THE GENERALIZATION OF GAUSS'S THEOREM.

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WITH one partial exception, mentioned below, theorists have hitherto demonstrated Gauss's theorem, so far as my knowledge goes, only for the case of a single homogeneous isotropic medium filling all space, and have then (implicitly) assumed it to hold true in all cases. The object of this paper is to point out the omission just mentioned and to establish in a logical manner the generality of the theorem.

First, consider the theorem as applied to electrostatics. In this case it may be stated as follows: *The electric flux outward across any closed surface surrounding a charge* q (rational units) *is equal to* q*;* or, for a single tube of induction, *the electric flux across every diaphragm of a given tube of induction is the same and equal to the charge at its ends.*

1. When the medium in which the charges are situated consists of a single infinite homogeneous isotropic dielectric, the law of inverse squares holds good in its simplest form, and the theorem follows logically, as in the common demonstration.

2. Now, by altering the charges and their configuration in the infinite dielectric just considered, the form of any tube of induction may be altered in any manner without altering the flux along it, called for brevity its *strength.* That is, the strength of a tube of induction is independent of its form.

3. If now dielectrics of different permittivity from that of the original dielectric, or conductors, or both are introduced into the field, the form of the portions of the tubes traversing what is left of the original dielectric will be altered; but, as just shown, this cannot alter their strength. Hence Gauss's theorem is valid throughout *any* region which contains a single homogeneous isotropic dielectric.

The validity of the theorem for the very important particular case in which the field contains any number of conductors and a single homogeneous isotropic dielectric can also be demonstrated as follows: Consider any number of equipotential surfaces, charged or uncharged, but *enclosing* no charges in an infinite homogeneous, isotropic dielectric. The surfaces may have any shapes, and any charges consistent with their being equipotentials. Throughout the field including these surfaces Gauss's theorem holds, by § 1. When applied to the region enclosed within any of the equipotentials, the theorem shows that the electric intensity at every point in this region is zero. Hence the dielectric enclosed by any or all of the equipotentials can be replaced by any other dielectric or by a *conducting substance* without in any way affecting the external (and only) electric field or the validity of Gauss's theorem. This establishes the validity of the theorem for the case considered.

4. To establish the complete generality of the theorem for a region containing any number of homogeneous isotropic substances it is necessary to show in addition only that in passing from one dielectric into another a tube of induction does not alter in strength. To do this, consider the electric field between the plates A and B of a closed condenser containing two dielectrics, *1* and *2*, *1* being in contact with A only, and *2* in contact with B only. If the charge of A is q, that of B is $-q$, and there is no charge upon the interface between the dielectrics *1* and *2* (the charges due to contact being equal and opposite at any point). Applying Gauss's theorem to the region *1*, we find the total strength of all the tubes emanating from A to be q; and likewise, in the region *2*, we find the total strength of all the tubes terminating upon B to be q. Thus the total strength of all the tubes is unchanged in passing from A to B. And since this result is absolutely independent of the size or shape of the dielectrics or conductors, it follows that, however the field is divided up into tubes, the strength of *every* tube remains unaltered in crossing the interface.

5. By an obvious extension of what precedes, Gauss's theorem is seen to be valid also when the permittivity of the dielectric varies continuously from point to point instead of changing suddenly at distinct interfaces.

6. To extend the theorem to æolotropic media, we have only to remember that when the electric induction has the direction of any one of the three principal axes the phenomena are in all respects precisely the same as if the dielectric were isotropic. Hence for induction in these three directions Gauss's theorem holds good. But the induction at any point is always the vector sum of three such component inductions. Hence the theorem is valid for any direction of the induction.

In magnetostatics Gauss's theorem, equating the magnetic flux into the surrounding medium from a pole to the pole strength, is valid only for the ideal case of a concentrated pole, and then only when the equal flux *to* the pole through the magnet is neglected. For this ideal case the theorem may be generalized in almost the same manner as in electrostatics. In general the flux from a magnetic pole is very different from its strength.

The theorem is always assumed to be valid also in the general electromagnetic field, which may contain moving charges and poles and varying intensities. The justification of this assumption is of course the agreement of its important consequences with experiment.

In gravitation the ordinary proof of the theorem is general, since the gravitational "permittivity" or "inductivity" appears to be identical for all kinds of matter.

The very inadequate generalization of Gauss's theorem to which reference was made in the first paragraph above is contained in Pellat's *Électrostatique non Fondé sur les Lois de Coulomb, etc.*, §33.[1] The generalization of the theorem given in this paper is wholly independent of Pellat's work, and was largely developed long before I had seen his article.

[1] Ann. de Chim. et de Phys. (7), V., 1, 1895.