Geometry on an Ellipsoid. *By* Prof. Clifford.

(*Read December* 12*th*, 1872.)

The metric properties of an ellipsoid are entirely determined by the four points in which it is met by the imaginary circle at infinity. I shall start, therefore, by assuming the existence of these four co-planar points o_1, o_2, o_3, o_4, which, taken all together, I call the absolute.

1. To represent the ellipsoid on a plane we require also two fixed points i, j; the plane sections of the ellipsoid are then represented by conics through these points, and the generating lines by lines on the plane through them. In fact, if we take a fixed point a on the ellipsoid E_2, and draw a line through a and a variable point x on the ellipsoid, this line will meet a plane L in one point y, which is the representative of x; the points i, j will then represent the generators through a. If we take the points i, j to be the absolute of the plane L, then all the plane sections will be represented by circles, the lines through i will represent one system of generating lines, and the lines through j the other. We shall have then, in addition, to consider the four points o; and the geometry of the ellipsoid will be merely the geometry of the plane considered in relation to these four points, which are concyclic.

2. We know, then, that the antipoints of the o lie upon three new circles, orthotomic of each other and of the first. These correspond to the principal sections of the ellipsoid. The antipoints themselves represent umbilici, four of which are real; I call these u_1, u_2, u_3, u_4, and we may now take the points u as our absolute instead of the points o.

3. What now are the directions of the lines of curvature at any point x of the ellipsoid? First, the indicatrix at x is represented by the point-circle at its corresponding point y, so that conjugate directions at x correspond to rectangular directions at y. Next, the tangent plane at x meets the plane at infinity in a line, say β. Through the points o can be drawn two conics to touch β, say at p, q. The lines xp, xq are tangents to the lines of curvature at x, since the points p, q are conjugates both of the section of the ellipsoid at infinity and of the imaginary circle. But now let o_1o_2 and o_3o_4 meet β in r, s respectively. Then the involution made by rs and the points where β meets the imaginary circle have p, q for double points. The interpretation of this on the plane is, that the directions through y corresponding to xp, xq make equal angles with the circles yo_1o_2, yo_3o_4. Hence—

> The lines of curvature of the ellipsoid are represented by confocal anallagmatics having the u for foci.
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> Sections made by two conjugate planes of the ellipsoid are represented by orthotomic circles.

4. A straight line γ in space may be denoted by the two points c_1, c_2, where it meets the ellipsoid. The sections drawn through this line will be represented by the series of coaxal circles through c_1, c_2. The sections through δ, the polar of γ, will therefore be represented by the series of coaxal circles through d_1, d_2, the antipoints of c_1 c_2. Thus, a straight line being represented by a pair of points, its polar is represented by their antipoints, as is otherwise obvious.

I denote further the principal sections and the plane at infinity by XYZU, which notation will serve also for the circles which represent them. Now, in general, a section of E_2 passing through a fixed point of space is represented by a circle orthotomic of a fixed circle. In particular, the points o_1 o_2, o_3 o_4; o_1 o_3, o_4 o_2; o_1 o_4, o_2 o_3, correspond in this way to the circles XYZ. I want now to find the interpretation on the plane of the rectangularity of the lines γ and δ. The planes joining them to the point o_1o_2, o_3o_4 are harmonic of the lines o_1o_2, o_3o_4. Hence the circles c_1c_2X, d_1d_2X are harmonic of the circles coaxal with them and passing through o_1o_2, o_3o_4 respectively. This is to be true when X and Y are interchanged: the conditions may finally be written

$$\frac{c_1c_2\text{YZ}}{d_1d_2\text{YZ}} = \frac{c_1c_2\text{ZX}}{d_1d_2\text{ZX}} = \frac{c_1c_2\text{XY}}{d_1d_2\text{XY}}.$$

If now for d_1, d_2 we may substitute c_1, c_1' where c_1' is indefinitely near to c_1 *in any direction*, c_1c_2 represents the normal at c_1.

5. A circle P, orthotomic of U, represents a diametral section. Let the pole of this section be called p; p is a point at infinity. We know that it is always possible to find another point q at infinity, which is conjugate to p with respect both to the ellipsoid and to the imaginary circle. We may then endeavour to find the circle Q, of which q is the pole. Further, lines λ, μ can be drawn through p, q respectively, which are at right angles, and also conjugate polars of the ellipsoid. To represent these we must find a pair of points on P which have their antipoints on Q. These circles cut orthogonally; on each of them, then, there is a singly infinite number of point-pairs representing axes of the quadric, viz., the point-pairs determined by diameters of the other circle. That is to say, any circle P, orthotomic of U, being given, there can always be found a point q^0, such that the lines through q^0 determine on P point-pairs representing axes of the quadric.

The determination of q^0 depends on the position of the projecting-point a. The generators through a meet the diametral section Q in two points; the remaining generators through these intersect on the representative of q^0.

6. I now proceed to construct Q when P is given. In the first place, Q has to be orthotomic of P and U. Next, if we draw through P and Q two new circles, one of which has o_1 o_2 for harmonics, and the other

o_3 o_4, these must be harmonics of P and Q. But a circle orthotomic of U and having o_1 o_2 for harmonics, must have them for inverse points, and therefore have its centre on o_1 o_2. Hence the line joining the centres of P and Q is cut harmonically by the lines o_1o_2, o_3o_4. Similarly, it is cut harmonically by o_1o_3, o_4o_2, and by oo_4, o_2o_3. Hence the centres of P and Q are polar opposites in regard to the quadrangle o_1 o_2 o_3 o_4. They are therefore conjugate points in regard to the circle U.

7. Intersections of the ellipsoid by spheres are represented by anallagmatics *passing through* the four points o_1, o_2, o_3, o_4. There are two systems of real circles passing through pairs of them; these represent the circular sections. Sphero-conics are represented by such of these anallagmatics as have XYZU for focal circles. To find the axes of any circle P of the U system we must then draw two such anallagmatics having double contact with P; the point of contact in pairs will represent the axes of the corresponding section.

January 9th, 1873.

Dr. HIRST, F.R.S., President, in the Chair.

The following gentlemen were elected Members of the Society:—Mr. G. B. Finch, M.A., Mr. T. O. Harding, and Mr. John Macleod; the last named gentleman was subsequently admitted into the Society. Prof. R. Stawell Ball, of Dublin, and Dr. J. Hopkinson, B.A., Fellow of Trinity College, Cambridge, were proposed for election.

Papers were read by Mr. S. Roberts, V.P., "On Parallel Surfaces;" Prof. H. J. S. Smith, F.R.S., "On the greatest common divisors of the minor determinants of a rectangular matrix, of which the constituents are integral numbers," and "On an arithmetical demonstration of a theorem in the Integral Calculus;" Prof. Wolstenholme, "On the Summation of certain series."

The following presents were received:

"Reale Istituto Lombardo—Rendiconti," serie ii., vol. ii., fasc. xi.—xvii.; and "Memorie," vol. xi., ii della serie iii., fasc. ii. Milano, 1869.

"Monatsbericht," Aug., 1872.

A Catalogue of a collection of models of Ruled Surfaces, constructed by M. Fabre de Lagrange, with an appendix containing an account of the application of analysis to their investigation and classification, by C. W. Merrifield, 1872: from the author.