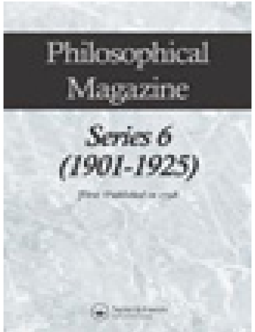

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LXVII. On vector differentials

Frank Lauren Hitchcock

disintegrate the molecule into positive and negative ions. The apparent masses of the ions therefore depend on the ratio of the electric force to the pressure.

10. The results of these experiments throw some light on the constitution of molecules of gases. We are led to conclude that:—It is possible to detach a particle from the molecule of a gas which is small, as regards mass and linear dimensions, compared with the molecule of hydrogen, also the particles produced from molecules of different gases are identically the same.

The mass of the negative ion coming from a zinc plate was previously shown by Professor Thomson* to be small compared with the molecule of hydrogen. The method which he adopted did not involve any of the principles underlying the present investigations.

A considerable number of phenomena connected with the electric discharge in gases may be explained in a general way by taking into consideration the physical properties of these negative ions. Thus some of the effects of variation of pressure, electric force, and distance between the plates can be accounted for. Also the high conductivity of gases under rapidly alternating forces may be due to the fact that the negative ions traverse a long distance before they are discharged by the electrodes. There are, however, many phenomena for which these physical properties supply no explanations: such as the appearance, at the electrodes, of the constituents of compound gases.

The experiments with ultra-violet light show that in carbonic acid the conductivity may arise from the genesis of small negative ions. I am at present continuing the researches with other gases and vapours, so as to obtain some additional evidence on this point, as it is to be expected that similar phenomena may occur with other compound gases.

LXVII. *On Vector Differentials.*

By FRANK LAUREN HITCHCOCK †.

1. IN studying physical quantities we are led to make a distinction between those which have by their very nature a direction in space, and those which, on the other hand, may be thought of as mere numbers. Directed quantities are conveniently called *vectors*, and non-directed ones *scalars*.

* J. J. Thomson, Phil. Mag. December 1900.

† Communicated by the Author.

The mathematical connexion between these two kinds of magnitudes is extremely intimate: if we have any scalar function continuously distributed through a portion of space, there is a vector function immediately derivable from it by the operator ∇, which derived vector was called by Maxwell the space-variation of the original scalar.

The object of the present paper is to study briefly the differentiation of vectors, a subject inseparably bound up with the quaternion operators ∇ and ϕ. I shall assume that the reader has some slight acquaintance with the calculus of Hamilton, and shall occasionally refer to Tait's 'Quaternions,' 3rd edition, 1890.

2. From the definition

$$d(\mathrm{F}q) = \lim_{n=\infty} n \left\{ \mathrm{F}\left(q + \frac{dq}{n}\right) - \mathrm{F}q \right\}$$

follows the very general proposition that a differential is a *linear* function: both q and $\mathrm{F}q$ are, in general, quaternions; but one or both habitually "degenerate" into vectors or scalars. In any case $d(\mathrm{F}q)$ is linear in dq.

It follows that if P is any scalar function of a point ρ, then $d\mathrm{P}$ is linear in $d\rho$.

Now every possible scalar term linear in $d\rho$ may, by very elementary transformations, be put in the form $\mathrm{S}\lambda d\rho$, where λ is of course a vector function of ρ. If there are several such terms we may *assume* that $\Sigma\lambda = -\nabla\mathrm{P}$, where the minus sign is introduced in order that our results may agree with Hamilton's original definition of ∇. Therefore

$$d\mathrm{P} = -\mathrm{S}d\rho\nabla\mathrm{P}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is a fundamental equation.

From this, remembering that $d\rho$ may, like any other vector, be thought of as the product of a scalar and a unit-vector, we have

$$\frac{d\mathrm{P}}{dh_{\epsilon}} = -\mathrm{S}\epsilon\nabla\mathrm{P}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

by writing $d\rho = \epsilon dh$ and then dividing both sides by dh. Here $\frac{d}{dh}$ may be thought of as an operator. It signifies differentiation with regard to any direction whatever in space, and ϵ is the corresponding unit-vector, either a constant or a function of ρ.

We have also

$$\begin{aligned} dP &= -S d\rho \nabla P \\ &= +S d\rho (iSi\nabla P + jSj\nabla P + kSk\nabla P) \\ &= -S d\rho \left(i\frac{dP}{dx} + j\frac{dP}{dy} + k\frac{dP}{dz} \right), \text{ by (2)}, \\ &= \frac{dP}{dx}dx + \frac{dP}{dy}dy + \frac{dP}{dz}dz, \end{aligned}$$

because $dx = -Sid\rho$, &c.

From (1) it appears that if dP be given, ∇P can be written by inspection.

3. Taking next σ any vector function of ρ, we have

$$d\sigma = \phi d\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

where ϕ is a linear and vector function. And, directly,

$$\frac{d\sigma}{dh_\epsilon} = \phi\epsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

For a fascinating account of the various types of these functions, see the last chapter of Kelland and Tait's 'Introduction to Quaternions.' The function ϕ is there considered as a homogeneous strain, and it seems convenient so to speak of it, even in those cases where it could not exist in a physical sense; for example, when the sum of the roots of the strain-cubic is zero.

To show that $\nabla\sigma$ may be written by inspection when $d\sigma$ is given, we may put

$$q = i\phi i + j\phi j + k\phi k;$$

and, if ϕ consists of several terms, we may consider each of them as a separate linear and vector function, call them ϕ_1, ϕ_2, &c.; to these will correspond q_1, q_2, . . ., whose sum, since q is linear in ϕ, must give the q of the whole function ϕ.

But $\nabla\sigma = q$, by (4); thus we can write down $\nabla\sigma$ if we know the part of q contributed by each term of ϕ.

Taking special cases, a term of the form $\beta S\alpha\lambda$, which we may call ϕ_1, and where λ is any vector whatever, gives

$$\begin{aligned} q_1 &= i\beta S\alpha i + j\beta S\alpha j + k\beta S\alpha k \\ &= -\alpha\beta; \end{aligned}$$

and in a similar manner the forms $V\alpha\lambda$, $V\alpha\lambda\beta$, and $g\lambda$ give, in order, 2α, $S\alpha\beta$, and $-3g$. Any other terms that may occur are to be treated in this way, and the sum of the results taken.

We have thus the means of finding the effect of ∇ on any function, scalar or vector, by merely differentiating it.

4. The following useful formulæ will be familiar to students of Tait:—

$$\nabla(\mathrm{FP}) = \frac{d(\mathrm{FP})}{d\mathrm{P}}\nabla \mathrm{P},$$

and

$$\nabla(\mathrm{PP}_1) = \mathrm{P}_1\nabla\mathrm{P} + \mathrm{P}\nabla\mathrm{P}_1,$$

in which the order is not important, and also

$$\nabla(\mathrm{P}\sigma) = \nabla\mathrm{P}\,.\,\sigma + \mathrm{P}\nabla\sigma,$$

where the order is vital. Here P and P_1 are scalars, FP is a scalar function of P, and σ is a vector.

To find the effect of ∇ on the product of any two vectors σ and τ we may adopt the notation $d\sigma = \phi d\rho$ and $d\tau = \theta d\rho$; whence

$$\begin{aligned} d(\sigma\tau) &= d\sigma\,.\,\tau + \sigma\,.\,d\tau \\ &= \phi d\rho\,.\,\tau + \sigma\,.\,\theta d\rho. \end{aligned}$$

From the scalar part of this differential we have

$$d\mathrm{S}\sigma\tau = \mathrm{S}d\rho(\phi'\tau + \theta'\sigma),$$

whence by (1),

$$\nabla\mathrm{S}\sigma\tau = -\phi'\tau - \theta'\sigma; \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

and from the vector part,

$$d\mathrm{V}\sigma\tau = \mathrm{V}\sigma\theta d\rho - \mathrm{V}\tau\phi d\rho,$$

each term of which, by the last article, contributes its portion of $\nabla\mathrm{V}\sigma\tau$. If we take $\phi_1 = \mathrm{V}\sigma\theta$, we have

$$\begin{aligned} q_1 &= i\mathrm{V}\sigma\theta i + j\mathrm{V}\sigma\theta j + k\mathrm{V}\sigma\theta k \\ &= -\mathrm{S}\,.\,\sigma\nabla\tau - \sigma\mathrm{S}\nabla\tau - \theta\sigma, \text{ (Tait, §§ 89, 90)} \end{aligned}$$

by the ordinary transformations.

Similarly the part corresponding to $-\mathrm{V}\tau\phi d\rho$ is

$$+\mathrm{S}\,.\,\tau\nabla\sigma + \tau\mathrm{S}\nabla\sigma + \phi\tau;$$

by adding the vector parts of these two quaternions we have

$$\mathrm{V}\nabla\mathrm{V}\sigma\tau = \tau\mathrm{S}\nabla\sigma - \sigma\mathrm{S}\nabla\tau + \phi\tau - \theta\sigma, \quad . \quad . \quad . \quad . \qquad (6)$$

and by adding the scalar parts,

$$\mathrm{S}\nabla(\sigma\tau) = \mathrm{S}\,.\,\tau\nabla\sigma - \mathrm{S}\,.\,\sigma\nabla\tau; \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

we have thus the three parts of $\nabla(\sigma\tau)$. Combining them gives

$$\nabla(\sigma\tau) = \tau S\nabla\sigma + (\phi - \phi')\tau + S\,.\,\tau\nabla\sigma$$
$$- \sigma S\nabla\tau - (\theta + \theta')\sigma - S\,.\,\sigma\nabla\tau\,;$$

but we have, identically,

$$V\,.\,V\nabla\sigma\,.\,\tau = (\phi - \phi')\tau, \quad . \quad . \quad . \quad . \quad (8)$$

by Tait, § 186; accordingly the first three terms of $\nabla(\sigma\tau)$ reduce to $\nabla\sigma\,.\,\tau$, and the last three, similarly, to $-\sigma\,.\,\nabla\tau - 2\theta\sigma$; whence, finally,

$$\nabla(\sigma\tau) = \nabla\sigma\,.\,\tau - \sigma\nabla\tau - 2\theta\sigma. \quad . \quad . \quad . \quad . \quad (9)$$

It may be noticed that $-\theta\sigma$ is the same as $S\sigma\nabla\,.\,\tau$.

If r and q are any two quaternion functions of ρ we have

$$\nabla(qr) = \nabla q\,.\,r - q\nabla r + 2Sq\,.\,\nabla r + 2S(Vq\nabla)r, \qquad (10)$$

which follows on combining (9) with $\nabla(P\sigma)$, &c , and which the reader may verify with ease.

5. It is convenient to classify vectors by the effect of ∇ upon them: if $V\nabla\sigma$ vanishes, σ is derivable from a scalar potential and its distribution is *irrotational*; if $S\nabla\sigma$ vanishes, σ is derivable from a vector potential, and its distribution is *solenoidal*; while if both these conditions are fulfilled at once, so that $\nabla\sigma = 0$, then the distribution is *Laplacean*. These distinctions are of fundamental importance in Physics.

There are also vectors which, though they do not directly satisfy the equation $V\nabla\sigma = 0$, yet do so when multiplied by a variable scalar. Hamilton and Tait showed that we then have $S\sigma\nabla\sigma = 0$. The simplest example is a unit-vector normal to a series of surfaces, and capable, therefore, of being written $U\nabla P$.

Taking the two vectors ∇P and $U\nabla P$, we shall adopt the notation:

$$dU\nabla P = d\nu = \chi d\rho,$$
$$d\nabla P \;\; = d(t\nu) = \psi d\rho\,;$$

the operators ψ and χ are then vector differentials, functions of ρ, and always linear in $d\rho$: their properties appear to be of considerable interest.

If α and β are any two *constant* unit-vectors, we shall have

$$d\left(\frac{dP}{dh_\alpha}\right) = -dS\alpha\nabla P, \text{ by (2)},$$
$$= -S\alpha d\nabla P, \text{ because } \alpha \text{ is constant},$$
$$= -S\alpha\psi d\rho\,;$$

whence by putting βdh_β for $d\rho$,

$$\frac{d^2\mathrm{P}}{dh_\beta dh_\alpha} = -\mathrm{S}\alpha\psi\beta = -\mathrm{S}\beta\psi\alpha, \quad . \quad . \quad . \quad (11)$$

where α and β are perfectly interchangeable, because either of them is any constant unit-vector whatever.

One consequence is that

$$\nabla\frac{d\mathrm{P}}{dh_\alpha} = \psi\alpha = \frac{d}{dh_\alpha}\nabla\mathrm{P}, \quad . \quad . \quad . \quad . \quad (12)$$

which may be extended to a vector by the usual method (Tait, § 149). Thus the operators ∇ and $\frac{d}{dh}$ are commutative, provided the direction h is constant. A single case of the same kind will presently be exhibited where the direction of differentiation is *not* constant.

6. The function χ, found by differentiating $\mathrm{U}\nabla\mathrm{P}$, or ν, owes most of its peculiarities to the fact that the differential of a unit-vector is always at right angles to the unit-vector itself (Tait, § 140, (2)); this is expressed by the equation

$$\mathrm{S}\nu\chi\epsilon = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where ϵ is any direction whatever. Thus the strain χ turns every vector into the tangent plane to the surface $\mathrm{P} = \mathrm{const.}$

If we form the strain-cubic in the usual manner we find that the absolute term vanishes, so that

$$\chi(\chi^2 - m_2\chi + m_1) = 0$$

for any direction whatever. Thus the cubic has a zero root; for another way of finding it we have, λ and μ being any vectors whatever,

$$\chi\mathrm{V}\chi'\lambda\chi'\mu = 0, \text{ Tait, § 157, (2).}$$

By interchanging χ and χ' in this last equation we obtain

$$\chi'\nu = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

for $\mathrm{V}\chi\lambda\chi\mu$ is parallel to ν, by (13). It appears from these results that that direction for which $\chi = 0$ is at right angles to the plane into which χ' turns every vector; and *vice versâ*.

Whence, by taking a special case of (8),

$$(\chi - \chi')\nu = \mathrm{V}(\mathrm{V}\nabla\nu)\nu,$$

the left side reduces to $\chi\nu$, that is, $\frac{d\nu}{dn}$; and remembering that ν satisfies the equation $\mathrm{S}\nu\nabla\nu = 0$, we see that ν, $\chi\nu$, and

$V\nabla\nu$ are mutually at right angles, while $T\chi\nu = TV\nabla\nu$. These facts are expressed by the equation

$$\nu\chi\nu = V\nabla\nu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Writing, as above, $\frac{d}{dn}$ for differentiation along the normal to the surface $P=\text{const.}$ we shall have

$$d\left(\frac{dP}{dn}\right) = -d(S\nu\nabla P)$$
$$= -S\nu\psi d\rho - S(t\nu)\chi d\rho\,;$$

the last term vanishes by (13), and ψ is self-conjugate by (11), hence

$$\nabla\frac{dP}{dn} = \psi\nu = \frac{d}{dn}\nabla P, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

an equation which should be compared with (5) and with (12), from the former of which it may be deduced by applying (14).

7. We are now able to examine the criterion that the vector ν shall, besides being derivable from a scalar potential by means of a scalar factor, be derivable from one particular scalar potential which shall satisfy Laplace's equation; to find, in other words, the condition that a scalar t can be found such that $\nabla(t\nu) = \nabla^2 P = 0$.

Remembering that $\frac{dP}{dn} = t$, we shall have

$$\psi d\rho = d(t\nu)$$
$$= \nu dt + t d\nu$$
$$= -\nu S d\rho \nabla t + t\chi d\rho$$
$$= -\nu S d\rho \frac{d(t\nu)}{dn} + t\chi d\rho, \text{ by (16)},$$
$$= -\frac{dt}{dn}\nu S\nu d\rho - t\nu S\chi\nu d\rho + t\chi d\rho$$
$$= -\frac{dt}{dn}\nu S\nu d\rho - t\chi\nu S\nu d\rho + t\chi' d\rho,$$

where the last step follows because ψ is self-conjugate.

By inspection of this result it is evident that, upon any vector in the tangent plane, the strain ψ has the same effect as χ', with the sole difference that ψ allongates the vector by the factor t. There are important geometrical applications of this fact, some of which will be found in the examples at the end of this paper. But we are now concerned to get an expression for $\nabla(t\nu)$. It is proved above that

$$\psi d\rho = -\frac{dt}{dn}\nu S\nu d\rho - t\nu S\chi\nu d\rho + t\chi d\rho,$$

where, by Art. 3, the first two terms give $-\frac{dt}{dn}$ and $-t\nu\chi\nu$, and the last term gives $t\nabla\nu$. Thus if P satisfies Laplace's equation, then

$$-\frac{dt}{dn} - t\nu\chi\nu + t\nabla\nu = 0;$$

the vector part gives an independent proof of (15); the scalar part is

$$\frac{dt}{dn} = tS\nabla\nu,$$

and since it has already been proved that, in general,

$$\nabla t = \frac{d(t\nu)}{dn} = \nu\frac{dt}{dn} + t\chi\nu,$$

we have, provided P satisfies Laplace's equation,

$$\nabla t = t(\nu S\nabla\nu + \chi\nu). \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The vector $\nu S\nabla\nu + \chi\nu$ may be written $\nabla\nu \,.\, \nu$; and because $\nabla^2 t$ is a scalar,

$$\begin{aligned} V\nabla(t\nabla\nu \,.\, \nu) &= 0 \\ &= V \,.\, \nabla t(\nabla\nu \,.\, \nu) + tV\nabla(\nabla\nu \,.\, \nu); \end{aligned}$$

which reduces at once to

$$V\nabla(\nabla\nu \,.\, \nu) = 0;$$

from (10), putting $\nabla\nu$ for q and ν for r and taking vectors,

$$V(\nabla^2\nu)\nu - V(\nabla\nu)^2 + 2S\nabla\nu \,.\, V\nabla\nu - 2\chi V\nabla\nu = 0,$$

where the second and third terms destroy each other, so that finally

$$V\nu\nabla^2\nu + 2\chi V\nabla\nu = 0, \quad . \quad . \quad . \quad . \quad (18\,a)$$

which is the required condition.

The same essential fact is expressed by saying that $\nabla\nu \,.\, \nu$ must be integrable *without* a factor, or that there must exist a scalar—call it u—such that

$$u = \nabla^{-1}(\nabla\nu \,.\, \nu). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

8. To examine the properties of $\chi V\nabla\nu$, we may write, as a

special case of (8),

$$(\chi-\chi')\mathrm{V}\nabla\nu=0,$$

which means that χ and its conjugate have the same effect on $\mathrm{V}\nabla\nu$. But it was shown in Art. 6 that χ turns every vector into a certain plane, and χ' turns every vector into another plane; hence $\chi\mathrm{V}\nabla\nu$ lies along the line of intersection of these two planes.

If ν' be a unit-vector such that $\chi\nu'=0$, it follows that

$$\chi\mathrm{V}\nabla\nu=x\mathrm{V}\nu\nu';$$

to determine the unknown scalar x, take ϵ and η two unit-vectors such that $\chi\epsilon=g\epsilon$ and $\chi\eta=g_1\eta$; it may be easily shown that ϵ, η, and ν will then form a rectangular system (see Ex. 2); and they may be taken so that $\epsilon\eta=\nu$. It is then legitimate to write $\frac{d\nu}{dn}$ in the following form (Tait, § 176):

$$\mathrm{S}\nu\nu'\,.\,\chi\nu=-g\epsilon\mathrm{S}\epsilon\nu'-g_1\eta\mathrm{S}\eta\nu';$$

operate by $\mathrm{V}\nu$,

$$\mathrm{S}\nu\nu'\mathrm{V}\nabla\nu=-g\eta\mathrm{S}\epsilon\nu'+g_1\epsilon\mathrm{S}\eta\nu',$$

and by using again the same form of χ,

$$-\chi\mathrm{V}\nabla\nu=\frac{gg_1}{\mathrm{S}\nu\nu'}\{-\epsilon\mathrm{S}\eta\nu'+\eta\mathrm{S}\epsilon\nu'\}$$

$$=\frac{m_1}{\mathrm{S}\nu'\nu}\mathrm{V}\nu'\nu,$$

where m_1 is the coefficient of χ in the strain-cubic. Thus if l be the angle between ν and ν', the tensor of $\chi\mathrm{V}\nabla\nu$ is $m_1\tan l$.

9. If, further, σ be any vector in the tangent plane, so that at all points $\mathrm{S}\sigma\nu=0$, then by (8)

$$(\phi-\phi')\nu+\mathrm{V}\nu\mathrm{V}\nabla\sigma=0;$$

here $\phi\nu$ may be written $\frac{d\sigma}{dn}$; by (5) we obtain

$$\nabla\mathrm{S}\sigma\nu=0=-\chi'\sigma-\phi'\nu;$$

the values of $\phi\nu$ and $\phi'\nu$ give by substituting,—

$$\frac{d}{dn}+\chi'+\mathrm{V}\nu\mathrm{V}\nabla=0, \quad . \quad . \quad . \quad . \quad . \quad (20)$$

provided the operand be at right angles to ν.

Operating on $V\nabla\nu$ and substituting the result in (18*a*) gives

$$V\,.\,\nu\nabla S\nabla\nu=\left(\frac{d}{dn}-\chi\right)V\nabla\nu. \quad . \quad . \quad . \quad (18\,b)$$

Again, by using the value of $V\nabla\nu$ from (15),

$$\frac{d}{dn}\,V\nabla\nu=V\nu\frac{d^2\nu}{dn^2},$$

and this, combined with the result of the last article, gives

$$V\,.\,\nu\nabla S\nabla\nu=V\nu\left(\frac{d^2\nu}{dn^2}-m_1\nu' S^{-1}\nu'\nu\right). \quad . \quad . \quad (18\,c)$$

One other transformation is obtained from the $\chi\nu$ of the last article by putting $\omega\nu'=-g_1\eta S\eta\nu'-g\epsilon S\epsilon\nu'$, so that the component of ν' at right angles to ν is $S\nu'\nu\,.\,\omega^{-1}\chi\nu$, and this gives, by substituting in (18*c*),—

$$V\nu\left(\frac{d^2\nu}{dn^2}-m_1\omega^{-1}\frac{d\nu}{dn}-\nabla S\nabla\nu\right)=0\,; \quad . \quad . \quad (18\,d)$$

that is, the vector in parentheses is normal to the surface $P=\text{const.}$ Here it is noteworthy that both the vector $V\nu\nabla S\nabla\nu$ and the linear and vector function $m_1\omega^{-1}$ are numerically determinate all over a given surface $P=P_0$. Thus (18) shows the character of ν, provided $\nabla^2P=0$, in the immediate neighbourhood of the given surface.

If ν be so given as to satisfy (18), P may be written $\nabla^{-1}(t\nu)$, and is determined by (19), since $u=\log t$ by (17).

Examples.

1. Give in terms of χ the curvature of a normal section of the surface $P=\text{const.}$ (Tait, § 350, where ν is the $t\nu$ of this paper.)

2. Show that two of the roots of the cubic in χ correspond to the sections of greatest and least curvature.

3. If ν' correspond to the other root, show that if ν, ν' and $\chi\nu$ are coplanar, $\chi^2\nu$ is parallel to $\chi\nu$. Of what class of surfaces is this a property?

4. Show that if P is a homogeneous function of x, y, and z, any straight line through the origin cuts the surfaces denoted by P at a constant angle.

5. Show that if P is a homogeneous spherical harmonic, $\nabla^{-1}(\nabla\nu\,.\,\nu)=\text{const.}$ is the equation of a cone.

6. For what class of surfaces *may* ν' lie in the tangent plane?

7. For what class of surfaces is χ self-conjugate? (Tait, § 332).

8. Discuss the pure and the rotational parts of the strain χ.

9. Prove the identities:—

$$(a)\quad \chi^2\nu = \mathrm{V}\,.\,\nu\chi\mathrm{V}\nabla\nu - \mathrm{S}\nabla\nu\,.\,\chi\nu,$$

$$(b)\quad \chi'\chi\nu = \chi^2\nu - \nu(\chi\nu)^2.$$

10. Discuss the pure strain $\chi + \chi'$.

11. Interpret $\mathrm{TV}\nabla\nu$ and $\frac{d}{dn}\mathrm{UV}\nabla\nu$. (Tait, §§ 299, 300.)

12. Show by (11) of Art. 5 that ∇^2 is the negative of Laplace's operator.

13. Show that $\frac{d}{dn}$ and $\frac{d}{dh}$ are commutative when applied to P, h being parallel to $\mathrm{V}\nabla\nu$.

14. With everything as in Art. 4, prove

$$\nabla(\sigma\tau) = \nabla\sigma\,.\,\tau - \nabla\tau\,.\,\sigma - 2\theta'\sigma.$$

15. Show by (7) that $\mathrm{V}\nabla\chi\nu$ lies in the tangent plane.

16. Use (6) of the same article to find $\mathrm{V}\nabla(\nu\mathrm{S}\nabla\nu + \chi\nu)$.

LXVIII. *Some Experiments on Electric Waves in Short Wire Systems, and on the Specific Inductive Capacity of a Specimen of Glass.*

By J. A. Pollock, *Professor of Physics, and* O. U. Vonwiller, *Deas-Thomson Scholar in Physics in the University of Sydney**.

THE experiments described in the following paper include observations of the waves along free wires, and also of the vibrations in the two systems formed when the wires are bridged at various points. In the former case it is shown, that when the electrical vibrations of the wire system are forced, they are in that mode whose free period is near to that of the vibrator oscillation, and therefore the distance between the nodes along the wires does not vary continuously with change of the period of the condenser discharge In the other case an explanation is found which accounts for the varying heights of the maxima deflexions observed when the bridge is moved along the wires. A method for finding the specific inductive capacity of solid dielectrics with

* Communicated by the Authors.