The Reduction of a Linear Substitution to its Canonical Form. By A. C. Dixon.

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The following is another solution of the problem discussed by Prof. Burnside (*Proc. Lond. Math. Soc.*, Vol. xxx., pp. 180–194), namely, the reduction of a linear substitution to its canonical form.

Take the substitution S as at p. 183, namely,

$$x'_s = \Sigma a_{st} x_t \quad (s, t = 1, 2, \ldots, n).$$

Let $\theta^{(1)}$ be a root of the characteristic equation

$$\Delta = \begin{vmatrix} a_{11}-\theta, & a_{12}, & a_{13}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}-\theta, & & \ldots, & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1}, & a_{n2}, & & \ldots, & a_{nn}-\theta \end{vmatrix} = 0.$$

Then at least one set of quantities

$$x_s^{(1)} \qquad (s = 1, 2, \ldots, n)$$

can be found such that

$$\Sigma_t a_{st} x_t^{(1)} = \theta^{(1)} x_s^{(1)} \quad (s = 1, 2, \ldots, n),$$

and at least one set $y_s^{(1)}$ such that

$$\Sigma_t a_{ts} y_t^{(1)} = \theta^{(1)} y_s^{(1)} \quad (s = 1, 2, \ldots, n).$$

Take one such set in each of the two cases, and write

$$x_1 = X_1 x_1^{(1)}, \quad x_s = X_1 x_s^{(1)} + X_s \quad (s = 2, \ldots, n).*$$

The equations of the substitution S then become

$$X'_1 x_1^{(1)} = X_1 \Sigma a_{1t} x_t^{(1)} + \sum_{t=2}^{t=n} a_{1t} X_t$$

$$= \theta^{(1)} x_1^{(1)} X_1 + \sum_2^n a_{1t} X_t,$$

* We may suppose $x_1^{(1)} \neq 0$; if this is not so, change the order of the variables x.

$$X_1' x_s^{(1)} + X_s' = X_1 \Sigma a_{st} x_t^{(1)} + \sum_2^n a_{st} X_t$$

$$= \theta^{(1)} x_s^{(1)} X_1 + \sum_2^n a_{st} X_t \quad (s = 2, 3, \ldots, n);$$

whence
$$X_s' = \sum_2^n \left(a_{st} - \frac{x_s^{(1)}}{x_1^{(1)}} a_{1t} \right) X_t.$$

It also follows that

$$X_1' \Sigma x_s^{(1)} y_s^{(1)} + \sum_2^n X_s' y_s^{(1)} = \theta^{(1)} X_1 \Sigma x_s^{(1)} y_s^{(1)} + \sum_{t=2}^{t=n} \{ X_t \sum_s a_{st} y_s^{(1)} \}$$

$$= \theta^{(1)} \left[X_1 \Sigma x_s^{(1)} y_s^{(1)} + \sum_2^n X_t y_t^{(1)} \right].$$

There are now two cases according as $\Sigma x_s^{(1)} y_s^{(1)} = 0$ or not. Suppose, first, $\Sigma x_s^{(1)} y_s^{(1)} \neq 0$, and write ξ_1 for $X_1 \Sigma x_s^{(1)} y_s^{(1)} + \sum_2^n X_t y_t^{(1)}$, that is, $\Sigma y_s^{(1)} x_s$, ξ_s for X_s $(s = 2, 3, \ldots, n)$. Thus

$$\xi_1' = \theta^{(1)} \xi_1,$$

$$\xi_s' = \sum_2^n \beta_{st} \xi_t \quad (s = 2, \ldots, n),$$

where
$$\beta_{st} = a_{st} - \frac{x_s^{(1)}}{x_1^{(1)}} a_{1t},$$

that is, the substitution is broken up into two, one of which affects only ξ_1, and the other only the other $n-1$ variables. The same process of reduction may then be applied to this last substitution and repeated until the canonical form is at last reached, unless at some stage it happens that

$$\Sigma x_s^{(1)} y_s^{(1)} = 0.$$

If
$$\Sigma x_s^{(1)} y_s^{(1)} = 0,$$

the functions $\xi_1, \xi_2, \ldots, \xi_n$ as just given are not linearly independent.

We may, however, put*

$$\xi_n = \Sigma y_s^{(1)} x_s = \sum_2^n X_t y_t^{(1)},$$

$$\xi_s = X_s \quad (s = 1, 2, \ldots, n-1).$$

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* It is assumed that $y_n^{(1)}$ does not vanish.

Thus
$$\xi'_n = \theta^{(1)}\xi_n,$$
$$\xi'_1 = \theta^{(1)}\xi_1 + \sum_2^n \frac{a_{1t}}{x_1^{(1)}}\xi_t,$$
$$\xi'_s = \sum_2^n \gamma_{st}\xi_t \quad (s = 2, 3, \ldots, n-1).$$

The substitution has thus been transformed,* so that in the characteristic determinant Δ the first column and last row contain zeros only, except where they meet the dexter (or principal) diagonal, and the two constituents at the ends of this are each $\theta^{(1)}-\theta$.

If we suppose $\xi_n = 0$, we have a linear substitution affecting $\xi_2, \xi_3, \ldots, \xi_{n-1}$ only for which the characteristic determinant is formed by striking out the first and last rows and columns of the one just arrived at. This substitution in $n-2$ variables can be further reduced by one of the two processes given, and the restoration of ξ_n will only affect the last column of the determinant Δ.

Thus by successive reduction the substitution is brought to a form in which all constituents of the determinant below the dexter diagonal vanish, and possibly some above it.

The next step is to destroy as many as possible of the constituents above this diagonal.

The substitution may now be written
$$x'_s = \sum_s^n a_{st}x_t.$$

If $a_{11} \neq a_{22}$, we may destroy a_{12} by putting
$$X_1 = x_1 + \frac{a_{12}}{a_{11}-a_{22}}x_2, \; X = x_s \; (s = 2, \ldots, n).$$

Thus
$$X'_1 = a_{11}x_1 + \sum_2^n \left(a_{1t} + \frac{a_{12}a_{2t}}{a_{11}-a_{22}}\right)x_t$$
$$= a_{11}X_1 + \sum_3^n \left(a_{1t} + \frac{a_{12}a_{2t}}{a_{11}-a_{22}}\right)X_t,$$
$$X'_s = \sum_s^n a_{st}X_t \quad (s = 2, \ldots, n).$$

In this way any constituent just above the diagonal may be destroyed,

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* By the substitution
$$\xi_1 = x_1/x_1^{(1)}, \quad \xi_s = x_s - x_1\frac{x_s^{(1)}}{x_1^{(1)}}, \quad \xi_n = \Sigma y_t^{(1)}x_t \quad (s = 2, 3, \ldots, n-1).$$

unless the adjacent constituents in the diagonal are identically equal, that is, we may put

$$a_{s,s+1} = 0,$$

unless $$a_{ss} = a_{s+1,s+1}.$$

In the same way, going on to the next line parallel to the diagonal we may reduce

$$a_{s,s+2} \text{ to } 0,$$

unless $$a_{ss} = a_{s+2,s+2},$$

and so on; so that ultimately we destroy

$$a_{st},$$

unless $$a_{ss} = a_{tt}.$$

Then, by changing the order of the variables, we may gather together all the constituents in the diagonal that are identically the same, and so divide the variables into sets

$$\begin{gathered} x_1,\ x_2,\ \ldots,\ x_p, \\ x_{p+1},\ x_{p+2},\ \ldots,\ x_{p+q}, \\ x_{p+q+1},\ \ldots,\ x_{p+q+r}, \\ \ldots\quad \ldots\quad \ldots\quad \ldots \end{gathered}$$

in such a way that each set is transformed independently of the others, that is,

$$\begin{aligned} x'_s &= \lambda x_s + \sum_{s+1}^{p} a_{st} x_t \qquad (s = 1, 2, \ldots, p), \\ x'_{p+s} &= \mu x_{p+s} + \sum_{p+s+1}^{p+q} a_{p+s,t} x_t \quad (s = 1, 2, \ldots, q). \\ &\ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots \end{aligned}$$

Each set may now be considered separately. Take, for instance, $x_1, x_2, \ldots, x_p$. The first step is to bring any zeros there may be in the line just above the diagonal to the lower end of that line, and, in fact, to arrange that the number of zeros immediately following the diagonal constituent in any row shall be at least as great as in the row above, unless all the constituents of both rows vanish except those in the diagonal. It is the same thing to say that, if a_{it} $(t > i)$ is the first constituent in the i^{th} row not to vanish after the i^{th}, then

$$a_{i-1,r} \neq 0,$$

for some value of $r < t$, but $> i-1$.

Suppose this not to be the case; then the equations of substitution include

$$x'_{i-1} = \lambda x_{i-1} + \sum_{t}^{p} a_{i-1,s} x_s,$$

$$x'_i = \lambda x_i + \sum_{t}^{p} a_{is} x_s.$$

Put
$$X_{i-1} = x_i, \quad X_i = x_{i-1} - \frac{a_{i-1,t}}{a_{it}} x_i,$$

$$X_s = x_s \quad (s \neq i-1, i).$$

This transformation will not affect the rows below the i^{th}, and it will reduce a_{is} to zero; by applying it successively, and always to the lowest pair of rows in which the desired condition is not fulfilled, we get the result. The transformation giving this result is not unique, and, in fact, the arrangement will not be interfered with by any transformation, such as

$$X_i = \beta x_i + \gamma x_j \quad (j > i, \ \beta \neq 0),$$

$$X_s = x_s \quad (s \neq i).$$

Suppose then, leaving out the diagonal, that the first constituent not vanishing in the first row is the a^{th}, in the a^{th} row the a'^{th}, in the a'^{th} row the a''^{th}, and so on. The series 1, a, a', a'', ... will come to an end, since it can contain no number $>p$.

Let b be the first number not included among 1, a, a', a'', ..., and let the first constituent not vanishing in the b^{th} row be the b'^{th}, in the b'^{th} row the b''^{th}, and so on.

Let c be the first number not included among 1, a, a', ..., b, b', b'', ..., and form with it another series c, c', c'', ..., and so on until the numbers 1, 2, ..., p are all exhausted.

Then the transformation

$$\begin{array}{ll} X_1 = x_1, & X_b = x_b, \\ X_a = x'_1 - \lambda x_1, & X_{b'} = x'_b - \lambda x_b, \\ X_{a'} = X'_a - \lambda X_a, & X_{b''} = X'_{b'} - \lambda X_{b'}, \\ X_{a''} = X'_{a'} - \lambda X_{a'}, & \\ \&c., & \end{array}$$

reduces the substitution for $x_1, x_2, \ldots, x_p$ to its canonical form, except for the order of suffixes.

As an example take the substitution S on p. 191,

$$\begin{aligned} x_1' &= -2x_1 - x_2 - x_3 + 3x_4 + 2x_5, \\ x_2' &= -4x_1 + x_2 - x_3 + 3x_4 + 2x_5, \\ x_3' &= x_1 + x_2 \quad - 3x_4 - 2x_5, \\ x_4' &= -4x_1 - 2x_2 - x_3 + 5x_4 + x_5, \\ x_5' &= 4x_1 + x_2 + x_3 - 3x_4. \end{aligned}$$

The characteristic equation is

$$(\theta+1)^2(\theta-2)^3 = 0.$$

Take $\theta^{(1)} = 2$; then the five equations to be satisfied by $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_5^{(1)}$ reduce to three only, namely,

$$\begin{aligned} -4x_1 - x_2 - x_3 + 3x_4 + 2x_5 &= 0, \\ x_1 + x_2 - 2x_3 - 3x_4 - 2x_5 &= 0, \\ -4x_1 - 2x_2 - x_3 + 3x_4 + x_5 &= 0. \end{aligned}$$

We may then put

$$x_1^{(1)} = -x_3^{(1)} = x_4^{(1)} = 1, \quad x_2^{(1)} = x_5^{(1)} = 0.$$

In like manner, the equations for $y_1^{(1)} \ldots$ reduce to three only,

$$y_3 = 0, \quad y_4 = 0, \quad y_1 + y_2 - y_5 = 0.$$

Now it is desirable that, if possible, $\Sigma x_s^{(1)} y_s^{(1)}$ should not vanish; we therefore put

$$y_1^{(1)} = 1 = y_5^{(1)}, \quad y_2 = y_3 = y_4 = 0.$$

The first transformation is therefore

$$X_1 = x_1 + x_5, \quad X_2 = x_2, \quad X_3 = x_3 + x_1, \quad X_4 = x_4 - x_1, \quad X_5 = x_5.$$

The substitution S as transformed is

$$\begin{aligned} x_1' &= 2x_1, \\ x_2' &= x_2 - x_3 + 3x_4 + 2x_5, \\ x_3' &= -x_3, \\ x_4' &= -x_2 \quad + 2x_4 - x_5, \\ x_5' &= x_2 + x_3 - 3x_4. \end{aligned}$$

We now ignore x_1 and take the root -1 of the characteristic equation. Then $x_2^{(1)}, x_3^{(1)}, x_4^{(1)}, x_5^{(1)}$ satisfy the equations

$$2x_2 - x_3 + 3x_4 + 2x_5 = 0,$$
$$-x_2 \quad + 3x_4 - x_5 = 0,$$
$$x_2 + x_3 - 3x_4 + x_5 = 0;$$

so that $\qquad x_2^{(1)} = -x_5^{(1)} = 1, \quad x_3^{(1)} = x_4^{(1)} = 0.$

In like manner, $\qquad y_3^{(1)} = 1, \quad y_2^{(1)} = y_4^{(1)} = y_5^{(1)} = 0.$

The new substitution is then

$$X_1 = x_1, \quad X_2 = x_2, \quad X_3 = x_4, \quad X_4 = x_2 + x_5, \quad X_5 = x_3,$$

and S becomes, when transformed by it,

$$\begin{aligned} x_1' &= 2x_1, \\ x_2' &= -x_2 + 3x_3 + 2x_4 - x_5, \\ x_3' &= 2x_3 - x_4, \\ x_4' &= 2x_4 \\ x_5' &= -x_5. \end{aligned}$$

To reduce this finally, put

$$X_1 = x_1, \quad X_5 = x_2 - x_3 - x_4, \quad X_4 = -x_5, \quad X_3 = x_3, \quad X_2 = -x_4,$$

and it becomes

$$x_1' = 2x_1, \quad x_2' = 2x_2, \quad x_3' = 2x_3 + x_2, \quad x_4' = -x_4, \quad x_5' = -x_5 + x_4,$$

which is the canonical form.

The successive transformations used are

$$X_1^{(1)} = x_1 + x_5, \; X_2^{(1)} = x_2, \quad X_3^{(1)} = x_1 + x_3, \; X_4^{(1)} = x_4 - x_1, \quad X_5^{(1)} = x_5,$$
$$X_1^{(2)} = X_1^{(1)}, \quad X_2^{(2)} = X_2^{(1)}, \; X_3^{(2)} = X_4^{(1)}, \quad X_4^{(2)} = X_2^{(1)} + X_5^{(1)}, \; X_5^{(2)} = X_3^{(1)},$$
$$X_1 = X_1^{(2)}, \quad X_2 = -X_4^{(2)}, \; X_3 = X_3^{(2)}, \quad X_4 = -X_5^{(2)}, \quad X_5 = X_2^{(2)} - X_3^{(2)} - X_4$$

the resultant transformation being

$$X_1 = x_1 + x_5, \quad X_2 = -x_2 - x_5, \quad X_3 = x_4 - x_1, \quad X_4 = -x_1 - x_3,$$
$$X_5 = x_1 - x_4 - x_5.$$