On a Class of Automorphic Functions. By W. BURNSIDE. Received and Communicated November 12th, 1891.

1. *Introductory.*

In a series of memoirs published in the first, third, fourth and fifth volumes of the *Acta Mathematica*, M. Poincaré has developed the theory of discontinuous groups of linear substitutions and of the one-valued functions which are unaltered by the substitutions of such a group.

In this paper I shall adhere as closely as possible to the notation and nomenclature used by M. Poincaré, and it will be convenient to state these at once as regards the substitutions.

A substitution which changes z into $\frac{\alpha z+\beta}{\gamma z+\delta}$, written $\left(z, \frac{\alpha z+\beta}{\gamma z+\delta}\right)$, is in its normal form when $\alpha\delta-\beta\gamma=1$, and the real part of α is positive.

Two points will always be unchanged by the substitution. If these are different, it can be written in the form

$$\frac{t-a}{t-b} = K\frac{z-a}{z-b},$$

where a, b are the unchanged (or double) points of the substitution.

When K is real, the substitution is called hyperbolic; when mod K is equal to 1, it is called elliptic, and in all other cases loxodromic. If the double points coincide, the substitution is called parabolic, and can be written in the form

$$\frac{1}{t-a} = \frac{1}{z-a}+A.$$

If $\alpha-1$, β, γ, $\delta-1$ are all infinitesimal, the substitution is called an infinitesimal substitution; and a group of substitutions is called discontinuous when it contains no infinitesimal substitution. The theory of groups of a finite number of different substitutions has been completely dealt with by, among others, Prof. Cayley and Prof. F. Klein, the most detailed accounts of it being contained in the memoir "On the Schwarzian Derivative and the Polyhedral Functions" by the former, and the "Vorlesungen über das Ikosaëder" of the latter. These groups M. Poincaré leaves on one side, as also those groups the substitutions of which cannot all be derived from a finite number of fundamental substitutions.

If all the substitutions of a group preserve one circle unchanged (for simplicity this circle can always be taken to be the axis of x), and transform the regions inside and outside this circle each into itself, M. Poincaré speaks of it as a fuchsian group; in the other case the group is called kleinian. His method of dealing with groups of substitutions is a geometrical one. He shows that, corresponding to a discontinuous group, a division of the z-plane, or part of the plane, into an infinite number of regions can be made, with the following properties. To each substitution $\left(z, \frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)$ of the group will correspond a particular region R_i of the plane, in such a way that when z is within R_0, $\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}$ will be within R_i. The regions R_i fill either the whole plane or the particular part of it exactly once, *i.e.*, they neither overlap each other nor leave uncovered portions. The region R_0 (called the generating polygon), and therefore also the other regions, can always be chosen so that their boundaries are arcs of circles, which, if the group is fuchsian, intersect the unchanged circle at right angles. The question as to whether the regions R_i will cover the whole plane or only a portion of it (it being understood that R_0 does not consist of two or more detached areas) will depend upon the nature of the fundamental substitutions of the group, and this difference leads to a division of the groups, and of the functions which are unchanged by them, into two classes*; namely, a first class, in which the regions R_i cover the whole plane, and the corresponding functions exist in the whole plane, and a second class, in which the functions have what is called a "natural limit." M. Poincaré considers cases of both these classes in his memoirs, but deals at considerably the greater length with the second class. Prof. Klein, in his investigations on the subject, which are contained in most detailed form in a memoir entitled "Neue Beiträge zur Riemann'schen Functionentheorie," (*Math. Ann.*, Band 21), limits himself expressly to the second class of groups.

The analytical expressions of the functions which are unaltered by the substitutions of a group are obtained in the following way by M. Poincaré. He first shows that if $\left(z, \frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)$ be any substitution of the group in its normal form, then the series

$$\Sigma \bmod (\gamma_i z+\delta_i)^{-2m},$$

* M. Poincaré distinguishes seven families of groups; of these the first, second, and sixth form what I have ventured to call the second class, while the rest make up the first class.

where m is a positive integer greater than unity, is an absolutely convergent series, except for values of z which make one or more terms of the series infinite.

It follows that the series

$$\sum_i H\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)(\gamma_i z+\delta_i)^{-2m},$$

where $H(z)$ represents a rational integral function, is uniformly convergent, except for particular values of z, and therefore that it defines a one-valued continuous function of z. Such a function M. Poincaré calls a theta-fuchsian or theta-kleinian function. Its fundamental property is shown to be the following, viz.: if

$$\Theta(z) = \sum_i H\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)(\gamma_i z+\delta_i)^{-2m},$$

then
$$\Theta\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) = (\gamma z+\delta)^{2m}\,\Theta(z),$$

$\left(z, \frac{\alpha z+\beta}{\gamma z+\delta}\right)$ being any substitution of the group.

If now $\Theta(z)$, $\Theta_1(z)$ are two such functions, formed with the same m but different functions $H(z)$, $H_1(z)$, then their ratio, if it is not a constant, will be a function which is unaltered by the substitutions of the group.

As regards the theorem that $\sum_i \operatorname{mod}(\gamma_i z+\delta_i)^{-2m}$ is a convergent series for integral values of m greater than unity, M. Poincaré does not, in his memoir on fuchsian functions, give any reasons for not dealing with the case $m=1$; but in a later memoir ("Sur les Groupes des Équations Linéaires," *Acta Math.*, Vol. IV., p. 308), he says: "Toujours dans le cas d'un groupe fuchsien, la série

$$\sum \operatorname{mod}(\gamma_i z+\delta_i)^{-2}$$

n'est pas convergent."

That this statement is not universally true, may be seen at once by considering the group arising from the repetition of a single hyperbolic or loxodromic substitution. This may be written in the form

$$\frac{t-a}{t-b} = K\frac{z-a}{z-b} \quad (\operatorname{mod} K > 1),$$

or
$$t = \frac{(bK-a)\,z-ab\,(K-1)}{(K-1)\,z-(aK-b)};$$

so that
$$\gamma_i z+\delta_i = \frac{(K^{\frac{1}{2}i}-K^{-\frac{1}{2}i})z-(aK^{\frac{1}{2}i}-bK^{-\frac{1}{2}i})}{b-a},$$

and the series in this case is

$$(b-a)^2\sum_{-\infty}^{\infty} \text{mod}\left(\frac{1}{K^{\frac{1}{2}i}(z-a)-K^{-\frac{1}{2}i}(z-b)}\right)^2,$$

which certainly is convergent.

M. Poincaré's statement with respect to the divergency of

$$\sum_i \text{mod}\,(\gamma_i z+\delta_i)^{-2},$$

then, clearly requires some limitation. I have endeavoured to show that, in the case of the first class of groups, this series is convergent, but at present I have not obtained a general proof. I shall offer two partial proofs of the convergency; one of which applies only to the case of fuchsian groups, and for that case is general, while the other will also apply to kleinian groups, but only when certain relations of inequality are satisfied. It follows, from the convergency of

$$\sum_i \text{mod}\,(\gamma_i z+\delta_i)^{-2},$$

that for the first class of groups, to which all my results are limited, functions of the form

$$\sum_i H\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)(\gamma_i z+\delta_i)^{-2}$$

exist; and the existence of these functions is shown to lead to a mode of formation of fuchsian* and kleinian functions (*i.e.*, functions which are unaltered by the substitutions of the groups) which is in a certain way simpler than M. Poincaré's, and which moreover directly connects these functions with an already well-explored branch of analysis.

I shall best explain the nature of the method aimed at, and the point of view taken, by means of a particular case, which is also the one I have treated in greatest detail. Suppose, then, that the generating polygon of the group is the space outside $2n$ circles, each of which is external to all the others. The n fundamental substitutions of the group, which must be hyperbolic or loxodromic if the group is discontinuous, will make these circles correspond in pairs; and if, by bending and stretching, without tearing, the corresponding points on the pairs of circles are brought to coincidence, the polygon will be

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* I have used the phrase "automorphic function," as introduced by Professor Klein, to denote generally any function which is unchanged by the substitutions of a discontinuous group, whatever be the nature of the group.

turned into a closed $(n+1)$-ply connected surface. On such a surface n everywhere-finite complex functions of position must exist, such that when a variable point describes any path on the surface, and returns to its original position, any one of these functions will only be altered by the addition of integral multiples of certain definite periods. [Considering the closed surface as a Riemann's surface extended in space, these functions are the n integrals of the first species upon it.] Also, in terms of n-ple θ-functions, with these n functions as their arguments, any rational algebraical function on the surface can be expressed.

Corresponding to these n integrals of the first species on the surface, there must be, on the z-plane, n independent functions which are everywhere finite (except at the singular points of the group, *i.e.*, the double points of its substitutions), and which increase by integral multiples of certain constant quantities when a variable point passes from a position z to any one of the corresponding positions $\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}$; or when it describes a closed path surrounding one or more of the circles. When once these functions are found, it must follow, from the correspondence between the z-plane and the before-constructed Riemann's surface, that any functions which are unaltered by the substitutions of the group must be capable of representation by means of multiple θ-functions, with these particular functions for their arguments. Following M. Poincaré's method, I form the everywhere-finite theta-fuchsian or theta-kleinian functions of the form

$$\Sigma H\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)(\gamma_i z+\delta_i)^{-2};$$

i.e., the functions which are only infinite at the singular points of the group; and it appears that there are n such functions, and that their integrals are the n required functions which on the z-plane have properties analogous to the n integrals of the first species on the Riemann's surface. I then show how to form functions which bear exactly the same relations to normal integrals of the second and third kinds, on the Riemann's surface, that the n functions just referred to bear to the integrals of the first kind. All the methods, then, by which uniform functions of position on the Riemann's surface are expressed in terms of the Abelian integrals on it, may be applied at once to the expression of automorphic functions by means of the three classes of functions which are the analogues of the integrals of the first, second, and third kinds.

2. *On the Convergency of* $\sum_i$ *mod* $(\gamma_i z+\delta_i)^{-2}$ *in certain cases.*

First Proof.—The following proof, which is modelled on the lines of M. Poincaré's first proof of the convergency of $\sum_i \text{mod}\,(\gamma_i z+\delta_i)^{-4}$, applies only to the case of fuchsian groups of the first class. I suppose that the real axis is taken for the unchanged circle, and that the point $z=\infty$ is not a singular point of the group. These suppositions involve no real loss of generality. Since, by supposition, the group is of the first class, the generating polygon may be taken to enclose that part of the real axis which includes the point at infinity. Describe any circle $PQAB$, such that the area bounded by it lies entirely within the generating polygon, and suppose that it meets the real axis in the points A and B. Let $P_iQ_iA_iB_i$ be the circle into which the substitution $\left(z, \dfrac{a_iz+\beta_i}{\gamma_iz+\delta_i}\right)$ transforms the circle $PQAB$, A_iB_i being the segment of the real axis corresponding to AB, and P_iQ_i the circular arc corresponding to PQ.

Then it follows that :—

(i.) ΣA_iB_i is finite, for it is less than that part of the real axis which lies outside the generating polygon ;

(ii.) that therefore the sum of the circumferences of all the circles is finite, for they all cut the real axis at the same angle ;

(iii.) and therefore that ΣP_iQ_i is finite.

The point $-\delta_i/\gamma_i$, being the homologue of the point at infinity in the substitution $\left(z, \dfrac{a_iz+\beta_i}{\gamma_iz+\delta_i}\right)$, is a point J_i, within the polygon R_i, and lying on the real axis. When z lies on the arc PQ, the ratio of the greatest to the least value of mod $(\gamma_iz+\delta_i)^{-2}$ is the square of the ratio of the distances between J_i and the furthest and nearest points from it on PQ. Now the points J_i all lie on a finite portion of the axis of x ; and therefore, if d_1 and d_2 be respectively the absolutely greatest and least distances of any of the points J_i from the arc PQ, the ratio of the greatest to the least value of mod $(\gamma_iz+\delta_i)^{-2}$, when z lies on the arc PQ, is less than $(d_1/d_2)^2$, whatever i may be.

If z is the point P, mod $(\gamma_iz+\delta_i)^{-2}$ is the ratio of an infinitesimal linear element at P_i to the corresponding element at P, and hence, when z is any point on the arc PQ,

$$\text{mod}\,(\gamma_iz+\delta_i)^{-2} < \left(\frac{d_1}{d_2}\right)^2 \frac{P_iQ_i}{PQ}\,;$$

therefore $$\sum_i \bmod (\gamma_i z+\delta_i)^{-2} < \left(\frac{d_1}{d_2}\right)^2 \frac{1}{PQ} \Sigma P_i Q_i;$$

or, since $\Sigma P_i Q_i$ is finite, the series is absolutely convergent.

Second Proof.—The proof just given applies only to fuchsian groups; depending directly, as it does, on the fact that there is a circle (or straight line) which remains unchanged by all the substitutions. I now go on to give a proof which will apply, when certain relations of inequality are satisfied, to kleinian groups similar to the one referred to in the introduction.

Suppose, then, that the generating polygon is the space outside $2n$ given circles, each of which is external to all the others. The fundamental substitutions of the group will be n substitutions, which make the circles correspond in pairs, and which, since the circles are external to each other, and the group is discontinuous, are necessarily either hyperbolic or loxodromic. The various points $-\delta_i/\gamma_i$ are all inside the given circles, and when z does not coincide with one of these points, a quantity M may be found which is less than $\bmod \left(z+\frac{\delta_i}{\gamma_i}\right)$, whatever i may be.

It follows that

$$\sum_i \bmod (\gamma_i z+\delta_i)^{-2} < \sum_i M^{-2} \bmod \gamma_i^{-2};$$

and therefore that it is sufficient to consider the series $\sum_i \bmod \gamma_i^{-2}$.

For shortness of statement I deal with the case $n=2$, but it will be clear that the same reasoning will apply, whatever be the value of n.

Let the S, S' be the two fundamental substitutions of the group.

All the substitutions of the group may then be arranged in the following way, viz.:—

$S,\ S',\ S^{-1},\ S'^{-1}$, 4 substitutions involving one operation;

$SS,\ SS',\ SS'^{-1}:\ S^{-1}S^{-1},\ S^{-1}S'^{-1},\ S^{-1}S':$

$S'S',\ S'S,\ S'S^{-1}:\ S'^{-1}S'^{-1},\ S'^{-1}S^{-1},\ S'^{-1}S:$

4×3 substitutions involving two operations;

then 4×3^2 substitutions involving three operations; and so on.

Now, since the fundamental substitutions are necessarily either hyperbolic or loxodromic, they will be of the forms

$$S \equiv \left(z, \frac{\alpha z+\beta}{\gamma z+\delta}\right), \quad \gamma = \frac{K^{\frac{1}{2}}-K^{-\frac{1}{2}}}{b-a}, \quad \delta = \frac{bK^{-\frac{1}{2}}-aK^{\frac{1}{2}}}{b-a};$$

$$S' \equiv \left(z, \frac{\alpha' z+\beta'}{\gamma' z+\delta'}\right), \quad \gamma' = \frac{K'^{\frac{1}{2}}-K'^{-\frac{1}{2}}}{b'-a'}, \quad \delta' = \frac{b'K'^{-\frac{1}{2}}-a'K'^{\frac{1}{2}}}{b'-a'}.$$

Where mod K and mod K' are both greater than unity.

If
$$S_n \equiv \left(z, \frac{\alpha_n z+\beta_n}{\gamma_n z+\delta_n}\right)$$

is any substitution of the n^{th} set, then

$$SS_n \equiv \left(z, \frac{\alpha_{n+1} z+\beta_{n+1}}{\gamma_{n+1} z+\delta_{n+1}}\right),$$

where
$$\alpha_{n+1} = \alpha\alpha_n+\beta\gamma_n, \quad \beta_{n+1} = \alpha\beta_n+\beta\delta_n,$$
$$\gamma_{n+1} = \gamma\alpha_n+\delta\gamma_n, \quad \delta_{n+1} = \gamma\beta_n+\delta\delta_n,$$

and
$$\frac{\gamma_{n+1}}{\gamma_n} = \gamma\frac{\alpha_n}{\gamma_n}+\delta = K^{\frac{1}{2}}\frac{\frac{\alpha_n}{\gamma_n}-a}{b-a}+K^{-\frac{1}{2}}\frac{b-\frac{\alpha_n}{\gamma_n}}{b-a}.$$

If SS_n is a substitution of the $\overline{n+1}^{\text{th}}$ set, the last substitution in S_n must *not* be S^{-1}; and I shall now show that this involves that $\frac{\alpha_n}{\gamma_n}$ and a are not within the same one of the four circles bounding the generating polygon.

Call the circles A, B, A', B', the four points a, b, a', b' being supposed inside the circles with the same letters. Then S changes the outside of A into the inside of B; S' changes the outside of A' into the inside of B'; and S'^{-1} changes the outside of B' into the inside of A'. Hence, if α_n/γ_n is inside A, S must be the last substitution of S_n, and (writing $S_n = SS_{n-1}$) $\alpha_{n-1}/\gamma_{n-1}$ must be inside A. The same reasoning may be repeated continually; but the point α/γ is inside B, and therefore the point α_n/γ_n cannot be inside A, and must be inside either B, A' or B'. A lower limit, independent of n, can therefore be found for the modulus of each of the fractions

$$\frac{\frac{\alpha_n}{\gamma_n}-a}{b-a}$$

which occur in the various ratios γ_{n+1}/γ_n; and, therefore, if mod K

and mod K' are sufficiently great as compared with these limits, the inequality

$$\text{mod}\,\frac{\gamma_{n+1}}{\gamma_n} > k,$$

where k is an arbitrarily chosen quantity, can be satisfied, whichever of the $\overline{n+1}^{\text{th}}$ set of the substitutions S_{n+1} may be, and also whatever n may be.

Hence, if $\bar{\gamma}$ be the greater of the two quantities γ and γ',

$$\Sigma\,\gamma_i^{-2} < \bar{\gamma}^{-2}\left[4+\frac{4\,.\,3}{k^2}+\frac{4\,.\,3^2}{k^4}+\ldots\right];$$

or the series is convergent if

$$k^2 > 3.$$

When the double points of the fundamental substitutions are given, the series in question can clearly be made convergent by taking the moduli of the multipliers sufficiently great; but, again, when the moduli of the multipliers are sufficiently great, the group given by the substitutions is certainly a discontinuous one. Hence this second proof certainly establishes the existence of kleinian groups of the kind considered, for which $\sum_i \text{mod}\,(\gamma_i z+\delta_i)^{-2}$ is a convergent series, though it does not prove that the series is convergent for every such kleinian group.

If one or more of the fundamental substitutions become parabolic, in such a way that the corresponding pairs of circles touch each other externally, the preceding proof may very simply be modified to show again that there exist, in such a case, kleinian groups for which the series in question is convergent; but it does not seem to me to be possible to apply it to the cases in which the groups contain elliptic substitutions. Since, however, the first proof applies to the case of fuchsian groups containing elliptic substitutions, and, by a suitable infinitesimal change of the parameters of the fundamental substitutions, such a group will become kleinian, it would appear very probable that there must also be kleinian groups of the first class containing elliptic substitutions for which $\Sigma\,\text{mod}\,(\gamma_i z+\delta_i)^{-2}$ is convergent. By limiting the investigation of this section to the series $\Sigma\,\text{mod}\,(\gamma_i z+\delta_i)^{-2}$, I do not intend to imply that the series $\sum_i \text{mod}\,(\gamma_i z+\delta_i)^{-1}$ is necessarily divergent. Indeed, for the group formed by a single hyperbolic or loxodromic substitution, it is clearly convergent; and the second proof just given, for the convergency of $\sum_i \text{mod}\,(\gamma_i z+\delta_i)^{-2}$, may evidently be altered to show the convergence of $\sum_i \text{mod}\,(\gamma_i z+\delta_i)^{-1}$ for certain groups derived from more than one fundamental substitution. It is not necessary, however, in the sequel, to consider functions whose existence depends on the convergency of this latter series.

3. *On the Functions $\theta(z, a)$ for Groups of the First Class.*

Having established the convergency of the series $\sum_i \text{mod}\,(\gamma_i z+\delta_i)^{-2}$ for all fuchsian and certain kleinian groups of the first class, I now propose to consider the properties of the corresponding theta-fuchsian and theta-kleinian series. In the first place, I deal with such a group as that described in the introduction; that is to say, a group derived from n fundamental, hyperbolic or loxodromic, substitutions. For such a group it follows that the series

$$\sum_i H\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)(\gamma_i z+\delta_i)^{-2}$$

is a uniformly convergent series, except for certain particular values of z, and therefore that it defines a one-valued continuous function.

It will be convenient to use a special symbol for this series in the case when $H(z)$ is a function of the first degree, and for this purpose I define $\theta(z, a)$ by the equation

$$\theta(z, a) = \sum_i \frac{(\gamma_i z+\delta_i)^{-2}}{\dfrac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}-a}.$$

If $\left(z, \dfrac{\alpha z+\beta}{\gamma z+\delta}\right)$ is any particular substitution of the group,

$$\theta\left(\frac{\alpha z+\beta}{\gamma z+\delta}, a\right) = \sum_i \frac{\left[\gamma_i(\alpha z+\beta)+\delta_i(\gamma z+\delta)\right]^{-2}}{\dfrac{\alpha_i(\alpha z+\beta)+\beta_i(\gamma z+\delta)}{\gamma_i(\alpha z+\beta)+\delta_i(\gamma z+\delta)}-a}(\gamma z+\delta)^2$$

$$= (\gamma z+\delta)^2 \sum_j \frac{(\gamma_j z+\delta_j)^{-2}}{\dfrac{\alpha_j z+\beta_j}{\gamma_j z+\delta^j}-a},$$

where
$$\left.\begin{aligned} a_j &= aa_i+\gamma\beta_i \\ \beta_j &= \beta a_i+\delta\beta_i \\ \gamma_j &= a\gamma_i+\gamma\delta_i \\ \delta_j &= \beta\gamma_i+\delta\delta_i \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

are again the coefficients of any substitution of the group.

Therefore
$$\theta\left(\frac{az+\beta}{\gamma z+\delta}, a\right) = (\gamma z+\delta)^2\,\theta(z, a),$$

which is M. Poincaré's general property of all such functions.

The function $\theta(z, a)$ has two simple infinities inside any one of the regions R_i, namely, the homologues of a and ∞ that lie inside R_i. The same statement is true of $\theta\left(z, \frac{aa+\beta}{\gamma a+\delta}\right)$, $\left(z, \frac{az+\beta}{\gamma z+\delta}\right)$ being again any particular substitution of the group. It follows that constant multiples of the two functions can be chosen such that their difference shall be independent of a.

Now
$$\theta\left(z, \frac{aa+\beta}{\gamma a+\delta}\right) = \sum_i \frac{(\gamma a+\delta)(\gamma_i z+\delta_i)^{-1}}{(a_i z+\beta_i)(\gamma a+\delta)-(\gamma_i z+\delta_i)(aa+\beta)};$$

put
$$\delta a_i-\beta\gamma_i = a_j,\quad a\gamma_i-\gamma a_i = \gamma_j,$$
$$\delta\beta_i-\beta\delta_i = \beta_j,\quad a\delta_i-\gamma\beta_i = \delta_j,$$

so that $a_j, \beta_j, \gamma_j, \delta_j$ are the coefficients of any substitution of the group, and
$$a_i = aa_j+\beta\gamma_j,\quad \gamma_i = \gamma a_j+\delta\gamma_j,$$
$$\beta_i = a\beta_j+\beta\delta_j,\quad \delta_i = \gamma\beta_j+\delta\delta_j.$$

Then
$$\theta\left(z, \frac{aa+\beta}{\gamma a+\delta}\right) = \sum_j \frac{(\gamma a+\delta)\left[\gamma(a_j z+\beta_j)+\delta(\gamma_j z+\delta_j)\right]^{-1}}{a_j z+\beta_j-a(\gamma_j z+\delta_j)}$$
$$= \left(a+\frac{\delta}{\gamma}\right)\sum_j \frac{(\gamma_j z+\delta_j)^{-2}}{\left(\frac{a_j z+\beta_j}{\gamma_j z+\delta_j}-a\right)\left(\frac{a_j z+\beta_j}{\gamma_j z+\delta_j}+\frac{\delta}{\gamma}\right)}$$
$$= \theta(z, a)-\theta\left(z, -\frac{\delta}{\gamma}\right),$$

or
$$\theta(z, a)-\theta\left(z, \frac{aa+\beta}{\gamma a+\delta}\right) = \theta\left(z, -\frac{\delta}{\gamma}\right),$$

where the right-hand side is independent of a.

The function $\theta\left(z, -\frac{\delta}{\gamma}\right)$, thus introduced, has no poles, its only singular points being the singular points of the group. For

$$\theta\left(z, -\frac{\delta}{\gamma}\right) = \sum_i \frac{\gamma}{(\gamma_i z+\delta_i)\left[\gamma(a_i z+\beta_i)+\delta(\gamma_i z+\delta_i)\right]}$$

$$= \sum_i \left[\frac{\gamma a_i+\delta\gamma_i}{(\gamma a_i+\delta\gamma_i)z+\gamma\beta_i+\delta\delta_i} - \frac{\gamma_i}{\gamma_i z+\delta_i}\right];$$

and in this form it is clear that the residue of the function for $z = -\delta/\gamma$ is zero.

The point $-\delta/\gamma$ is one of the homologues of infinity. Denoting any one, a_i/γ_i, of these by J_i, the function $\theta(z, J_i)$ is finite everywhere, except at the singular points of the group. There are clearly an infinite number of such functions, but, as I shall next show, only n of them are linearly independent. Thus

$$\theta(z, J_p)-\theta(z, J_q) = \theta\left(z, \frac{a_p}{\gamma_p}\right)-\theta\left(z, \frac{a_q}{\gamma_q}\right)$$

$$= \sum \frac{a_p\gamma_q - a_q\gamma_p}{\left[\gamma_p(a_i z+\beta_i)-a_p(\gamma_i z+\delta_i)\right]\left[\gamma_q(a_i z+\beta_i)-a_q(\gamma_i z+\delta_i)\right]};$$

and writing, as before,

$$\delta_p a_i-\beta_p\gamma_i = a_j, \quad a_p\gamma_i-\gamma_p a_i = \gamma_j,$$

$$\delta_p\beta_i-\beta_p\delta_i = \beta_j, \quad a_p\delta_i-\gamma_p\beta_i = \delta_j,$$

$$\theta(z, J_p)-\theta(z, J_q)$$

$$= -\sum_j \frac{a_p\gamma_q-a_q\gamma_p}{(\gamma_j z+\delta_j)\left[(a_p\gamma_q-a_q\gamma_p)(a_j z+\beta_j)+(\beta_p\gamma_q-\delta_p a_q)(\gamma_j z+\delta')\right]}$$

$$= -\sum_j \frac{(\gamma_j z+\delta_j)^{-2}}{\dfrac{a_j z+\beta_j}{\gamma_j z+\delta_j} - \dfrac{\delta_p a_q-\beta_p\gamma_q}{-\gamma_p a_q+a_p\gamma_q}}$$

$$= -\theta\left(z, \frac{\delta_p a_q-\beta_p\gamma_q}{-\gamma_p a_q+a_p\gamma^q}\right)$$

$$= -\theta(z, J_{p^{-1}q}).$$

The same process will give

$$\theta(z, J_p) = -\theta(z, J_{p^{-1}}).$$

Hence, if S_p, S_q are two of the fundamental substitutions, and J the homologue of infinity formed by any combination of these two, then $\theta(z, J)$ can be expressed in the form

$$m\theta(z, J_p) + n\theta(z, J_q),$$

where m and n are positive or negative integers.

It follows from this at once that if $S_1, S_2, \ldots S_n$ are the n fundamental substitutions, then, whatever homologue of infinity J may be, $\theta(z, J)$ can always be expressed in the form

$$m_1\theta(z, J_1) + m_2\theta(z, J_2) + \ldots + m_n\theta(z, J_n),$$

where $m_1, m_2, \ldots m_n$ are positive or negative integers.

Thus, as stated, not more than n of these functions are linearly independent. That the n functions written above are really independent will be proved later on.

4. *On the Integrals of the Functions $\theta(z, J)$.*

The functions $\theta(z, J)$, being everywhere finite and continuous, except at the singular points of the group, can be integrated along any path which avoids these singular points, and their integrals along any such paths of finite length will be finite.

Thus
$$\int_{z_0}^{z} \theta(z, J)\, dz = \sum_i \log \frac{\dfrac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i} - J}{\dfrac{\alpha_i z_0 + \beta_i}{\gamma_i z_0 + \delta_i} - J}$$
$$= \log \prod_i \frac{\dfrac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i} - J}{\dfrac{\alpha_i z_0 + \beta^i}{\gamma_i z_0 + \delta_i} - J}$$

will represent, when the path from z_0 to z is assigned (keeping clear of the singular points), a definite finite quantity, and the given path will determine the branch of the logarithm that is to be taken.

I first consider the integral when the path is confined to the generating polygon; the relation between which and the fundamental substitutions is to be taken as follows.

The generating polygon consists of the space outside n pairs of circles C_1 and C_1', C_2 and C_2', ... C_n and C_n', each external to all the others. The substitution S_p transforms the circle C_p' into the circle C_p, and the space outside the circle C_p' into the space inside the circle C_p, so that the point J_p is inside C_p, and the point J_{p-1} is inside C_p'.

An element of the integral of $\theta(z, J_p)$ may be expressed as follows:

$$\theta(z, J_p)\,dz = \sum_i d\,.\log\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i} - J_p\right)$$

$$= \sum_i d\,.\log \frac{\alpha_i\left(z - \dfrac{\delta_i J_p - \beta_i}{-\gamma_i J_p + \alpha_i}\right)}{\gamma_i\left(z + \dfrac{\delta_i}{\gamma_i}\right)}$$

$$= \sum d\,.\log \frac{\alpha_i(z - J_{i-1\,p})}{\gamma_i(z - J_{i-1})}.$$

When S_i is the identical substitution (z, z), the corresponding term on the right-hand side is of a slightly different form, namely, $d\,.\log(z-J_p)$; hence

$$\theta(z, J_p)\,dz = d\,.\log(z-J_p) + \sum_i d\,.\log\frac{\alpha_i(z-J_{i-1\,p})}{\gamma_i(z-J_{i-1})},$$

where now the identical substitution is not included under the sign of summation.

Now, as before stated, J_p is inside C_p, and it is easy to see that J_{i-1} and $J_{i-1\,p}$ are either both inside or both outside C_p; hence

$$\int_{C_p} \theta(z, J_p)\,dz = 2\pi i,$$

where the integral is taken in the positive direction round a closed curve within the generating polygon surrounding C_p once, and surrounding no other circle.

Again, J_p is outside C'_p, and J_{i-1}, $J_{i-1\,p}$ are either both inside or both outside C'_p, except when $i = p$, and then J_{i-1} is inside and $J_{i-1\,p}$ is outside; hence

$$\int_{C'_p} \theta(z, J_p) = -2\pi i,$$

where the integral is taken positively round C'_p.

For any other circle C_q or C'_q, J_p is outside and J_{i-1} and $J_{i-1\,p}$ are always either both inside or both outside.

Hence $$\int_{C_q} \theta(z, J_p) = \int_{C'_q} \theta(z, J_p) = 0.$$

These results prove incidentally that the n functions

$$\theta(z, J_1),\quad \theta(z, J_2)\ \ldots\ldots\ \theta(z, J_n)$$

are linearly independent of each other, and that they are not mere constants.

If A_p, A'_p be two corresponding points on C_p and C'_p, and B_p, B'_p any other pair of corresponding points on the same two circles, and if $A_pA'_p$, $B_pB'_p$ be joined by paths $A_pMA'_p$, $B_pNB'_p$, which do not enclose any of the circles, then

$$\int \theta(z, J)\, dz,$$

round $A_pMA'_pB'_pNB_pA_p$, vanishes, since the integrand is finite and continuous at all points within the contour. Therefore

$$\left(\int_{A_pMA'_p} - \int_{B_pNB'_p} + \int_{A'_pB'_p} - \int_{A_pB_p}\right) \theta(z, J)\, dz = 0.$$

Now, if z_1, z_2 are corresponding points on the circular arcs $A'_pB'_p$, A_pB_p,

$$z_2 = \frac{\alpha_p z_1 + \beta_p}{\gamma_p z_1 + \delta_p};$$

therefore $$\theta(z_2, J) = (\gamma_p z_1 + \delta_p)^2\, \theta(z_1, J),$$

and $$dz_2 = (\gamma_p z_1 + \delta_p)^{-2}\, dz_1;$$

therefore $$\theta(z_2, J)\, dz_2 = \theta(z_1, J)\, dz_1.$$

It follows that

$$\int_{A_pB_p} \theta(z, J)\, dz = \int_{A'_pB'_p} \theta(z, J)\, dz,$$

the integrals being taken along the circular arcs, and therefore

$$\int_{B_pNB'_p} \theta(z, J)\, dz = \int_{A_pMA'_p} \theta(z, J)\, dz.$$

Hence the integral

$$\int_{z_0}^{(\alpha_p z + \beta_p)/(\gamma_p z_0 + \delta_p)} \theta(z, J)\, dz,$$

where z_0 is any point on the circle C'_p, is independent of the particular position of the point, the path varying continuously without passing outside the fundamental polygon.

Now let the points $A_1A'_1$, $A_2A'_2$, &c. ... be joined by lines of any form, which neither cut themselves nor each other, and which do not leave the generating polygons; and regard these lines, as well as the circles, as part of the boundary of the polygon. In the figure so formed the integral of $\theta(z, J)$, when the lower limit has an assigned value, is a one-valued finite continuous function.

Let
$$\int_{A'_p A_p} \theta(z, J_q)\, dz = a_{qp},$$
and
$$\int_{A'_p A_p} \theta(z, J_p)\, dz = a_{pp},$$
so that the quantities a_{pp}, a_{pq}, &c., are the constant values of the integrals just discussed. When the variable paths between the corresponding points are chosen so as to be reconcilable with the barriers $A_1A'_1$, $A_2A'_2$ ..., I shall show that
$$a_{pq} = a_{qp}.$$

For this purpose, let definite lower limits be chosen for the integrals (they need not yet be expressed), and write
$$\phi_p = \int^z \theta(z, J_p)\, dz.$$
Then, as stated, within the polygon with the barriers,
$$\phi_1, \phi_2 \dots \phi_n$$
are everywhere one-valued, finite and continuous.

If the positive side of the barrier A'_pA_p is that which one would first reach by starting from any point of the circle C_p and moving round it in the direction opposite to that of the hands of a watch, then the value of ϕ_p at any point on the positive side of A'_pA_p is greater by $2\pi i$ than its value at the corresponding point on the left-hand side; while the values of ϕ_q at corresponding points on the right and left-hand sides are the same.

Since ϕ_p, ϕ_q are finite and continuous everywhere inside the polygon with barriers, it follows that
$$\int \phi_p d\phi_q,$$
extended round the complete boundary, is zero.

The complete boundary consists of the n contours made up of the n pairs of circles and the two sides of the barrier connecting each pair.

The values ϕ_p, $d\phi_q$, at corresponding points on C_r, C'_r, are given by
$$\phi_p(C_r) = a_{pr} + \phi_p(C'_r),$$
$$d\phi_q(C_r) = -d\phi_q(C'_r),$$

and, at corresponding points on either side of $A'_r A_r$, ϕ_p and $d\phi_q$ have the same values. Hence the integral round the contour formed of these lines is

$$a_{pr}\int_{C'_r} d\phi_q = 0.$$

The contour formed by C_p, C'_p and $A'_p A_p$ gives

$$-2\pi i\int_{A'_p A_p} d\phi_q = -2\pi i\, a_{qp},$$

and the contour C_q, C'_q and $A'_q A_q$ gives

$$-a_{pq}\int_{C'_p} d\phi_q = 2\pi i\, a_{pq}.$$

Hence, finally, adding the various parts of the integral,

$$a_{pq}-a_{qp} = 0.$$

This proof will be seen to be the same as Prof. C. Neumann's proof of the corresponding property of Abelian integrals of the first species.

Suppose, now, that the path of integration passes outside the generating polygon, and let the homologues of the barriers be drawn for all the polygons. Then it is at once obvious that, inside any polygon with its barriers, the integrals are one-valued, finite and continuous, and also that they are continuous in crossing from one polygon to another. They are therefore one-valued and finite in the infinite plane, and continuous except at the barriers, and at corresponding points on the opposite sides of the barriers they have constant differences $\pm 2\pi i$ or zero. The value of the integrals at any point in the infinite plane, with the barriers as boundaries, may be determined as follows:—

Let O be any point in the generating polygon R_0 and O_i its homologue in R_i. It has just been shown that

$$\phi_p(O_i)-\phi_p(O),$$

is independent of the shape of the path between O and O_i, so long as it does not cut the barriers. Join OO_i by any path whatever satisfying this condition, and suppose that it passes successively through the polygons $R_0, R_a, R_b \ldots R_l, R_i$. Let O_0 be the point where it meets the boundary between R_0 and R_a; find O_a on the boundary between R_a and R_b, so that O_a is a homologue of O_0; then O_b on the boundary between R_b and R_c so that it is a homologue of O_a; and so on. Take now $OO_0O_aO_b \dots O_lO_i$ for the path of integration. The points O and O_i, O_0 and O_l are respectively homologues, and the paths OO_0, O_lO_i lie each entirely within one polygon. Therefore

$$\int_{OO_0} d\phi_p = \int_{O_iO_l} d\phi_p,$$

and the contributions to the whole value of the integral from OO_0 and O_lO_i destroy each other. Each of the portions O_0O_a, $O_aO_b \dots$ is reconcilable with the homologue of one of the original barriers $A_1'A_1$, $A_2'A_2 \dots$, taken either positively or negatively. Hence, finally, if among these homologues that of $A_1'A_1$ occurs n_1 times, that of $A_2'A_2$ n_2 times, and so on,

$$\phi_p\left(\frac{a_i z+\beta_i}{\gamma_i z+\delta_i}\right)-\phi_p(z) = n_1 a_{p1}+n_2 a_{p2}+\dots+n_n a_{pn};$$

the positive or negative integers here are definite, depending only on the particular polygon R_i; the suffix p may have any value from 1 to n.

When, finally, the barriers are dispensed with, and the now many-valued integrals are considered in the infinite plane bounded only by the singular points, any path of integration whatever can be made up by combining the path just considered with a number of closed loops. Each time such a loop cuts a barrier, $\pm 2\pi i$ or zero will be added to the value of the integral. Hence, finally, whatever be the path of integration,

$$\phi_p\left(\frac{a_i z+\beta_i}{\gamma_i z+\delta_i}\right)-\phi_p(z) = 2m_p\pi i+n_1 a_{p1}+n_2 a_{p2}+\dots+n_n a_{pn},$$

where $$p = 1, 2, \dots n.$$

It is now clear that the functions ϕ are the analogues in the infinite plane of the integrals of the first species on the $\overline{n+1}$-ply connected Riemann's surface into which the generating (or any one) polygon can be deformed by bending and stretching till corresponding points on its surface are brought to coincide. Also, if each of the ϕ's is multiplied by $-\frac{1}{2}$, they will directly represent the n independent normal integrals of the first species; the $2n$ sections on the surface being the n curves obtained by bringing the pairs of corresponding circles to coincidence and the n closed curves into which this process transforms the barriers.

To complete the analogy, and make the introduction of the θ-functions possible, it must still be shown that the real part of

$$\Sigma n_p^2 a_{pp} + 2\Sigma n_p n_q a_{pq}$$

is essentially positive.

The proof of this, again, is essentially the same as that of the corresponding theorem with respect to Abelian integrals, and may be stated shortly.

If
$$\phi_p = u_p + iv_p,$$

then, since $\Sigma n_p \phi_p$ is finite, continuous, and one-valued in the generating polygon with barriers,

$$\int \Sigma n_p u_p \,.\, \Sigma n_p \, dv_p,$$

taken in the positive direction round its boundary, is necessarily positive. The separate terms give at once

$$\int u_p dv_p = 2\pi a_{pp},$$

$$\int u_p dv_q = \int u_q dv_p = 2\pi a_{pq},$$

where a_{pp} and a_{pq} are the real parts of a_{pp} and a_{pq}; and from these equations the result follows at once.

5. *On the analogues of the Abelian integrals of the second and third kinds.*

The function
$$-\sum_i \frac{(\gamma_i z + \delta_i)^{-2}}{\left(\dfrac{a_i z + \beta_i}{\gamma_i z + \delta_i} - a\right)^2}$$

has a double infinity at the point a and its homologues; while the homologues of $z = \infty$ are not infinities of the function. Its integral will therefore be a one-valued function, finite and continuous everywhere except at the point a and its homologues, and at these points it will have a simple infinity. This function will be denoted by $\psi_a(z)$, so that

$$\psi_a(z) = \sum_i \left(\frac{1}{\dfrac{a_i z + \beta_i}{\gamma_i z + \delta_i} - a} - \frac{1}{\dfrac{a_i z_0 + \beta_i}{\gamma_i z_0 + \delta_i} - a} \right),$$

where z_0 represents the origin of the integration, which can be chosen at convenience. The expression in brackets can be transformed as follows:—

$$\psi_a(z) = \sum_i \frac{z_0 - z}{[a_i z + \beta_i - a(\gamma_i z + \delta_i)][a_i z_0 + \beta_i - a(\gamma_i z_0 + \delta_i)]}$$

$$= \sum_i \frac{(-\gamma_i a + \delta_i)^{-2}(z_0 - z)}{\left(\dfrac{\delta_i a - \beta_i}{-\gamma_i a + a_i} - z\right)\left(\dfrac{\delta_i a - \beta_i}{-\gamma_i a + a_i} - z_0\right)}$$

$$= \sum_j (\gamma_j a + \delta_j)^{-2} \left(\frac{1}{\dfrac{a_j a + \beta_j}{\gamma_j a + \delta_j} - z_0} - \frac{1}{\dfrac{a_j a + \beta_j}{\gamma_j a + \delta_j} - z} \right)$$

$$= \theta(a, z_0) - \theta(a, z).$$

It follows at once that

$$\psi_a\left(\frac{a_p z + \beta_p}{\gamma_p z + \delta_p}\right) - \psi_a(z) = \theta(a, z) - \theta\left(a, \frac{a_p z + \beta_p}{\gamma_p z + \delta_p}\right)$$

$$= -\theta(a, J_p)$$

$$= -\left(\frac{d\phi_p}{dz}\right)_{z=a};$$

and also that

$$\psi_a\left(\frac{a_i z + \beta_i}{\gamma_i z + \delta_i}\right) = \psi_a(z) - \sum_1^n n_p \left(\frac{d\phi_p}{dz}\right)_{z=a};$$

where the integers n_p are the same as in the general formula for $\phi\left(\dfrac{a_i z + \beta_i}{\gamma_i z + \delta_i}\right)$. The functions $\psi(z, a)$ are thus the exact analogues of the normal integrals of the second kind on the Riemann's surface corresponding to the division of the z-plane; namely, they have one infinity inside each region, they are one-valued (*i.e.*, they do not change when the variable describes a closed path which cuts the barriers), and, when one of the substitutions of the group is performed on the variable, they increase by integral multiples of definite constants, which bear the same relation to the quasi-periods of the ϕ's that the periods of the normal integrals of the second kind bear to those of the first.

Again, the function

$$\sum_i (\gamma_i z + \delta_i)^{-2} \left(\frac{1}{\dfrac{a_i z + \beta_i}{\gamma_i z + \delta_i} - a} - \frac{1}{\dfrac{a_i z + \beta_i}{\gamma_i z + \delta_i} - b} \right)$$

is finite everywhere except at a and b and their homologues, which points are simple infinities. It may be supposed, without loss of generality for what follows, that a and b are both within the generating polygon.

If a and b be joined by a barrier, and the homologues of this barrier be drawn, it is easy to see that the integral of the function in question will be one-valued, finite, and continuous, except at these new barriers, having values differing by $2\pi i$ at corresponding points on opposite sides of the new barriers, and, in particular, being continuous at the original barriers. If the integral from an arbitrary origin be written $\chi_{a,b}(z)$, then, in the infinite plane, as bounded by the new barriers and the singular points of the group,

$$\chi_{a,b}(z) = \sum_i \log \frac{\left(\dfrac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}-a\right)\left(\dfrac{\alpha_i z_0+\beta_i}{\gamma_i z_0+\delta_i}-b\right)}{\left(\dfrac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}-b\right)\left(\dfrac{\alpha_i z_0+\beta_i}{\gamma_i z_0+\delta_i}-a\right)},$$

where the branch of the logarithm is that which makes

$$\chi_{a,b}(z_0) = 0.$$

The expression on the right-hand side can be transformed, as in the case of $\psi(z, a)$, so as to give

$$\chi_{a,b}(z) = \sum_j \log \frac{\left(\dfrac{\alpha_j a+\beta_j}{\gamma_j a+\delta_j}-z\right)\left(\dfrac{\alpha_j b+\beta_j}{\gamma_j b+\delta_j}-z_0\right)}{\left(\dfrac{\alpha_j a+\beta_j}{\gamma_j a+\delta_j}-z_0\right)\left(\dfrac{\alpha_j b+\beta_j}{\gamma_j b+\delta_j}-z\right)}.$$

Hence $$\chi_{a,b}\left(\frac{\alpha_p z+\beta_p}{\gamma_p z+\delta_p}\right)-\chi_{a,b}(z) = \phi_p(b)-\phi_p(a),$$

and, for any substitution,

$$\chi_{a,b}\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right) = \chi_{a,b}(z)+\sum_1^n n_p\left[\phi_p(b)-\phi_p(a)\right].$$

It is also clear, by comparing the two expressions for $\chi_{a,b}(z)$, that, if the points z_1, z_2 be inside the same polygon, then

$$\chi_{a,b}(z_2)-\chi_{a,b}(z_1) = \chi_{z_2,z_1}(a)-\chi_{z_2,z_1}(b),$$

which is the equivalent of the interchange of argument and parameter for integrals of the third kind.

Just, then, as the functions $\psi(z, a)$ answer in all respects to the normal Abelian integrals of the second kind, so the functions $\chi_{a,b}(z)$ are the exact analogues of the normal integrals of the third kind.

6. *On the Functions which are unchanged by the substitutions of a group.*

It has now been shown that the functions $\phi_p(z)$, $\psi_a(z)$, $\chi_{a,b}(z)$ behave, on the z-plane, with respect to closed paths surrounding singular points of the groups and open paths passing from any point to any one of its homologues, in exactly the same way that the normal integrals of the first, second and third species on a Riemann's surface behave when the variable describes closed curves reconcilable with the a-sections and the b-sections on the surface; *i.e.*, when the variable describes any closed curve whatever on the surface. It follows therefore that the different methods of representing algebraic functions on the Riemann's surface, in terms of the normal integrals, may be applied at once to form automorphic functions on the z-plane, in terms of the functions ϕ, ψ, and χ.

Thus, if $z_1, z_2 \dots z_m$ are m points, such that no one is a homologue of any of the others (or, more simply, m different points in the same polygon), the function

$$C_0 + \sum_1^m C_r \psi_{z_r}(z)$$

will be unchanged by all the substitutions of the group if the n equations

$$\sum_1^m C_r \theta(z_r, J_p) = 0 \qquad (p = 1, 2 \dots n)$$

are satisfied. When the positions of the m given points are arbitrary, these n equations will, in general, be independent, and can therefore only be satisfied if m is equal to, or greater than, $n+1$. Hence, in general, a function which is unaltered by the substitutions of the group must take every value $n+1$ times at least in each polygon.

Again, if $f(z)$ be a function which is unaltered by the substitutions of the group, and $a_1, a_2 \dots a_m$ its zeros, $b_1, b_2 \dots b_m$ its infinities inside the generating polygon, it can be shown that

$$\sum_1^m \left[\phi_p(a_r) - \phi_p(b_r)\right] = 2m_p \pi i + \sum_{q=1}^{q=n} n'_q a_{pq}, \quad (p = 1, 2 \dots n),$$

where m'_p, n'_q are integers.

For consider the integral

$$\int \phi_p \frac{f'(z)}{f(z)} dz,$$

extended round the boundary of the generating polygon, including the barriers. It is equal to

$$2\pi i \int_{A'_p A_p} \frac{f'(z)}{f(z)}\,dz - \sum_q a_{pq} \int_{C_q} \frac{f'(z)}{f(z)}\,dz\,;$$

or, since
$$f(A_p) = f(A'_p),$$

and the path of the second integral is closed, the result is

$$2\pi i\,[2m'_p \pi i + \Sigma n'_q a_{pq}],$$

where m'_p, n'_q are positive or negative integers.

In the generating polygon with barriers, ϕ_p is everywhere finite, one-valued, and continuous, and $\dfrac{f'(z)}{f(z)}$ is so also, except at the points $a_1, a_2 \dots$, $b_1, b_2 \dots b_m$, and these points are simple infinities with residues $+1$ and -1 respectively. The integral in question, then, may be evaluated again by taking it round infinitely small contours surrounding these points in the positive direction, which gives for its value

$$2\pi i \sum_1^m [\phi_p(a_r) - \phi_p(b_r)],$$

and, on equating the two values and dividing by $2\pi i$, the given result follows.

From this, the expression of an automorphic function, in terms of the functions χ, may be at once deduced. For suppose that $a_1, a_2 \dots a_m$, $b_1, b_2 \dots b_m$ are $2m$ points satisfying the n conditions just investigated.

Then
$$\exp\left[\sum_1^m \chi_{a_r, b_r}(z) + \sum_1^n n'_p \phi_p\right],$$

is a function which in the generating polygon has the simple zeros $a_1, a_2 \dots a_m$, and the simple infinities $b_1, b_2 \dots b_m$; it is everywhere one-valued, and, when $\dfrac{a_p z + \beta_p}{\gamma_p z + \delta_p}$ is written for z, it becomes multiplied by the factor

$$\exp\left\{\sum_1^m [\phi_p(b_r) - \phi_p(a_r)] + \sum_1^n n'_p a_{pq}\right\},$$

which is the same as $\exp[-2m'_p \pi i]$, or unity. It is, therefore, unchanged by all the substitutions of the group.

Finally, writing, for convenience,

$$-\tfrac{1}{2}\phi_p = u_p, \quad -\tfrac{1}{2}a_{pp} = b_{pp}, \quad -\tfrac{1}{2}a_{pq} = b_{pq},$$

and forming the θ-function

$$\theta(x_1, x_2, \dots x_n)$$
$$= \left(\sum_{-\infty}^{\infty}\right)^n \exp\left[b_{11}n_1^2 + 2b_{12}n_1n_2 + \dots + b_{pp}n_p^2 + 2x_1n_1 + \dots + 2x_nn_n\right],$$

the previous results show that the conditions are satisfied for its convergence so long as the arguments x_p are finite; and if now these are replaced by the functions $u_p - c_p$, where the c_p's are any constants, the behaviour of the θ-function, when any substitution of the group is performed on the variable, will be exactly the same as that of a θ-function, whose arguments are the integrals of the first kind, on a Riemann's surface when the variable describes any closed path on the surface.

It follows that all the various known theorems, with respect to the representation of uniform functions of position on a Riemann's surface by means of θ-functions, may be now directly applied to the formation of automorphic functions in terms of the θ-functions just constructed, with the u_p's as their arguments.

7. *On the Symmetrical Case.*

When each of the n pairs of circles C_1' and C_1, C_2' and C_2, &c., which form the boundary of the generating polygon, are inverses of each other with respect to another circle C_0, and the fundamental substitutions which interchange C_1' and C_1, C_2' and C_2, &c., are all hyperbolic, M. Poincaré calls the group symmetrical. In this case the substitution S_p, which changes C_p' into C_p, is equivalent to a pair of inversions performed successively at C_p' and C_0, and any substitution of the group is equivalent to an even number of inversions with respect to the $n+1$ circles $C_0, C_1', C_2' \dots C_n'$ (or $C_0, C_1, C_2 \dots C_n$). Conversely, an even number of inversions with respect to any of the circles $C_0, C_1' \dots C_n'$ (or $C_0, C_1 \dots C_n$), is equivalent to some substitution of the group.

I shall suppose at first that the circle C_0 coincides with the real axis, and shall show further on that the results so obtained may be applied to any symmetrical group whatever.

If C_0 coincides with the axis of x, the fundamental substitutions of the group can be put in the form

$$\left(\frac{z-a_p}{z-a_p'},\ K_p\frac{z-a_p}{z-a_p'}\right),$$

where K_p is real, and a_p, a'_p are conjugate imaginaries; and it therefore follows that the substitutions of the group may be taken in pairs

$$\left(z, \frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right) \text{ and } \left(z, \frac{\alpha'_i z+\beta'_i}{\gamma'_i z+\delta'_i}\right),$$

such that $\alpha_i, \beta_i, \gamma_i, \delta_i$ and $\alpha'_i, \beta'_i, \gamma'_i, \delta'_i$ are respectively conjugate imaginaries.

If z, z' and a, a' are conjugate imaginaries, so also are

$$\frac{(\gamma_i z+\delta_i)^{-2}}{\dfrac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}-a} \text{ and } \frac{(\gamma'_i z'+\delta'_i)^{-2}}{\dfrac{\alpha'_i z'+\beta'_i}{\gamma'_i z'+\delta'_i}-a'},$$

and therefore also $\quad \theta(z, a)$ and $\theta(z', a')$.

Now J_p and J_{p-1} are in this case conjugate imaginaries, and it has been shown that in any case

$$\theta(z, J_{p-1}) = -\theta(z, J_p),$$

whence it follows that

$$\theta(z, J_p) \text{ and } -\theta(z', J_p),$$

are conjugate imaginaries.

Hence, when z is real, $\theta(z, J_p)$ is a pure imaginary.

Now suppose that z' is any point on C'_q, so that z is the corresponding point on C_q; then

$$\text{mod } dz = \text{mod } dz',$$

and

$$\arg dz = -\arg dz' = \phi,$$

where ϕ is the angle that the tangent at z to the circle C_q makes with the axis of x; but

$$z = \frac{\alpha_q z'+\beta_q}{\gamma_q z'+\delta_q},$$

and therefore

$$dz = \frac{dz'}{(\gamma_q z'+\delta_q)^2};$$

and

$$(\gamma_q z'+\delta_q)^2 = e^{-2i\phi}.$$

Also, in consequence of the relation between z and z',

$$\theta(z, J_p) = (\gamma_q z'+\delta_q)^2\,\theta(z', J_p)$$
$$= e^{-2i\phi}\,\theta(z', J_p),$$

or $$\arg\theta(z, J_p) = -2\phi + \arg\theta(z', J_p);$$

but, since $\theta(z, J_p)$ and $-\theta(z', J_p)$ are conjugate imaginaries,

$$\arg\theta(z, J_p) = \pi - \arg\theta(z', J_p);$$

therefore $$\arg\theta(z, J_p) = \frac{\pi}{2} - \phi.$$

The ratio of any two of the functions $\theta(z, J)$ is therefore real at each of the circles C_1', C_1, &c., as well as at C_0.

The barriers in this case may be taken as straight lines perpendicular to the axis of x; and since it has been shown that, when z and z' are conjugate imaginaries, so also are $\theta(z, J_p)$ and $-\theta(z', J_p)$, it follows at once that

$$\int_{A_Q' A_Q} \theta(z, J_p)\, dz \,(= a_{pq})$$

is real, where a_{pq} is any one of the $n(n-1)$ constants.

Finally, since for points on any one of the bounding circles, including A_0,

$$\arg\theta(z, J_p) = \frac{\pi}{2} - \phi,$$

and $$\arg dz = \phi,$$

the variable part of $\int \theta(z, J_p)\, dz$, or of ϕ_p, is a pure imaginary at all the circles.

To extend these results to the case of any symmetrical group whatever, a slight digression is necessary, on the connexion between the functions of a group S_i and those of the group $\Sigma^{-1}S_i\Sigma$, where Σ is any arbitrary substitution.

If $\left(z, \dfrac{Az+B}{Cz+D}\right)$ be any substitution not contained in the group $\left(z, \dfrac{a_i z+\beta_i}{\gamma_i z+\delta_i}\right)$, then the various substitutions given by

$$\left\{\frac{Az+B}{Cz+D},\ \frac{a_i\dfrac{Az+B}{Cz+D}+\beta_i}{\gamma_i\dfrac{Az+B}{Cz+D}+\delta_i}\right\}$$

will also form a discontinuous group, that may be represented by the accented symbols

$$\left(z, \frac{a_i' z+\beta_i'}{\gamma_i' z+\delta_i'}\right).$$

A division of the infinite plane into polygons, for this latter group, is obtained by transforming the polygons of the first group by the substitution

$$\left(z, \frac{Az+B}{Cz+D}\right).$$

By an algebraical process very similar to that used already in dealing with the function $\theta(z, a)$, the result of making linear substitutions for z and a in $\theta(z, a)$ is easily obtained. Thus, if

$$z = \frac{Az'+B}{Cz'+D}, \quad a = \frac{Aa'+B}{Ca'+D},$$

then, with the notation just given,

$$\theta(z, a) = (Cz'+D)^2 \sum_i (\gamma_i' z'+\delta_i')^{-2} \left(\frac{1}{\frac{\alpha_i' z'+\beta_i'}{\gamma_i' z'+\delta_i'} - a'} - \frac{1}{\frac{\alpha_i' z'+\beta_i'}{\gamma_i' z'+\delta_i'} + \frac{D}{C}} \right)$$

$$= (Cz'+D)^2 \left[\theta'(z', a') - \theta'\left(z', -\frac{D}{C}\right) \right],$$

where θ' is a function formed with the substitutions of the new group.

To the infinities $\frac{\alpha_i a+\beta_i}{\gamma_i a+\delta_i}$ and $\frac{\alpha_i}{\gamma_i}$ of $\theta(z, a)$, correspond $\frac{\alpha_i' a'+\beta_i'}{\gamma_i' a'+\delta_i}$ and $\frac{\alpha_i' D-\beta_i' C}{\gamma_i' D-\delta_i' C}$; while the homologues of infinity in the new group are not infinities of the right-hand side.

It follows that

$$\theta(z, J_p) = (Cz'+D)^2 \theta'(z', J_p');$$

and, since

$$dz = (Cz'+D)^{-2} dz',$$

$$\phi_p(z) = \phi_p'(z'),$$

the integrals being taken along homologous paths. In particular, the constants a_{pp}, a_{pq} for the two groups are identical.

In precisely the same way it may be shown that

$$\psi_a(z) = (Ca'+D)^2 \psi_{a'}'(z'),$$

and

$$\chi_{a,b}(z) = \chi_{a',b'}'(z').$$

If the generating polygon of the group in which C_0 is the axis of x, be transformed by any linear substitution, the new polygon so formed will be the generating polygon of a symmetrical group, the substitutions of which can be formed from those of the original one by the process just investigated; and in this manner any symmetrical polygon, and therefore the corresponding group, can be formed.

Hence for any symmetric group the constants a_{pp}, a_{pq} are real, the functions φ_p have their real parts constant at the circumferences of the $n+1$ circles, and the ratio of any two of the functions $\theta(z, J)$ is real at the circumferences of the circles.

When the symmetrical group is also fuchsian, the n pairs of circles are all cut at right angles by one circle, viz., the circle which is unchanged by the substitutions of the group.

The simplest form in which to consider the symmetrical fuchsian group is that in which the unchanged circle and A_0 are taken as two straight lines at right angles; in particular, as the axes Ox and Oy.

If S be any substitution of the group, and A represent an inversion with respect to Oy, then ASA will also be a substitution, and these two will be of the forms

$$\left(z, \frac{az+\beta}{\gamma z+\delta}\right) \quad \text{and} \quad \left(z, \frac{az-\beta}{-\gamma z+\delta}\right),$$

where a, β, γ, δ are all real; hence, pairing the substitutions in this way,

$$\theta(z, a) = \frac{1}{z-a} + \sum_i \left(\frac{(\gamma_i z+\delta_i)^{-2}}{\dfrac{a_i z+\beta_i}{\gamma_i z+\delta_i} - a} + \frac{(-\gamma_i z+\delta_i)^2}{\dfrac{a_i z-\beta_i}{-\gamma_i z+\delta_i} - a} \right),$$

$$\theta(-z, -a) = \frac{1}{-z+a} + \sum_i \left(\frac{(-\gamma_i z+\delta_i)^2}{\dfrac{a_i z-\beta_i}{\gamma_i z-\delta_i} + a} + \frac{(\gamma_i z+\delta_i)^2}{-\dfrac{a_i z+\beta_i}{\gamma_i z+\delta_i} + a} \right);$$

whence $$\theta(-z, -a) = -\theta(z, a).$$

Now $$J_{p-1} = -J_p,$$

and $$\theta(z, J_{p-1}) = -\theta(z, J_p);$$

therefore $$\theta(-z, J_p) = \theta(z, J_p).$$

Also, since J_p is real, $\theta(z, J_p)$ and $\theta(z', J_p)$ are conjugate imaginaries, z and z' being conjugate imaginaries themselves.

Hence, if $\theta(z, J_p)$ vanishes when $z = a+ib$, it vanishes for the four values $z = \pm a \pm ib$.

Again, in this case, the function

$$\psi_a(z) - \psi_{-a}(z),$$

which is clearly not identically zero, is a fuchsian function; for

$$\psi_a\left(\frac{a_p z+\beta_p}{\gamma_p z+\delta_p}\right)-\psi_{-a}\left(\frac{a_p z+\beta_p}{\gamma_p z+\delta_p}\right)-[\psi_a(z)-\psi_{-a}(z)]$$
$$=\theta(a, J_p)-\theta(-a, J_p)$$
$$=0.$$

Hence, in the case of symmetrical fuchsian groups, automorphic functions exist which take every value twice only in the generating polygon. It follows that the corresponding Abelian integrals are of the hyper-elliptic class.

8. *On the Zeros of* $\theta(z, J)$.

The functions $\theta(z, J)$ have no infinities, except at the singular points of the group, and hence the integral

$$\frac{1}{2\pi i}\int d\log\theta(z, J),$$

taken in the positive direction round the generating polygon, is equal to the number of simple zeros of $\theta(z, J)$ therein contained.

The barriers clearly give nothing towards the value of the integral; also, if z is a point of C'_p, and z_p the corresponding point of C_p,

$$\theta(z_p, J)=(\gamma_p z+\delta_p)^2\,\theta(z, J);$$

and, therefore, when the integrals are taken round the circles in the directions of watch-hands, so as to make a positive circuit of the generating polygon,

$$\int_{C_p} d\log\theta(z, J)+\int_{C'_p} d\log\theta(z, J)$$
$$=\int_{C'}\left[d\log\theta(z, J)-d\log\theta(z_p, J)\right]$$
$$=-2\int_{C'} d\log(\gamma_p z+\delta_p)=4\pi i.$$

A similar result holds for each pair of circles, and the total number of zeros in the generating polygon is therefore $2n$. Of these zeros the form of the function shows that two are at infinity; for

$$\theta\left(z, \frac{a_p}{\gamma_p}\right)=\frac{1}{z-\frac{a_p}{\gamma}}+\frac{(\gamma_p z+\delta_p)^{-2}}{\frac{a_p z+\delta_p}{\gamma_p z+\delta_p}-\frac{a_p}{\gamma_p}}+\sum_i\frac{(\gamma_i z+\delta_i)^{-2}}{\frac{a_i z+\beta_i}{\gamma_i z+\delta_i}-\frac{a_p}{\gamma_p}}$$
$$=\frac{1}{z-\frac{a_p}{\gamma_p}}-\frac{1}{z+\frac{\delta_p}{\gamma_p}}+\sum_i\frac{(\gamma_i z+\delta_i)^{-2}}{\frac{a_i z+\beta_i}{\gamma_i z+\delta_i}-\frac{a_p}{\gamma_p}};$$

and the terms which would give a simple zero at $z=\infty$, obviously cancel each other.

The relation
$$\theta\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}, J\right)=(\gamma_i z+\delta_i)^2\,\theta(z, J)$$
shows that generally, if z is a zero of $\theta(z, J)$, so also are its homologues; but that this is not true for $z=\infty$. Hence the functions $\theta(z, J)$ have, in any region R_i, $2n-2$ simple zeros, while in the generating polygon there is, in addition to these, a double zero at $z=\infty$.

The ratio of any two linear homogeneous functions of the $\theta(z, J)$'s will be an automorphic function, which generally will take every value $2n-2$ times in the generating or any other polygon. By a proper choice of the constants it is clearly possible always to form in this way a function which will take every value less than $n+1$ times, and on the other hand, the known theory of uniform functions, on a multiply-connected surface, indicates that any function which takes every value less than $n+1$ times should be capable of being represented in the way considered.

It is easy to verify that, in the case of the symmetrical fuchsian group, functions taking every value twice may be found in this way; for consider the function
$$\frac{a_2\,\theta(z, J_2)+a_3\,\theta(z, J_3)+\ldots+a_n\,\theta(z, J_n)}{\theta(z, J_1)}.$$

The double zeros of the numerator and denominator, at $z=\infty$, destroy each other. The $2n-2$ finite zeros of the numerator depend on the $n-2$ ratios of the constants, *i.e.*, on $2n-4$ real constants. If z_0 is a zero of the denominator, two linear relations between the $2n-4$ real constants must be satisfied in order that z_0 may also be a zero of the numerator, and, since
$$\theta(-z, J)=\theta(z, J),$$
these two relations ensure that the numerator and denominator shall have two zeros in common. Hence the available constants will enable $2n-4$ zeros of the numerator to become coincident with zeros of the denominator, so that when the constants are thus chosen, the function is one which becomes infinite (and therefore has every other value) twice only in any polygon.

The calculation that has been applied to $\theta(z, J)$ will hold similarly with any function of the form
$$\sum_i f\left(\frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)(\gamma_i z+\delta_i)^{-2},$$

where $f(z)$ is a rational function of z; so that the number of zeros of any such function in the generating polygon exceeds the number of infinities by $2n$.

9. *On Groups of the First Class which contain Elliptic Substitutions.*

I now go on to consider shortly the case in which a group contains elliptic substitutions, so that throughout this section it will be implied that the group is fuchsian, as the convergence of

$$\sum_i \text{mod}\,(\gamma_i z+\delta_i)^{-2}$$

for kleinian groups, with elliptic substitutions, has not been proved.

The essential point in which this case differs from that hitherto treated, lies in the fact that now some of the quantities

$$\theta(z, J_p) \qquad\qquad (p = 1,\ 2\ldots n)$$

will vanish identically.

That this must be so, if the previously given theory be correct, is obvious at once from geometrical considerations; for it is clear that the closed surface formed by bending and deforming the generating polygon till corresponding points of the boundary are brought to coincidence will no longer be $n+1$-ply connected, when some of the n fundamental substitutions are elliptic. There will, therefore, be less than n everywhere-finite integrals upon it, and, therefore, less than n ϕ-functions in connexion with the group.

Suppose, now, that an elliptic substitution S is one of the fundamental substitutions of the group; and consider the corresponding $\theta(z, J)$, defined by

$$\theta(z, J) = \sum_i \frac{(\gamma_i z+\delta_i)^{-2}}{\dfrac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}-J}.$$

The term written on the right-hand side has a simple infinity at $-\dfrac{\delta_i}{\gamma_i}$, and, if

$$\frac{(\gamma_j z+\delta_j)^{-2}}{\dfrac{\alpha_j z+\beta_j}{\gamma_j z+\delta_j}-J}$$

is the term that cancels this, then

$$\frac{\alpha_j\left(-\dfrac{\delta_i}{\gamma_i}\right)+\beta_j}{\gamma_j\left(-\dfrac{\delta_i}{\gamma_i}\right)+\delta_j}=J;$$

or $$S_i^{-1}S_j = S;$$

i.e., $$S_j = S_i S.$$

If, again, the term with suffix k is that which destroys the simple infinity $-\frac{\delta_j}{\gamma_j}$, introduced by the term with suffix j, then

$$S_k = S_j S = S_i S^2.$$

Now, if m is the period of the elliptic substitution S,

$$S^m = 1;$$

and therefore the terms in $\theta(z, J)$, corresponding to the substitutions

$$S_i, \quad S_i S, \quad S_i S^2, \dots S_i S^{m-1},$$

are such that at any point where one of the terms becomes infinite another takes an equal and opposite infinite value.

Hence the sum of these terms, being a rational function of z which is nowhere infinite, must be a constant, and this constant is zero, as may be seen by making z infinite. It follows at once that, since $\theta(z, J)$ is in any case a uniformly convergent series, it is in this case identically zero.

[It may be interesting to give the result of a similar grouping of the terms of a $\theta(z, J)$, which corresponds to a hyperbolic substitution. The series of terms

$$\dots S_i S^{-m}, \quad S_i S^{-m+1}, \dots S_i, \quad S_i S, \dots S_i S^m, \dots$$

is then infinite, and their sum is easily shown to be equal to

$$\frac{1}{z - S_i\alpha} - \frac{1}{z - S_i\beta},$$

where α and β are the double points of the hyperbolic substitution.]

Not only will all the functions $\theta(z, J)$ which correspond to elliptic substitutions, vanish identically, but it will generally happen that some of those corresponding to hyperbolic substitutions, will also do so. For when some of the fundamental substitutions are elliptic, there will generally be certain identical relations, of the form

$$1 = S_\alpha^m S_\beta^n \dots,$$

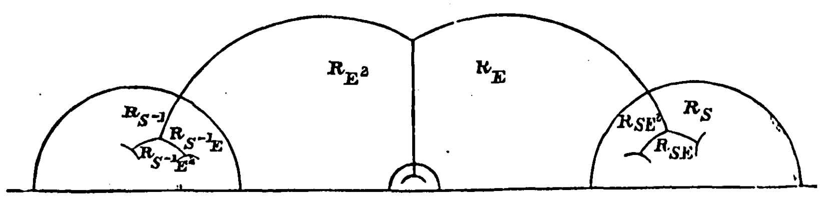
connecting them; and when these are used in the equations of the form

$$\theta(z, J_p) + \theta(z, J_{q-1}) + \theta(z, J_{p-1q}) = 0,$$

established in Section 3, it will be found that the result stated will follow.

Owing to the infinite variety of cases that can occur, it seems almost impossible to consider this matter generally, but the following simple example will illustrate the property in question.

Consider the fuchsian group formed from a single hyperbolic substitution S and a single elliptic substitution E of period 3. Taking the generating polygon R_0, as previously, to contain the point at infinity (and for simplicity assuming the unchanged circle to be the real axis), the division of the z-plane into regions by the group is given by the accompanying figure (the upper half only being drawn).



An inspection of the figure shows at once that SE^2 must be an elliptic substitution of period 2, or that

$$(SE^2)^2 = 1.$$

Now $$\theta(z, J_{S^{-1}}) + \theta(z, J_{E^{-1}}) + \theta(z, J_{SE^2}) = 0,$$

by the previously quoted equation; but, since E^{-1} and SE^2 are elliptic substitutions, the two latter terms of the equation are identically zero; hence, also, the first term must be. In this case, then, both the functions $\theta(z, J)$, corresponding to the fundamental substitutions, vanish; and this is in proper correspondence with the fact that the closed surface, formed by bringing corresponding points of the boundary of R_0 to coincidence, is simply-connected.

Instead of attempting to treat generally the case of a group with elliptic substitutions, I propose to continue the discussion of the special case in which all the $\theta(z, J)$'s vanish identically.

The ϕ-functions, being integrals of the $\theta(z, J)$'s, vanish identically, or rather, do not exist.

The integrals of the second kind, $\psi_a(z)$, become in this case automorphic functions, which take every value once in each polygon; and hence it immediately follows that they can all be expressed as linear functions of any one of them. It is both interesting, and will serve as some verification of the general accuracy of the preceding investigation, to prove this result by direct calculation.

For this purpose I transform the product $\theta(a, z)\,\theta(a', z)$ in the following manner:—

$$\theta(a, z)\theta(a', z) = \sum_i\sum_j \frac{(\gamma_i a+\delta_i)^{-2}(\gamma_j a'+\delta_j)^{-2}}{\left(\frac{a_i a+\beta_i}{\gamma_i a+\delta_i}-z\right)\left(\frac{a_j a'+\beta_j}{\gamma_j a'+\delta_j}-z\right)}$$

$$= \sum_i\sum_j \frac{(\gamma_i a+\delta_i)^{-2}(\gamma_j a'+\delta_j)^{-2}}{\frac{a_i a+\beta_i}{\gamma_i a+\delta_i}-\frac{a_j a'+\beta_j}{\gamma_j a'+\delta_j}}\left(\frac{1}{\frac{a_j a'+\beta_j}{\gamma_j a'+\delta_j}-z}-\frac{1}{\frac{a_i a+\beta_i}{\gamma_i a+\delta_i}-z}\right).$$

Now
$$\sum_i \frac{(\gamma_i a+\delta_i)^{-2}}{\frac{a_i a+\beta_i}{\gamma a+\delta_i}-\frac{a_j a'+\beta_j}{\gamma_j a'+\delta_j}} = \theta\left(a, \frac{a_j a'+\beta_j}{\gamma_j a'+\delta_j}\right)$$
$$= \theta(a, a'),$$

since
$$\theta(a, J) \equiv 0;$$

hence
$$\theta(a, z)\,\theta(a', z) = \theta(a, a')\,\theta(a', z) + \theta(a', a)\,\theta(a, z).$$

In reference to this formula it is to be noticed that the zero of $\theta(a, z)$ in the generating polygon is at infinity, and therefore independent of a.

It has already been shown, in Section 5, that

$$\psi_a(z) = \theta(a, z_0) - \theta(a, z),$$

and therefore the formula just obtained is equivalent to a lineo-linear relation between any two ψ-functions. If the zero of the function be represented in the symbol by writing $\psi_a(z, z_0)$, instead of $\psi_a(z)$, the explicit relation required will be

$$\psi_{a'}(z, z_0') = \psi_{a'}(a, z_0')\,\frac{\psi_a(z, z_0) - \psi_a(z_0', z_0)}{\psi_a(z, z_0) - \psi_a(a', z_0)}.$$

The ψ-function in this case corresponds to what Prof. Klein calls a *fundamental function* on a simply-connected Riemann's surface; in terms of it any automorphic function, with a finite number of infinities, may be expressed rationally. Its analytical form is that of an infinite series.

But a function with precisely the same properties may be formed at once from the integrals of the third kind, *i.e.*, from the functions

$\chi_{a,b}(z)$. For, since the ϕ-functions are non-existent, it follows at once, from their previously proved properties, that

$$e^{\chi_{a,b}(z)} = \prod_i \frac{\left(\frac{a_i z+\beta_i}{\gamma_i z+\delta_i}-a\right)\left(\frac{a_i z_0+\beta_i}{\gamma_i z_0+\delta_i}-b\right)}{\left(\frac{a_i z+\beta_i}{\gamma_i z+\delta_i}-b\right)\left(\frac{a_i z_0+\beta_i}{\gamma_i z_0+\delta_i}-a\right)}$$

$$= \prod_i \frac{\left(\frac{a_i a+\beta_i}{\gamma_i a+\delta_i}-z\right)\left(\frac{a_i b+\beta_i}{\gamma_i b+\delta_i}-z_0\right)}{\left(\frac{a_i b+\beta_i}{\gamma_i b+\delta_i}-z\right)\left(\frac{a_i a+\beta_i}{\gamma_i a+\delta_i}-z_0\right)};$$

is an automorphic function with the single zero a and the single infinity b in the generating polygon. The expressions here quoted are convergent infinite products, whatever the value of z_0, the point $z = z_0$ being that at which the function takes the value unity. If then z_0 is taken infinite, the function

$$C \prod_i \frac{\frac{a_i z+\beta_i}{\gamma_i z+\delta_i}-a}{\frac{a_i z+\beta_i}{\gamma_i z+\delta_i}-b};$$

is the general expression for an automorphic function, which takes every value once in the generating polygon, in the form of an infinite product.

The passage from the infinite series to the infinite product form may be carried out as follows :—

Let $$x = \psi_a(z, b);$$

then $$\frac{dx}{dz} = \sum_i \frac{(\gamma_i z+\delta_i)^{-2}}{\left(\frac{a_i z+\beta_i}{\gamma_i z+\delta_i}-a\right)^2},$$

and $\frac{1}{x}\frac{dx}{dz}\,[=f(z)]$ has simple infinities at a and b, and a zero of the second order at ∞, these being its only zeros and infinities in the generating polygon; while also

$$f\left(\frac{a_i z+\beta_i}{\gamma_i z+\delta_i}\right) = (\gamma_i z+\delta_i)^2 f(z).$$

Hence it at once follows that

$$\frac{1}{x}\frac{dx}{dz} = \sum_i \left(\frac{1}{\dfrac{a_i z+\beta_i}{\gamma_i z+\delta_i}-b} - \frac{1}{\dfrac{a_i z+\beta_i}{\gamma_i z+\delta_i}-a} \right) (\gamma_i z+\delta_i)^{-2}$$

$$= \frac{d}{dz}\left[\chi_{b,a}(z)\right];$$

or, on integration,

$$x = \text{const.} \times e^{\chi_{b,a}(z)}$$

$$= C \prod_i \frac{\dfrac{a_i z+\beta_i}{\gamma_i z+\delta_i}-b}{\dfrac{a_i z+\beta_i}{\gamma_i z+\delta_i}-a}.$$

Returning for a moment to the more general case, if the closed surface formed as before by deforming the generating polygon is $n'+1$-ply connected, it is to be expected that $n-n'$ of the $\theta(z, J)$'s will vanish identically, and that, from the remaining n', a theory can be constructed in all respects similar to that of Section 4. That this is so in the case of any given group may be verified directly. If, among the fundamental substitutions of the group, there are one or more parabolic substitutions, then, in order that the group may be discontinuous, the double points of these substitutions, which are singular points of the group, must be vertices of the generating polygon. It may be shown, in a manner similar to that used for the elliptic substitutions, that the functions $\theta(z, J)$, corresponding to a parabolic substitution, vanish identically, and the preceding theory is generally applicable; except that the vertices themselves, which are the double points of the parabolic substitutions, must be reckoned as not belonging to the polygons, for, being singular points of the group, they are essentially singular points of all the functions considered.

10. *Conclusion.*

The investigations of this paper arose in an attempt to extend to the case of three or more circles the problems solved in a previous paper "On Functions determined by their Discontinuities and by a certain form of Boundary Condition," which was printed in Vol. XXII. of the Society's *Proceedings*. By using the method and notation of that paper, a series, defined by the equation

$$w = \frac{A}{z-a} + \frac{A_0}{z-a_0} + \frac{A_1}{z-a_1} + \ldots + \frac{A_n}{z-a_n} + \frac{A_{01}}{z-a_{01}} + \frac{A_{02}}{z-a_{02}}$$

$$+ \ldots + \frac{A_{012}}{z-a_{012}} + \ldots + \ldots,$$

may be formed, which, if convergent, will represent in the space external to $n+1$ circles, each of which is external to all the others, a function with a single infinity at a, and whose imaginary part has constant values at the circles.

If $\left(z, \frac{\alpha_1 z+\beta_1}{\gamma_1 z+\delta_1}\right) \ldots \left(z, \frac{\alpha_n z+\beta_n}{\gamma_n z+\delta_n}\right)$ are the n linear substitutions equivalent to pairs of inversions at the circles 0 and 1, 0 and 2, ... 0 and n, respectively, and $\left(z, \frac{\alpha_i z+\beta_i}{\gamma_i z+\delta_i}\right)$ any substitution of the group formed from these as fundamental substitutions, it may be directly verified that the above series is

$$w = A \sum_i \frac{(\gamma_i a+\delta_i)^{-2}}{z - \frac{\alpha_i a+\beta_i}{\gamma_i a+\delta_i}} + A_0 \sum_i \frac{(\gamma_i a_0+\delta_i)^{-2}}{z - \frac{\alpha_i a_0+\beta_i}{\gamma_i a_0+\delta_i}};$$

and that it is therefore a convergent series, and does in fact represent the function in question. An indirect proof of the accuracy of this statement will be now given, so that the direct one, which, though simple, cannot be made very short, may be omitted.

Consider, in the first case of the symmetrical group, namely, that for which the $(n+1)^{\text{th}}$ circle is taken as the real axis, the function

$$A\psi_a(z) + A'\psi_{a'}(z),$$

where A, A', and a, a' are conjugate imaginaries.

Regarded as a function of z, this expression has entirely real coefficients, and therefore will take conjugate imaginary values when z does.

Now let z be any point on the circle C_p, so that z' is the corresponding point on $C_{p'}$. Then, if

$$A\psi_a(z) + A'\psi_{a'}(z) = P+iQ,$$

$$A\psi_a(z') + A'\psi_{a'}(z') = P-iQ;$$

but $$z = \frac{\alpha_p z' + \beta_p}{\gamma_p z' + \delta_p},$$

and therefore

$$A\psi_a(z)+A'\psi_{a'}(z)-\left[A\psi_a(z')+A'\psi_{a'}(z')\right]=A\theta(a, J_p)+A'\theta(a', J_p),$$

or
$$iQ=\tfrac{1}{2}A\theta(a, J_p)+\tfrac{1}{2}A'\theta(a', J_p).$$

It follows that, at each separate circular bounding curve of the generating polygon, the imaginary part of

$$A\psi_a(z)+A'\psi_{a'}(z)$$

has constant values.

If, now, the transformation described in Section 7 be applied to this function, it becomes

$$A(\gamma b+\delta)^2\psi_b(z)+A'(\gamma b_0+\delta)^2\psi_{b_0}(z),$$

where the ψ-functions are formed with the substitutions of the new group, while b, b_0 are inverse points with respect to the $(n+1)^{\text{th}}$ circle, which itself is the result of transforming the real axis by the substitution $\left(z, \dfrac{az+\beta}{\gamma z+\delta}\right)$ of the transformation.

The relations between b, b_0 and a, a' are

$$a=\frac{ab+\beta}{\gamma b+\delta},\quad a'=\frac{ab_0+\beta}{\gamma b_0+\delta};$$

or, since a, a' are conjugate imaginaries,

$$\frac{ab_0+\beta}{\gamma b_0+\delta}=\frac{a'b'+\beta'}{\gamma'b'+\delta'};$$

and therefore
$$\frac{db_0}{(\gamma b_0+\delta)^2}=\frac{db'}{(\gamma'b'+\delta')^2}.$$

Finally, if
$$A(\gamma b+\delta)^2=B,$$

then
$$A'(\gamma b_0+\delta)^2=B'\left(\frac{\gamma b_0+\delta}{\gamma'b'+\delta'}\right)^2=B'\frac{db_0}{db'};$$

and therefore, for any symmetric group,

$$B\psi_b(z)+B'\frac{db_0}{db'}\psi_{b_0}(z)$$

is a function whose imaginary part has constant values at $n+1$ given circles (each external to all the others) and which has a single infinity, with given residue B, in the space bounded by them.

On substituting for the ψ-functions by means of the equation

$$\psi_b(z) = \theta(b, z_0) - \theta(b, z),$$

and re-writing a and a_0 for b and b_0, the expression just obtained is at once seen to differ only by a constant from the previous function w. A reference to p. 352 of my former paper, already quoted, will show that $A_0 = A' \frac{da_0}{da'}$, which is the necessary relation between the coefficients.

If the point a is at infinity, so that a_0 is the centre of the $(n+1)^{\text{th}}$ circle,

$$w = A\psi_\infty(z) + A' r_0^2 \psi_{a_0}(z),$$

or, written at length,

$$w = \sum_i \left(A \left\{ \frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i} - \frac{\alpha_i z_0 + \beta_i}{\gamma_i z_0 + \delta_i} \right\} + A' r_0^2 \left\{ \frac{1}{\frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i} - a_0} - \frac{1}{\frac{\alpha_i z_0 + \beta_i}{\gamma_i z_0 + \delta_i} - a_0} \right\} \right);$$

and the real and imaginary parts of w will be respectively the velocity-potential and stream-function for a uniform streaming motion past the $n+1$ circles, the speed of the stream at an infinite distance, and the angle it makes with the real axis, being mod. A and arg. A.

The n functions $\phi_1, \phi_2 \ldots \phi_n$, the group being still considered symmetrical, are the functions of z whose imaginary and real parts give the velocity-potentials and the stream-functions for the n independent circulating motions that can take place about the $n+1$ circles, on the supposition that the circulation in any circuit enclosing all the $n+1$ circles is zero.

For, if $\kappa_1, \kappa_2 \ldots \kappa_{n+1}$ are the circulation constants for the $n+1$ circles, connected by the relation

$$\kappa_1 + \kappa_2 + \ldots + \kappa_{n+1} = 0,$$

then the function

$$\kappa_1\phi_1 + \kappa_2\phi_2 + \ldots + \kappa_n\phi_n$$

is everywhere finite in the space external to the $n+1$ circles, while its real part is constant at each circle, and increases by κ_r when the variable describes a closed path which surrounds the r^{th} circle once. But these are the conditions that $u+iv$ should satisfy if u is the stream-function and $-v$ the velocity-potential of the proposed circulating motion.

The two examples just given will serve to show that the theory of automorphic functions of the first class may serve to elucidate considerably many two-dimensional physical problems. With respect to the general problem dealt with in the previously quoted paper, viz., the formation of a function w with given infinities, such that the imaginary part of $we^{i\theta_r}$ is constant at the circumference of the circle C_r ($r = 1, 2, \ldots n+1$), it will be found that, even when the differences $\theta_r - \theta_s$ are commensurable with π, the terms in the infinite series for w which contain the same exponential do not arise from one of them by the operations of any sub-group of the original group; and therefore, that w *cannot* be represented as the sum of a finite number of ψ-functions formed from the substitutions of sub-groups in a manner analogous to that proved to be possible when the group arose from a single fundamental substitution.

Note on the Motion of a Fluid Ellipsoid under its own Attraction.

By M. J. M. Hill, M.A., Sc.D., Professor of Mathematics at University College, London. Communicated in Abstract, June 11th, 1891. Received January 27th, 1892.

1. In regard to this case of Fluid Motion, it was shown by Dirichlet that the particles which, at any instant, lie on an ellipsoid concentric, similar, and similarly situated to the external free surface, always do so; or, one family of surfaces, which always contain the same particles, is a family of ellipsoids which move so as always to be similar and similarly situated to the boundary.

The following additional results regarding this case of Fluid Motion were recently communicated to me by Mr. A. E. H. Love.

2. The particles which, at any instant, lie on a tangent plane to one of the ellipsoids concentric, similar, and similarly situated to the free surface, always lie on a tangent plane to this moving ellipsoid, and the same particle is always at the point of contact.

3. The enveloping cylinders of the ellipsoids mentioned above, whose generators are parallel to the vortex lines, always contain the same particles. Hence there are two families of surfaces, viz., the ellipsoids and the enveloping cylinders, which move so as to contain the same particles, and are such that one particular member of one always touches one particular member of the other.