Generalised Form of Certain Series. By J. W. L. GLAISHER, M.A., F.R.S.

[*Read May 9th*, 1878.]

1. The results referred to in the title are consequences of the identical equation

$$\left(1+\frac{p}{p}x+\frac{p\,.\,p+2}{p\,.\,p+1}\frac{x^2}{2!}+\frac{p\,.\,p+2\,.\,p+4}{p\,.\,p+1\,.\,p+2}\frac{x^3}{3!}+\&c.\right)e^{-x}$$

$$=1+\frac{1}{p+1}\frac{x^2}{2}+\frac{1}{p+1\,.\,p+3}\frac{x^4}{2^2\,.\,2!}+\frac{1}{p+1\,.\,p+3\,.\,p+5}\frac{x^6}{2^3\,.\,3!}+\&c. \quad \ldots\ldots(1);$$

and it is convenient to proceed to them at once, leaving the demonstration of (1) to § 4.

It may be observed that, in (1), if p is equal to a negative integer, say if $p=-2i$, then in the series that multiplies e^{-x} there will be i zero terms; for the coefficient of x^{i+1} involves the factor $p+2i$ in the numerator, and therefore vanishes, as also does every term up to x^{2i+1}, in which the factor $p+2i$ first makes its appearance in a denominator, and can therefore be divided out from the numerator and denominator; this factor also can be divided out from all the succeeding terms. Thus the terms in x^{i+1}, x^{i+2}, ... x^{2i} disappear. It follows, therefore, that in the product

$$\left(1+\frac{1}{p+1}\frac{x^2}{2}+\frac{1}{p+1\,.\,p+3}\frac{x^4}{2^2\,.\,2!}+\frac{1}{p+1\,.\,p+3\,.\,p+5}\frac{x^6}{2^3\,.\,3!}+\&c.\right)e^{x},$$

if $p=-2i$, there are no terms in x^{i+1}, x^{i+2}, ... x^{2i}.

If p = a negative uneven integer, both sides of (1) become infinite: this case is supposed to be excluded in all that follows.

2. Since the right-hand side of (1) only involves x^2, it follows that

$$\left(1+x+\frac{p+2}{p+1}\frac{x^2}{2!}+\frac{p+2\,.\,p+4}{p+1\,.\,p+2}\frac{x^3}{3!}+\&c.\right)e^{-x}$$

$$=\left(1-x+\frac{p+2}{p+1}\frac{x^2}{2!}-\frac{p+2\,.\,p+4}{p+1\,.\,p+2}\frac{x^3}{3!}+\&c.\right)e^{x},$$

where p has any value; whence

$$e^{2x}=\frac{e^x}{e^{-x}}=\frac{1+x+\dfrac{p+2}{p+1}\dfrac{x^2}{2!}+\dfrac{p+2\,.\,p+4}{p+1\,.\,p+2}\dfrac{x^3}{3!}+\&c.}{1-x+\dfrac{p+2}{p+1}\dfrac{x^2}{2!}-\dfrac{p+2\,.\,p+4}{p+1\,.\,p+2}\dfrac{x^3}{3!}+\&c.} \quad \ldots\ldots\ldots (2).$$

Putting $p = 0$ in this expression, the numerator $= \frac{1}{2}(1+e^{2x})$, and the denominator $= \frac{1}{2}(1+e^{-2x})$. Putting $p = \infty$, the numerator and denominator become respectively $= e^x$ and e^{-x}.

From (2) we have

$$\frac{e^x-e^{-x}}{e^x+e^{-x}} = \frac{x+\dfrac{p+2\,.\,p+4}{p+1\,.\,p+2}\dfrac{x^3}{3!}+\dfrac{p+2\,\ldots\,p+8}{p+1\,\ldots\,p+4}\dfrac{x^5}{5!}+\&c.}{1+\dfrac{p+2}{p+1}\dfrac{x^2}{2!}+\dfrac{p+2\,.\,p+4\,.\,p+6}{p+1\,.\,p+2\,.\,p+3}\dfrac{x^4}{4!}+\&c.};$$

therefore

$$\tan x = \frac{x-\dfrac{p+2\,.\,p+4}{p+1\,.\,p+2}\dfrac{x^3}{3!}+\dfrac{p+2\,\ldots\,p+8}{p+1\,\ldots\,p+4}\dfrac{x^5}{5!}-\&c.}{1-\dfrac{p+2}{p+1}\dfrac{x^2}{2!}+\dfrac{p+2\,.\,p+4\,.\,p+6}{p+1\,.\,p+2\,.\,p+3}\dfrac{x^4}{4!}-\&c.}\ \ldots\ldots\ldots(3),$$

which is true for all values of p. The cases of $p = 0$ and $p = \infty$ correspond respectively to the formulæ

$$\tan x = \frac{\frac{1}{2}\sin 2x}{\frac{1}{2}(1+\cos 2x)}, \qquad \tan x = \frac{\sin x}{\cos x}.$$

For $p = 1$, (3) gives

$$\tan x = \frac{x-\dfrac{3\,.\,5}{2\,.\,3}\dfrac{x^3}{3!}+\dfrac{3\,.\,5\,.\,7\,.\,9}{2\,.\,3\,.\,4\,.\,5}\dfrac{x^5}{5!}-\&c.}{1-\dfrac{3}{2}\dfrac{x^2}{2!}+\dfrac{3\,.\,5\,.\,7}{2\,.\,3\,.\,4}\dfrac{x^4}{4!}-\&c.};$$

whence, putting $2x$ for x,

$$\tan 2x = \frac{\dfrac{2!}{(1!)^3}x-\dfrac{6!}{(3!)^3}x^3+\dfrac{10!}{(5!)^3}x^5-\&c.}{1-\dfrac{4!}{(2!)^3}x^2+\dfrac{8!}{(4!)^3}x^4-\&c.}.$$

3. From (2) it follows that

$$\frac{2x}{e^{2x}-1} = \frac{1-x+\dfrac{p+2}{p+1}\dfrac{x^2}{2!}-\dfrac{p+2\,.\,p+4}{p+1\,.\,p+2}\dfrac{x^3}{3!}+\&c.}{1+\dfrac{p+2\,.\,p+4}{p+1\,.\,p+2}\dfrac{x^2}{3!}+\dfrac{p+2\,\ldots\,p+8}{p+1\,\ldots\,p+4}\dfrac{x^4}{5!}+\&c.}$$

$$= 1-x+\frac{B_1}{2!}2^2x^2-\frac{B_2}{4!}2^4x^4+\&c.,$$

$B_1, B_2, B_3 \ldots$ being the Bernoullian numbers; whence

$$\frac{1+\dfrac{p+2}{p+1}\dfrac{x^2}{2!}+\dfrac{p+2\,.\,p+4\,.\,p+6}{p+1\,.\,p+2\,.\,p+3}\dfrac{x^4}{4!}+\&c.}{1+\dfrac{p+2\,.\,p+4}{p+1\,.\,p+2}\dfrac{x^2}{3!}+\dfrac{p+2\,\ldots\,p+8}{p+1\,\ldots\,p+4}\dfrac{x^4}{5!}+\&c.}$$

$$= 1 + \frac{B_1}{2!} 2^2 x^2 - \frac{B_2}{4!} 2^4 x^4 + \frac{B_3}{6!} 2^6 x^6 - \&c.$$

Multiplying up by the denominator of the left-hand expression, and equating the coefficients of x^{2n}, we have

$$2^{2n} B_n - \frac{p+2 \,.\, p+4}{p+1 \,.\, p+2} \frac{2n\,(2n-1)}{3!} 2^{2n-2} B_{n-1}$$
$$+ \frac{p+2 \ldots p+8}{p+1 \ldots p+4} \frac{2n \ldots 2n-3}{5!} 2^{2n-4} B_{n-2} \ldots\ldots$$
$$\ldots\ldots + (-)^{n+1} \frac{p+2 \,.\, p+4 \ldots p+4n}{p+1 \,.\, p+2 \ldots p+2n} \frac{1}{2n+1}$$
$$= (-)^{n+1} \frac{p+2 \,.\, p+4 \ldots p+4n-2}{p+1 \,.\, p+2 \ldots p+2n-1},$$

which is true independently of the value of p.

As an example, put $n = 2$, and this formula gives

$$2^4 B_2 - \frac{p+2 \,.\, p+4}{p+1 \,.\, p+2} 2^3 B_1 - \frac{p+2 \ldots p+8}{p+1 \ldots p+4} \,.\, \frac{1}{5} = -\frac{p+2 \,.\, p+4 \,.\, p+6}{p+1 \,.\, p+2 \,.\, p+3},$$

viz., $$2^4 B_2 \,.\, p+1 \,.\, p+3 - 2^3 B_1 \,.\, p+4 \,.\, p+3 - (p+6 \,.\, p+8)\, \tfrac{1}{5}$$
$$= -\,(p+4 \,.\, p+6),$$

viz., $2^4 B_2 (p^2 + 4p + 3) - 2^3 B_1 (p^2 + 7p + 12) + \frac{1}{5}(4p^2 + 36p + 72) = 0$;

whence, equating the coefficients of the powers of p to zero,

$$2^4 B_2 \quad - 2^3 B_1 \quad + \tfrac{4}{5} = 0,$$
$$2^4 B_2 \,.\, 4 - 2^3 B_1 \,.\, 7 + \tfrac{36}{5} = 0,$$
$$2^4 B_2 \,.\, 3 - 2^3 B_1 \,.\, 12 + \tfrac{72}{5} = 0;$$

which, since $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, are at once seen to be true.

4. In order to prove (1), we observe that the coefficient of x^n on the left-hand side

$$= (-)^n \left(\frac{1}{n!} - \frac{p}{p} \frac{1}{(n-1)!} + \frac{p \,.\, p+2}{p \,.\, p+1} \frac{1}{(n-2)!\,2!} \ldots\ldots \right.$$
$$\left. \ldots\ldots + (-)^n \frac{p \,.\, p+2 \ldots p+2n-2}{p \,.\, p+1 \ldots p+n-1} \frac{1}{n!} \right)$$
$$= (-)^n \left(\frac{1}{n!} - \frac{\frac{1}{2}p}{p} \frac{2}{(n-1)!} + \frac{\frac{1}{2}p \,.\, \frac{1}{2}p+1}{p \,.\, p+1} \frac{2^2}{(n-2)!\,2!} \ldots\ldots \right.$$
$$\left. \ldots\ldots + (-)^n \frac{\frac{1}{2}p \,.\, \frac{1}{2}p+1 \ldots \frac{1}{2}p+n-1}{p \,.\, p+1 \ldots p+n-1} \frac{2^n}{n!} \right)$$

$$= (-)^n \frac{1}{p \cdot p+1 \dots p+n-1}$$
$$\times \left(\frac{p \cdot p+1 \dots p+n-1}{n!} - \tfrac{1}{2}p \cdot 2 \cdot \frac{p+1 \dots p+n-1}{(n-1)!} \right.$$
$$+ \frac{\frac{1}{2}p \cdot \frac{1}{2}p+1}{2!} \cdot 2^2 \cdot \frac{p+2 \dots p+n-1}{(n-2)!} \dots\dots$$
$$\left. \dots\dots + (-)^n \frac{\frac{1}{2}p \cdot \frac{1}{2}p+1 \dots \frac{1}{2}p+n-1}{n!} 2^n \right);$$

and $(-)^n \times$ the expression in brackets = the coefficient of t^n in

$$(1+t)^{-p} + \tfrac{1}{2}p \cdot 2t(1+t)^{-p-1} + \frac{\frac{1}{2}p \cdot \frac{1}{2}p+1}{2!} 2^2 t^2 (1+t)^{-p-2} + \&c.,$$

viz., in $$(1+t)^{-p}\left(1 - \frac{2t}{1+t}\right)^{-\frac{1}{2}p}, = (1-t^2)^{-\frac{1}{2}p}.$$

Thus, if n be uneven, the coefficient of x^n is zero; and if n be even, the coefficient of x^n

$$= \frac{1}{p \cdot p+1 \dots p+n-1} \cdot \frac{\frac{1}{2}p \cdot \frac{1}{2}p+1 \dots \frac{1}{2}p+\frac{1}{2}n-1}{(\frac{1}{2}n)!}$$
$$= \frac{1}{p+1 \cdot p+3 \dots p+n-1} \frac{1}{2^{\frac{1}{2}n}(\frac{1}{2}n)!}.$$

This proof does not apply as it stands when $p = -2i$, but it can be extended so as to include this case by taking $p = -2i+h$, and making h indefinitely small. The equation (1) can, however, be proved for all values of p without the use of limits, as follows:—

Consider the coefficient of x^n in the product

$$\left(1 + \frac{1}{p+1}\frac{x^2}{2} + \frac{1}{p+1 \cdot p+3}\frac{x^4}{2^2 \cdot 2!} + \frac{1}{p+1 \cdot p+3 \cdot p+5}\frac{x^6}{2^3 \cdot 3!} + \&c.\right)e^x.$$

First, suppose n even $= 2m$; the coefficient of x^{2m}

$$= \frac{1}{(2m)!} + \frac{1}{p+1}\frac{1}{2}\frac{1}{(2m-2)!} + \frac{1}{p+1 \cdot p+3}\frac{1}{2^2 \cdot 2!}\frac{1}{(2m-4)!} \dots$$
$$\dots + \frac{1}{p+1 \cdot p+3 \dots p+2m-1}\frac{1}{2^m \cdot m!}$$

$$= \frac{1}{2m \cdot 2m-1 \dots m+1} \frac{1}{p+1 \cdot p+3 \dots p+2m-1}$$
$$\times \left(\frac{p+1 \cdot p+3 \dots p+2m-1}{m!} + \frac{p+3 \dots p+2m-1}{(m-1)!}\frac{2m \cdot 2m-1}{m}\frac{1}{2} \right.$$
$$+ \frac{p+5 \dots p+2m-1}{(m-2)!}\frac{2m \dots 2m-3}{m \cdot m-1}\frac{1}{2^2}\frac{1}{2!} \dots$$
$$\left. \dots + \frac{2m \cdot 2m-1 \dots 1}{m!}\frac{1}{2^m \cdot m!}\right),$$

and the expression in brackets

$$= 2^m \left(\frac{\frac{1}{2}(p+1) \cdot \frac{1}{2}(p+1)+1 \dots \frac{1}{2}(p+1)+m-1}{m\,!} \right.$$

$$+ \frac{\frac{1}{2}(p+3) \dots \frac{1}{2}(p+3)+m-2}{(m-1)\,!}\,(m-\tfrac{1}{2})$$

$$\left. + \frac{\frac{1}{2}(p+5) \dots \frac{1}{2}(p+5)+m-3}{(m-2)\,!}\,\frac{m-\frac{1}{2} \cdot m-\frac{3}{2}}{2\,!} \dots + \frac{m-\frac{1}{2} \cdot m-\frac{3}{2} \dots \frac{1}{2}}{m\,!} \right)$$

$= 2^m$. coefficient of t^m in

$$(1-t)^{-\frac{1}{2}(p+1)} + (m-\tfrac{1}{2})\,t\,(1-t)^{-\frac{1}{2}(p+3)} + \frac{m-\frac{1}{2} \cdot m-\frac{3}{2}}{2\,!}\,t^2\,(1-t)^{-\frac{1}{2}(p+5)} + \&c.,$$

viz., in $(1-t)^{-\frac{1}{2}(p+1)}\left(1+\frac{t}{1-t}\right)^{m-\frac{1}{2}}, \; = (1-t)^{-\frac{1}{2}(p+1)-m+\frac{1}{2}}, \; = (1-t)^{-\frac{1}{2}p-m}.$

Thus, 2^m . coefficient of t^m

$$= 2^m \frac{\frac{1}{2}p+m \cdot \frac{1}{2}p+m+1 \dots \frac{1}{2}p+2m-1}{m\,!}$$

$$= \frac{p+2m \cdot p+2m+2 \dots p+4m-2}{m\,!},$$

and therefore the coefficient of x^{2m}

$$= \frac{p+2m \cdot p+2m+2 \dots p+4m-2}{p+1 \cdot p+3 \dots p+2m-1}\,\frac{1}{(2m)\,!}$$

$$= \frac{p \cdot p+2 \cdot p+4 \dots p+4m-2}{p \cdot p+1 \cdot p+2 \dots p+2m-1}\,\frac{1}{(2m)\,!}.$$

Secondly, suppose n uneven $= 2m+1$, it can be shown, as above, that the coefficient of x^{2m+1}

$$= \frac{1}{2m+1 \cdot 2m \dots m+1} \cdot \frac{1}{p+1 \cdot p+3 \dots p+2m-1} \cdot 2^m \cdot T,$$

where $T =$ the coefficient of t^m in

$$(1-t)^{-\frac{1}{2}(p+1)} + (m+\tfrac{1}{2})\,t\,(1-t)^{-\frac{1}{2}(p+3)} + \frac{m+\frac{1}{2} \cdot m-\frac{1}{2}}{2\,!}\,t^2\,(1-t)^{-\frac{1}{2}(p+5)} + \&c.,$$

$$= (1-t)^{-\frac{1}{2}(p+1)}\left(1+\frac{t}{1-t}\right)^{m+\frac{1}{2}} = (1-t)^{-\frac{1}{2}p-m-1}.$$

Thus $$T = \frac{\frac{1}{2}p+m+1 \cdot \frac{1}{2}p+m+2 \dots \frac{1}{2}p+2m}{p+1 \cdot p+3 \dots p+2m-1},$$

whence the coefficient of x^{2m+1}

$$= \frac{p \cdot p+2 \dots p+4m}{p \cdot p+1 \dots p+2m}\,\frac{1}{(2m+1)\,!}.$$

5. I may mention that the right-hand side of (1) is a Bessel's function; for

$$J_n(z) = \frac{z^n}{2^n\Gamma(n+1)}\left(1 - \frac{z^2}{2.2n+2} + \frac{z^4}{2.4.2n+2.2n+4} - \&c.\right),$$

and we obtain the series in (1) on putting $n = \frac{1}{2}(p-1)$ and $z = xi$. It was, in fact, in connexion with the different particular integrals of the differential equation

$$\frac{d^2u}{dx^2} - a^2u = \frac{n.n+1}{x^2}u,$$

which is a transformation of the equation of Bessel's functions, that I met with the equation (1).

A Bessel's function is, as is known, a limiting form of a hypergeometric series, and the equation (1) may, in fact, be written

$$F\left(a, \tfrac{1}{2}p, p, \frac{2x}{a}\right)e^{-x} = F\left(\alpha, \beta, \tfrac{1}{2}p+\tfrac{1}{2}, \frac{x^2}{4\alpha\beta}\right),$$

where a, α, β are each of them infinite; or, if we please,

$$F\left(a, \tfrac{1}{2}p, p, \frac{2x}{a}\right)e^{-x} = F\left(a, a, \tfrac{1}{2}p+\tfrac{1}{2}, \frac{x^2}{4a^2}\right),$$

where a is infinite.

On a new method of determining the Differential Resolvents of Algebraical Equations. By ROBERT RAWSON, Associate of I.N.A., Hon. Member of the Literary and Philosophical Society, Manchester.

[*Read June 13th*, 1878.]

The theory of *differential resolvents* of algebraical equations appears to have originated in a short sketch of a theory of transcendental roots by Sir James Cockle, F.R.S., published in the "Philosophical Magazine" for August, 1860. Eighteen months later, the subject was considered in a series of suggestive papers on the theory of transcendental solutions, by the Rev. Robert Harley, F.R.S., published in the "Proceedings" of the Literary and Philosophical Society of Manchester, Vol. ii., pp. 181—184, 199—203, 237—241.

The leading idea is, to determine, from a given algebraical equation, a linear differential equation which is satisfied by the roots of the given