$$=\frac{\sin a}{\sin \xi}\sin \xi \sin Ixz$$

$$=\frac{\sin a}{\sin \xi}\sin\left(\frac{\pi}{2}-Iy\right)$$

$$=\frac{\sin a}{\sin \xi}\cos \eta$$

$$=\cos a \tan \tfrac{1}{2}\theta \cos \eta\,;$$

therefore $\cos\alpha : \cos\beta : \cos\gamma = 1 : \tan\frac{1}{2}\theta\cos\zeta : \tan\frac{1}{2}\theta\cos\eta$

$$=1:\nu:\mu.$$

Now, projecting the coordinates x, y, z, and then X, Y, Z of any point P along OA, it follows that

$$x\cos\alpha+y\cos\beta+z\cos(\pi-\gamma)=X\cos\alpha+Y\cos(\pi-\beta)+Z\cos\gamma;$$

therefore $$x+\nu y-\mu z = X-\nu Y+\mu Z,$$

which is the first of equations (1).

The second and third equations can be obtained in like manner.

A Property of Skew Determinants. By M. J. M. Hill, M.A., D.Sc., F.R.S., Professor of Mathematics at University College, London. Received and read May 9th, 1895.

It has been shown by Professor Cayley that the orthogonal transformation could be expressed thus

$$\left.\begin{array}{l} x_1 = a_{1,1}y_1+a_{1,2}y_2+\ldots+a_{1,n}y_n \\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots \\ x_n = a_{n,1}y_1+a_{n,2}y_2+\ldots+a_{n,n}y_n \end{array}\right\}\;\ldots\ldots\ldots\ldots(1),$$

where $$a_{r,r}=\frac{2\beta_{r,r}-\Delta}{\Delta}\;\ldots\ldots\ldots\ldots(2),$$

$$a_{r,s}=\frac{2\beta_{r,s}}{\Delta}\;\ldots\ldots\ldots\ldots(3),$$

where Δ is the skew determinant

$$\begin{vmatrix} b_{1,1} & \dots & b_{1,n} \\ \dots & \dots & \dots \\ b_{n,1} & \dots & b_{n,n} \end{vmatrix}$$

in which $b_{r,s} = -b_{s,r} \quad r \neq s$
but $b_{r,r} = 1$ ....(4),

and where $\beta_{r,s}$ is the co-factor of $b_{r,s}$.

To see directly that this is orthogonal, it is necessary to show that

$$\beta_{1,r}^2 + \beta_{2,r}^2 + \dots + \beta_{n,r}^2 = \Delta\beta_{r,r} \quad \text{....................(5)},$$

and $$2(\beta_{1,r}\beta_{1,s} + \beta_{2,r}\beta_{2,s} + \dots + \beta_{n,r}\beta_{n,s}) = \Delta(\beta_{r,s} + \beta_{s,r}) \quad \text{......(6)}.$$

As the latter equation includes the former, it is sufficient to prove it.

Let $$\Delta^2 = \begin{vmatrix} c_{1,1} & \dots & c_{1,n} \\ \dots & \dots & \dots \\ c_{n,1} & \dots & c_{n,n} \end{vmatrix} \quad \text{..............................(7)},$$

where $$c_{r,s} = b_{r,1}b_{s,1} + b_{r,2}b_{s,2} + \dots + b_{r,n}b_{s,n} \quad \text{...............(8)}.$$

It should be noticed that $c_{r,s}$ contains two terms from the diagonal of Δ, viz., $b_{r,r}$ and $b_{s,s}$ which occur in the terms $b_{r,r}b_{s,r}$ and $b_{r,s}b_{s,s}$, whose sum is equal to

$$b_{s,r} + b_{r,s} = 0.$$

They may therefore be omitted or replaced by

$$b_{r,r}b_{r,s} + b_{s,r}b_{s,s},$$

which also vanishes.

Hence $$c_{r,s} = b_{1,r}b_{1,s} + b_{2,r}b_{2,s} + \dots + b_{n,r}b_{n,s} \quad \text{.................(9)}.$$

If $r = s$, $$c_{r,r} = b_{r,1}^2 + b_{r,2}^2 + \dots + b_{r,n}^2 = b_{1,r}^2 + b_{2,r}^2 + \dots + b_{n,r}^2.$$

Also $$c_{r,s} = c_{s,r}.$$

Multiply now the matrices

$$\begin{Vmatrix} b_{1,1} & \dots & b_{1,n} \\ \dots & \dots & \dots \\ b_{r-1,1} & \dots & b_{r-1,n} \\ b_{r+1,1} & \dots & b_{r+1,n} \\ \dots & \dots & \dots \\ b_{n,1} & \dots & b_{n,n} \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} b_{1,1} & \dots & b_{1,n} \\ \dots & \dots & \dots \\ b_{s-1,1} & \dots & b_{s-1,n} \\ b_{s+1,1} & \dots & b_{s+1,n} \\ \dots & \dots & \dots \\ b_{n,1} & \dots & b_{n,n} \end{Vmatrix}.$$

The product is

$$\begin{vmatrix} c_{1,1} & \dots & c_{1,s-1} & c_{1,s+1} & \dots & c_{1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{r-1,1} & \dots & c_{r-1,s-1} & c_{r-1,s+1} & \dots & c_{r-1,n} \\ c_{r+1,1} & \dots & c_{r+1,s-1} & c_{r+1,s+1} & \dots & c_{r+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{n,1} & \dots & c_{n,s-1} & c_{n,s+1} & \dots & c_{n,n} \end{vmatrix},$$

and is also $\quad (-1)^{r+s}(\beta_{r,1}\beta_{s,1}+\beta_{r,2}\beta_{s,2}+\dots+\beta_{r,n}\beta_{s,n}).$

Again, $\qquad \Delta\,.\,\beta_{r,s}$

$$= \begin{vmatrix} b_{1,1} & \dots & b_{1,n} \\ \vdots & \vdots & \vdots \\ b_{n,1} & \dots & b_{n,n} \end{vmatrix} \begin{vmatrix} b_{1,1} & \dots & b_{1,s-1} & b_{1,s} & b_{1,s+1} & \dots & b_{1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{r-1,1} & \dots & b_{r-1,s-1} & b_{r-1,s} & b_{r-1,s+1} & \dots & b_{r-1,n} \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ b_{r+1,1} & \dots & b_{r+1,s-1} & b_{r+1,s} & b_{r+1,s+1} & \dots & b_{r+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n,1} & \dots & b_{n,s-1} & b_{n,s} & b_{n,s+1} & \dots & b_{n,n} \end{vmatrix}$$

$$= \begin{vmatrix} c_{1,1} & \dots & c_{1,r-1} & b_{1,s} & c_{1,r+1} & \dots & c_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ c_{n,1} & \dots & c_{n,r-1} & b_{n,s} & c_{n,r+1} & \dots & c_{n,n} \end{vmatrix},$$

as is seen by multiplying rows by rows, and using (8).

Also $$\Delta\,.\,\beta_{s,r}$$

$$= \begin{vmatrix} b_{1,1} & \dots & b_{1,n} \\ \vdots & \vdots & \vdots \\ b_{n,1} & \dots & b_{n,n} \end{vmatrix} \begin{vmatrix} b_{1,1} & \dots & b_{1,r-1} & 0 & b_{1,r+1} & \dots & b_{1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{s-1,1} & \dots & b_{s-1,r-1} & 0 & b_{s-1,r+1} & \dots & b_{s-1,n} \\ b_{s,1} & \dots & b_{s,r-1} & 1 & b_{s,r+1} & \dots & b_{s,n} \\ b_{s+1,1} & \dots & b_{s+1,r-1} & 0 & b_{s+1,r+1} & \dots & b_{s+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n,1} & \dots & b_{n,r-1} & 0 & b_{n,r+1} & \dots & b_{n,n} \end{vmatrix}$$

$$= \begin{vmatrix} c_{1,1} & \dots & c_{1,r-1} & b_{s,1} & c_{1,r+1} & \dots & c_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n,1} & \dots & c_{n,r-1} & b_{s,n} & c_{n,r+1} & \dots & c_{n,n} \end{vmatrix},$$

as is seen by multiplying columns by columns, and using (9).

Therefore $$\Delta\,(\beta_{r,s}+\beta_{s,r})$$

$$= \begin{vmatrix} c_{1,1} & \dots & c_{1,r-1} & (b_{1,s}+b_{s,1}) & c_{1,r+1} & \dots & c_{1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{s,1} & \dots & c_{s,r-1} & (b_{s,s}+b_{s,s}) & c_{s,r+1} & \dots & c_{s,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{n,1} & \dots & c_{n,r-1} & (b_{n,s}+b_{s,n}) & c_{n,r+1} & \dots & c_{n,n} \end{vmatrix}$$

$$= (-1)^{r+s}\,.\,2 \begin{vmatrix} c_{1,1} & \dots & c_{1,r-1} & c_{1,r+1} & \dots & c_{1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{s-1,1} & \dots & c_{s-1,r-1} & c_{s-1,r+1} & \dots & c_{s-1,n} \\ c_{s+1,1} & \dots & c_{s+1,r-1} & c_{s+1,r+1} & \dots & c_{s+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{n,1} & \dots & c_{n,r-1} & c_{n,r+1} & \dots & c_{n,n} \end{vmatrix}$$

$$= 2\,(\beta_{r,1}\,\beta_{s,1}+\dots+\beta_{r,n}\,\beta_{s,n}).$$

Putting $r = s$, $$\beta_{r,1}^2+\beta_{r,2}^2+\dots+\beta_{r,n}^2 = \Delta\beta_{r,r},$$

and also by symmetry

$$2\,(\beta_{1,r}\,\beta_{1,s}+\beta_{2,r}\,\beta_{2,s}+\ldots+\beta_{n,r}\,\beta_{n,s}) = \Delta\,(\beta_{r,s}+\beta_{s,r}),$$

$$\beta_{1,r}^2+\beta_{2,r}^2+\ldots+\beta_{n,r}^2 = \Delta\beta_{r,r}.$$

Researches in the Calculus of Variations.—Part VI., The Theory of Discontinuous or Compounded Solutions. By E. P. CULVERWELL, M.A., Fellow of Trinity College, Dublin. Communicated (with certain additional Critical Remarks) May 10th, 1894.

In the following pages I hope to place the theory of discontinuous, or as they may more appropriately be called compounded, solutions in the calculus of variations on a satisfactory basis. The theory also leads to a rule for ascertaining whether the continuous solution given by the ordinary equations of the calculus is, or is not, the only possible solution.

1. Discontinuity presents itself in two ways in the calculus of variations :—

(1) There may be *stationary* solutions, which involve discontinuity of some fluxion of y at some point or points of the integral.

(2) There may be maximum or minimum solutions, which are not *stationary*, *i.e.*, solutions in which δU, as well as $\delta^2 U$, is capable of only one sign.

Solutions of this class appear to arise in two principal ways :—

(*a*) The region of integration may be restricted so that, along a certain boundary, δy is not capable of either sign. The restriction may either be *explicit*, as when we are asked to find the shortest *sea* line from one bay to another, or *implicit*, as in the case of *least* action, where the fact that the values of the variables must be *real* gives rise to a boundary. This class of problem has been sufficiently treated of by Mr. Todhunter in his Adams Prize Essay, entitled "Researches in the Calculus of Variations," and it will be unnecessary here to discuss it.