For the third degree we should have, writing for brevity, 12 instead of $\begin{matrix}\mu, & \nu \\ 1, & 2\end{matrix}$

$$\begin{matrix}\lambda, \mu, \nu \\ 2, 3, 4\end{matrix} \quad \begin{matrix}\lambda, \mu, \nu \\ 3, 4, 1\end{matrix} \quad \begin{matrix}\lambda, \mu, \nu \\ 4, 1, 2\end{matrix} \quad \begin{matrix}\lambda, \mu, \nu \\ 1, 2, 3\end{matrix}$$

$$\begin{array}{l} = (\ \ . \quad \lambda_2 34 + \lambda_3 42 + \lambda_4 23) \\ \ \ \ (\lambda_1 34 \quad . \quad + \lambda_3 41 + \lambda_4 13) \\ \ \ \ (\lambda_1 24 + \lambda_2 41 \quad . \quad + \lambda_4 12) \\ \ \ \ (\lambda_1 23 + \lambda_2 31 + \lambda_3 12 \quad . \ ) \end{array} = \begin{array}{cccc} . & 34, & -24, & 23 \\ -34, & . & 14, & 31 \\ 24, & -14, & . & 12 \\ -23, & -31, & -12, & . \end{array} = -1 . 2 . 3 . 4$$

$$= (23 . 14 + 31 . 24 + 12 . 34)^2.$$

From this it is easily seen that such a product can always be reduced to a skew symmetrical determinant; and consequently that, when the number (n) of columns is odd, the product will vanish. When the number is even, the product will

$$= -1 . 2 \ldots n. \qquad \text{(XV.)}$$

On the Transformation of Gauss' Hypergeometric Series into a Continued Fraction. By Thomas Muir, M.A., F.R.S.E.

[*Read 10th February,* 1876.]

Gauss, in his *Disquisitio circa seriem infinitam**, viz., the series

$$1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)}{1 . 2}.\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{1 . 2 . 3}.\frac{\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \ldots,$$

or $F(\alpha, \beta, \gamma, x)$, established a simple proposition regarding it, and from this was able to express

$$\frac{F(\alpha, \beta+1, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)} \text{ in the form } \cfrac{1}{1 - \cfrac{a_1 x}{1 - \cfrac{a_2 x}{1 - \ldots}}}$$

where $a_1, a_2, \ldots$ are functions of α, β, γ; and the continued fraction so obtained for any particular series was found to be quite different in form from that got by using the previously established general method of Euler. The object of the present short paper is to place on as firm

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* *Vide* Abhandl. der Götting. Gesellsch. d. Wissensch. II., 1812, and Werke, t. iii., p. 125.

a basis the general theorem* (recently discovered by the author) of which this is a case, that is to say, the theorem in which the series concerned is any series with coefficients following no specified law.

Beginning with the series

$$B_0+B_1x+B_2x^2+B_3x^3+\ldots,\quad\text{or}\quad f_0,$$
$$C_0+C_1x+C_2x^2+C_3x^3+\ldots,\quad\text{or}\quad f_1,$$

and subtracting C_0 times the first from B_0 times the second, and dividing the result by x, we have

$$\begin{vmatrix} B_0 & B_1 \\ C_0 & C_1 \end{vmatrix} + \begin{vmatrix} B_0 & B_2 \\ C_0 & C_2 \end{vmatrix} x + \begin{vmatrix} B_0 & B_3 \\ C_0 & C_3 \end{vmatrix} x^2 + \ldots\ldots,\quad\text{or}\quad f_2.$$

Next, subtracting $\begin{vmatrix} B_0 & B_1 \\ C_0 & C_1 \end{vmatrix}$ times the second series from C_0 times this third series, and dividing by x, there results

$$\begin{vmatrix} B_0 & B_1 & B_2 \\ 0 & C_0 & C_1 \\ C_0 & C_1 & C_2 \end{vmatrix} + \begin{vmatrix} B_0 & B_1 & B_3 \\ 0 & C_0 & C_2 \\ C_0 & C_1 & C_3 \end{vmatrix} x + \begin{vmatrix} B_0 & B_1 & B_4 \\ 0 & C_0 & C_3 \\ C_0 & C_1 & C_4 \end{vmatrix} x^2 + \ldots\ldots,\quad\text{or}\quad f_3.$$

Again, subtracting $\begin{vmatrix} B_0 & B_1 & B_2 \\ 0 & C_0 & C_1 \\ C_0 & C_1 & C_2 \end{vmatrix}$ times the third series from $\begin{vmatrix} B_0 & B_1 \\ C_0 & C_1 \end{vmatrix}$ times the fourth, and dividing by C_0x, we find

$$\begin{vmatrix} B_0 & B_1 & B_2 & B_3 \\ 0 & B_0 & B_1 & B_2 \\ 0 & C_0 & C_1 & C_2 \\ C_0 & C_1 & C_2 & C_3 \end{vmatrix} + \begin{vmatrix} B_0 & B_1 & B_2 & B_4 \\ 0 & B_0 & B_1 & B_3 \\ 0 & C_0 & C_1 & C_3 \\ C_0 & C_1 & C_2 & C_4 \end{vmatrix} x + \begin{vmatrix} B_0 & B_1 & B_3 & B_5 \\ 0 & B_0 & B_1 & B_4 \\ 0 & C_0 & C_1 & C_4 \\ C_0 & C_1 & C_2 & C_5 \end{vmatrix} x^2 + \ldots,\quad\text{or}\quad f_4;$$

for, performing the operations, except the division by C_0x, upon the second terms of the series, we have

$$\begin{vmatrix} B_0 & B_1 \\ C_0 & C_1 \end{vmatrix} \cdot \begin{vmatrix} B_0 & B_1 & B_3 \\ 0 & C_0 & C_2 \\ C_0 & C_1 & C_3 \end{vmatrix} x - \begin{vmatrix} B_0 & B_1 & B_2 \\ 0 & C_0 & C_1 \\ C_0 & C_1 & C_2 \end{vmatrix} \cdot \begin{vmatrix} B_0 & B_2 \\ C_0 & C_2 \end{vmatrix} x,$$

$$\text{which} = \begin{vmatrix} B_0 & B_1 & 0 & 0 & B_2 \\ C_0 & C_1 & 0 & 0 & C_2 \\ 0 & B_2 & B_0 & B_1 & B_3 \\ 0 & C_1 & 0 & C_0 & C_2 \\ 0 & C_2 & C_0 & C_1 & C_3 \end{vmatrix} x = \begin{vmatrix} B_0 & B_1 & 0 & B_0 & B_2 \\ C_0 & C_1 & 0 & C_0 & C_2 \\ 0 & B_2 & B_0 & B_1 & B_3 \\ 0 & C_1 & 0 & C_0 & C_2 \\ 0 & C_2 & C_0 & C_1 & C_3 \end{vmatrix} x = C_0 \begin{vmatrix} B_0 & B_1 & B_2 & B_3 \\ 0 & B_0 & B_1 & B_2 \\ 0 & C_0 & C_1 & C_2 \\ C_0 & C_1 & C_2 & C_3 \end{vmatrix}$$

and the performance of the same operations upon the other terms of

* *Vide* Transactions of Roy. Soc. of Edin., 1875–76.

the series differs from this only in there being higher suffixes to the letters of the last column of all the determinants, except of course those which are the multipliers.

Continuing this process of deriving functions, the divisor next time being $\begin{vmatrix} B_0 & B_1 \\ C_0 & C_1 \end{vmatrix} x$, we have

$$\begin{vmatrix} B_0 & B_1 & B_2 & B_3 & B_4 \\ 0 & B_0 & B_1 & B_2 & B_3 \\ 0 & 0 & C_0 & C_1 & C_2 \\ 0 & C_0 & C_1 & C_2 & C_3 \\ C_0 & C_1 & C_2 & C_3 & C_4 \end{vmatrix} + \begin{vmatrix} B_0 & B_1 & B_2 & B_3 & B_5 \\ 0 & B_0 & B_1 & B_2 & B_4 \\ 0 & 0 & C_0 & C_1 & C_3 \\ 0 & C_0 & C_1 & C_2 & C_4 \\ C_0 & C_1 & C_2 & C_3 & C_5 \end{vmatrix} x + \begin{vmatrix} B_0 & B_1 & B_2 & B_3 & B_6 \\ 0 & B_0 & B_1 & B_2 & B_5 \\ 0 & 0 & C_0 & C_1 & C_4 \\ 0 & C_0 & C_1 & C_2 & C_5 \\ C_0 & C_1 & C_2 & C_3 & C_6 \end{vmatrix} x^2 + \ldots, \text{ or } f_5,$$

and so on.

Hence, denoting the first terms of the series by $\theta_0, \theta_1, \theta_2, \ldots$ respectively, we have the following equations:—

$$\theta_0 f_1 - \theta_1 f_0 = x f_2,$$
$$\theta_1 f_2 - \theta_2 f_1 = x f_3,$$
$$\theta_2 f_3 - \theta_3 f_2 = \theta_1 x f_4,$$
$$\theta_3 f_4 - \theta_4 f_3 = \theta_2 x f_5,$$
$$\ldots \quad \ldots \quad \ldots$$
$$\theta_{n-1} f_n - \theta_n f_{n-1} = \theta_{n-2} x f_{n+1};$$

whence

$$\frac{f_0}{f_1} = \frac{\theta_0}{\theta_1} - \frac{\frac{1}{\theta_1} x}{\frac{f_1}{f_2}},$$

$$\frac{f_1}{f_2} = \frac{\theta_1}{\theta_2} - \frac{\frac{1}{\theta_2} x}{\frac{f_2}{f_3}},$$

$$\frac{f_2}{f_3} = \frac{\theta_2}{\theta_3} - \frac{\frac{\theta_1}{\theta_3} x}{\frac{f_3}{f_4}},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{f_{n-1}}{f_n} = \frac{\theta_{n-1}}{\theta_n} - \frac{\frac{\theta_{n-2}}{\theta_n} x}{\frac{f_n}{f_{n+1}}};$$

and therefore

$$\frac{f_1}{f_0} = \cfrac{1}{\frac{\theta_0}{\theta} - \cfrac{\frac{1}{\theta_1}x}{\frac{\theta_1}{\theta_2} - \cfrac{\frac{1}{\theta_2}x}{\frac{\theta_2}{\theta_3} - \cfrac{\frac{\theta_1}{\theta_3}x}{\frac{\theta_3}{\theta_4} - \dots - \cfrac{\frac{\theta_{n-2}}{\theta_n}x}{\frac{f_n}{f_{n+1}}}}}}};$$

so that we have

$$\frac{C_0+C_1x+C_2x^2+\dots}{B_0+B_1x+B_2x^2+\dots} = \cfrac{\theta_1}{\theta_0 - \cfrac{\theta_2 x}{\theta_1 - \cfrac{\theta_3 x}{\theta_2 - \cfrac{\theta_1\theta_4 x}{\theta_3 - \dots - \cfrac{\theta_{n-3}\theta_n x}{\theta^{n-1} - \frac{\theta_{n-2}x f_{n+1}}{f_n}}}}}} \quad \text{.........(I.)},$$

where $\theta_0 = B_0$, $\theta_1 = C_0$, $\theta_2 = \begin{vmatrix} B_0 & B_1 \\ C_0 & C_1 \end{vmatrix}$, &c.

Putting $B_0 = 1$, $B_1 = B_2 = \dots\dots = 0$, we find that

$$\theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \theta_5,\ \theta_6, \dots\dots$$

become

$$C_0,\ C_1,\ \begin{vmatrix} C_0 & C_1 \\ C_1 & C_2 \end{vmatrix},\ \begin{vmatrix} C_1 & C_2 \\ C_2 & C_3 \end{vmatrix},\ \begin{vmatrix} C_0 & C_1 & C_2 \\ C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \end{vmatrix},\ \begin{vmatrix} C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \end{vmatrix},\ \dots\dots$$

and denoting these by $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \dots\dots$,
we have

$$C_0+C_1x+C_2x^2+C_3x^3+\dots = \cfrac{\gamma_0}{1 - \cfrac{\gamma_1 x}{\gamma_0 - \cfrac{\gamma_2 x}{\gamma_1 - \cfrac{\gamma_1\gamma_3 x}{\gamma_2 - \dots - \cfrac{\gamma_{n-2}\gamma_{n+1}x}{\gamma_n - \dots}}}}} \quad \text{.........(II.)}$$

Again, putting $C_0 = 1$, $C_1 = C_2 = \dots\dots = 0$ in (1), we find, on writing

$$\beta_3, \beta_4, \beta_5, \ldots\ldots \text{ for } \begin{vmatrix} B_1 & B_2 \\ B_2 & B_3 \end{vmatrix}, \begin{vmatrix} B_2 & B_3 \\ B_3 & B_4 \end{vmatrix}, \begin{vmatrix} B_1 & B_2 & B_3 \\ B_2 & B_3 & B_4 \\ B_3 & B_4 & B_5 \end{vmatrix}, \ldots\ldots$$

$$B_0+B_1x+B_2x^2+B_3x^3+\ldots = B_0+\cfrac{B_1x}{1-\cfrac{B_2x}{B_1-\cfrac{\beta_3x}{B_2-\cfrac{B_1\beta_4x}{\beta_3-\ldots-\cfrac{\beta_{n-2}\beta_{n+1}x}{\beta_n-\ldots}}}}} \quad \ldots\ldots\ldots\text{(III.)},$$

as is also easily seen from (II.).

These results involve many interesting particular examples not hitherto obtained, and some even of those which have been obtained it is more simple to consider as cases of (I.), (II.), or (III.), than to deal with them as at present is commonly done.

Gauss' proposition above referred to is

$$F(\alpha,\beta+1,\gamma+1,x)-F(\alpha,\beta,\gamma,x) = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}F(\alpha+1,\beta+1,\gamma+2,x) \quad \ldots\ldots\ldots(1);$$

and the result thence deduced is

$$\frac{F(\alpha,\beta+1,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)} =$$

$$= \cfrac{\gamma}{\gamma-\cfrac{\alpha(\gamma-\beta)x}{\gamma+1-\cfrac{(\beta+1)(\gamma+1-\alpha)x}{\gamma+2-\cfrac{(\alpha+1)(\gamma+1-\beta)x}{\gamma+3-\cfrac{(\beta+2)(\gamma+2-\alpha)x}{\gamma+4-\ldots}}}}} \quad \ldots\ldots\ldots\text{(A)}.$$

With this process and result of Gauss the following may be profitably compared:—

Putting $$F(\alpha,\beta,\gamma,x) = y,$$

the differential equation

$$(x^2-x)\frac{d^2y}{dx^2}+\left\{(\alpha+\beta+1)x-\gamma\right\}\frac{dy}{dx}+\alpha\beta y = 0$$

is easily established; and since

$$\frac{dy}{dx} = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1,x),$$

and $$\frac{d^2y}{dx^2} = \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}F(\alpha+2,\beta+2,\gamma+2,x),$$

it follows that

$$(\alpha+1)(\beta+1)(x^2-x)\,F(\alpha+2,\beta+2,\gamma+2,x)$$
$$+\{(\alpha+\beta+1)x-\gamma\}\,F(\alpha+1,\beta+1,\gamma+1,x)+\gamma(\gamma+1)\,F(\alpha,\beta,\gamma,x)=0 \quad \text{.........(2)},$$

which, being like (1) a relation between three of the functions, we can make use of in the same way. The result is

$$\frac{F(\alpha+1,\beta+1,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)}= \quad \text{.................(B)}$$

$$=\cfrac{\gamma}{\gamma-(\alpha+\beta+1)x+\cfrac{(\alpha+1)(\beta+1)(x-x^2)}{\gamma+1-(\alpha+\beta+3)x+\dots}}$$

$$\dots+\cfrac{(\alpha+n)(\beta+n)(x-x^2)}{(\gamma+n)-(\alpha+\beta+2n+1)x+\cfrac{(\alpha+n+1)(\beta+n+1)(x-x^2)}{(\gamma+n+1)\frac{F_{n+1}}{F_{n+2}}}},$$

where F_{n+1} stands for $F(\alpha+n+1,\beta+n+1,\gamma+n+1,x)$.

The expressions transformed into continued fractions in (A) and (B) are not the same, but, putting $\beta=0$ in (A), we have

$$F(\alpha,1,\gamma+1,x)= \quad \text{.........}(a);$$

$$=\cfrac{\gamma}{\gamma-\cfrac{\alpha\gamma x}{\gamma+1-\cfrac{1.(\gamma+1-\alpha)x}{\gamma+2-\cfrac{(\alpha+1)(\gamma+1)x}{\gamma+3-\cfrac{2(\gamma+2-\alpha)x}{\gamma+4-\dots}}}}}$$

and, putting $\beta=0$ in (B), and then changing $\alpha+1$ into α, we find the same expression

$$=\cfrac{\gamma}{\gamma-\alpha x+\cfrac{1.\alpha(x-x^2)}{\gamma+1-(\alpha+2)x+\cfrac{2(\alpha+1)(x-x^2)}{\gamma+2-(\alpha+4)x+\dots}}} \quad \text{.........}(b).$$

It is well known that the right-hand member of (A) may be continued *ad infinitum*, the correction to be added to any partial denominator in order to make the two numbers identical being of no account in the limit. The same, however, cannot be said regarding (B), and an instructive case worth noticing is where $\beta=\gamma$.

[It having been inadvertently stated, in the author's absence from the meeting at which the foregoing was read, that no law of formation was apparent in the determinants made use of for the expression of the continued fraction found as the equivalent of

$$\frac{C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots}{B_0 + B_1 x + B_2 x^2 + B_3 x^3 + \dots},$$

the determinant of the n^{th} order, denoted by θ_n in (I.), is herewith given, viz.:

$$\begin{vmatrix} B_0 & B_1 & B_2 & B_3 & \dots\dots & B_{n-1} \\ 0 & B_0 & B_1 & B_2 & \dots\dots & B_{n-2} \\ 0 & 0 & B_0 & B_1 & \dots\dots & B_{n-3} \\ \dots & \dots & \dots & \dots & \dots\dots & \dots \\ \dots & \dots & \dots & \dots & \dots\dots & \dots \\ 0 & 0 & C_0 & C_1 & \dots\dots & C_{n-3} \\ 0 & C_0 & C_1 & C_2 & \dots\dots & C_{n-2} \\ C_0 & C_1 & C_2 & C_3 & \dots\dots & C_{n-1} \end{vmatrix}$$

where the number of rows of B's is the same as the number of rows of C's, or greater than it by one, according as n is even or odd.

It is also worth while to add another continued fraction for comparison and contrast with that given in (I.) above. On using the method of Undetermined Coefficients to find the expansion of

$$\frac{C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots}{B_0 + B_1 x + B_2 x^2 + B_3 x^3 + \dots}$$

in ascending powers of x, the coefficients sought arise naturally as determinants from the solution of the equations to which we are led. Transforming this expansion into a continued fraction by Euler's method, we have

$$\frac{C_0 + C_1 x + C_2 x^2 + \dots}{B_0 + B_1 x + B_2 x^2 + \dots} = \cfrac{\delta_1}{B_0 - \cfrac{B_0 \delta_2 x}{B_0 \delta_1 + \delta_2 x - \cfrac{B_0 \delta_1 \delta_3 x}{B_0 \delta_2 + \delta_3 x - \cfrac{B_0 \delta_2 \delta_4 x}{B_0 \delta_3 + \delta_4 x - \dots}}}}$$

where δ_n is the determinant

$$\begin{vmatrix} B_0 & B_1 & B_2 & B_3 & \dots\dots & B_{n-1} \\ 0 & B_0 & B_1 & B_2 & \dots\dots & B_{n-2} \\ 0 & 0 & B_0 & B_1 & \dots\dots & B_{n-3} \\ \dots & \dots & \dots & \dots & \dots\dots & \dots \\ \dots & \dots & \dots & \dots & \dots\dots & \dots \\ 0 & 0 & 0 & 0 & \dots\dots & B_1 \\ C_0 & C_1 & C_2 & C_3 & \dots\dots & C_{n-1} \end{vmatrix}$$

22nd Sept., 1876.]