibus quae pro numero Observationum $n+1$ aequali numero incognitarum obtinentur, advocatis singulis *Combinationum Ponderibus* RR. Videmus ipsorum $\frac{1}{(P)}$ valorem quodammodo medium in parte dextra aequationis praecedentis formatum non ipsi $\frac{1}{\mathfrak{P}}$ aequari sicuti in Prop. II. §. antec. de incognitarum valoribus usu venit, sed ipsi $\frac{1}{\mathfrak{P}}$ multiplicato per $p+1-n$, hoc est per excessum Observationum numeri unitate aucti super numerum incognitarum. Quod bene quadrat, quia determinationum pondera cum Observationum numero crescunt.

Regiom. 17 Martii 1841.