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THE SEQUENCE OF THEOREMS IN SCHOOL GEOMETRY.*

BY PROF. T. P. NUNN, D.Sc.

THE subject of my address has recently been "ventilated" in more than one organ of educational opinion; and it is stated that a committee will shortly be assembled to explore the possibility of an escape from the present chaos to the sweet simplicity of an agreed and authoritative sequence.

I know nothing about the constitution of the committee nor about the proposals that are likely to be brought before it; nor had it been announced or foreshadowed when I suggested the subject for discussion to-night. My intention in suggesting that subject was merely to revive and develop further certain proposals brought forward in an address given to the Association† at a time when many of its members were engaged in a vastly more serious discussion elsewhere. I venture, however, to hope that the circumstances I have referred to may add materially to the usefulness of our debate.

I assume as common ground that the school course in geometry should show two main divisions: (1) a heuristic stage in which the chief purpose is to order and clarify the spatial experiences which the pupil has gained from his everyday intercourse with the physical world, to explore the more salient and interesting properties of figures, and to illustrate the useful applications of geometry, as in surveying and "Mongean" geometry; (2) a stage in which the chief purpose is to organise into some kind of logical system the knowledge gained in the earlier stage and to develop it further. In the first stage obvious truths (such as the transversal properties of parallel lines) are freely taken for granted, and deduction is employed mainly to derive from them important and striking truths (such as the constancy of the angle-sum of a triangle) which are not forced upon us by observation. The second stage is marked by an attempt, more or less thorough-going and "rigorous," to explore the connexions between geometrical truths and to exhibit them as the logical consequences of a few simple principles.

About the first stage I shall say nothing except to urge (i) that its range should be liberal, including the simpler truths of tri-dimensional geometry and the properties of figures, such as the conic sections, which were excluded from the Euclidean canon, and (ii) that it should occupy the pupil until he is mature enough really to profit by the second stage—which I interpret as

* The substance of a lecture to the Bristol Branch of the Mathematical Association, 17th March, 1922. Some replies on points raised in the discussion have been incorporated.

† At the Annual Meeting of January, 1917.

meaning that he is not more than two years from the General Schools (or Matriculation) examination. It is upon the second stage that I wish to concentrate attention.

The central purpose of this stage is, we have said, to develop the logic of geometry. A simple illustration may make that purpose clear. Let $ABCD$ be a quadrilateral in which the angles at A, B, C, are right angles by construction; then it may be proved that D is also a right angle. What does the word "prove" mean here? It does not mean that argument is needed to make a boy believe the statement, for there is hardly anything more obviously true. What it means is that the rectangularity of D can be shown not to be an isolated fact but a logical consequence of truths (the fundamental congruence-theorem and the parallel postulate) which he has already accepted. In other words, the argument is not for conviction, but to bring out the logical structure of this particular region of geometrical truth.* Indeed, the example illustrates not inaptly Russell and Whitehead's dictum that we have often more reason to believe our axioms true because true consequences flow from them than to believe in the consequences because they flow from the axioms.

Now, if a boy is to gain profit from logical geometry he must in the first place have a degree of mental maturity which is rarely reached before adolescence, and in the second place its purpose must be carefully explained to him. Good teachers, no doubt, always do explain it, though the requirement is ignored in most of the text-books. But if the explanation is to be really satisfactory it must, I submit, be more philosophical than is usually the case. Pray do not jib at the word philosophical. I mean nothing worse than this: that we should take a good deal of pains to make our pupils realise clearly the logical architecture of the geometrical system. Treated in a sufficiently broad and concrete way, the subject is a fascinating one, appealing strongly to a boy's curiosity and imagination. If he does not get a reasonable amount of intellectual satisfaction out of it, we have a clear indication that he ought not to be doing logical geometry at all.

What is the "logical architecture" of the ordinary geometrical system? It has three main features. The first consists of certain fundamental properties of points, lines and planes—for instance, the fact that a plane is determined by three non-collinear points and that two planes intersect in a straight line. These are the "foundations of geometry," and, as you know, have been the object of an immense amount of patient and subtle scrutiny in recent years. The second feature comprises the axioms and theorems about congruence, especially the congruence of triangles. The third is, in the Euclidean system, some form of the parallel postulate—now-a-days usually the axiom (improperly) called Playfair's. About the last feature I shall shortly make a proposal which is the *fons et origo* of this discourse. But let us proceed towards it in an orderly way.

The study of the foundations of geometry presupposes a logical faculty far more developed than it can generally be in the boy of fourteen or fifteen. It should be taken up, if at all, after matriculation, and in that place I shall briefly consider it. At the beginning of the logical stage it must suffice to call attention to the obvious properties of lines and planes and to point out that they are to be assumed in what follows. The usual theorems about the angles between intersecting lines form a natural appendix to the discussion.

In Euclid's *Elements* the theorems about the congruence of triangles occur at intervals in the first book. Modern text-books rightly bring them together and by that means emphasise their importance and their significance for the

* I do not, of course, deny that the logical coherence of a geometrical system fortifies our belief in the truth of all its parts; my point is merely that this result of a logical inquiry into geometry is not the main reason why we undertake it. To avoid another possible source of misunderstanding, I add that I deliberately ignore here, as too abstract for the school-boy, the standpoint of the truly "pure" geometer. The geometry I have in view is the scientific study of actual space.

geometrical argument. But it is still usual to follow Euclid in proving them by the method of superposition, notwithstanding the severity with which modern geometers have criticised it. Mr. Bertrand Russell, who asserts that it "strikes every intelligent child as a juggle," possibly overestimates the intelligence of children, but does not exaggerate its defects as a principle of proof. The following argument indicates the substance of his objections.

Let two houses be built of the same materials and from the same plans, one (say) in Bristol, the other in Melbourne, and suppose the genius of Aladdin's lamp, in sportive mood, to remove the Bristol house one night and to replace it by the one from Melbourne. Then, though the occupants would no doubt receive a severe shock next morning, passers by would be quite unaware that anything had happened. The argument from superposition says we may conclude that because the Melbourne house, when it reaches Bristol, exactly fills the place of the Bristol house, it filled a space of exactly the same size and shape before it left Melbourne. However true the conclusion may be, it certainly does not follow from the premiss. The fact that the Melbourne house replaces perfectly the one at Bristol proves at most that the same plans carried out twice over *on the same spot*, produce results geometrically indistinguishable; it cannot prove that the Melbourne house, while still in Australia, was "congruent" with its fellow at Bristol. Our belief in that congruence has, surely, an earlier and deeper foundation. Suppose that you were in Melbourne and in the confidence of the humorous demon. Then, if you knew that the two houses had been built from the same plans, you would be certain, *before the event*, that the Melbourne house would exactly replace the one at Bristol. And, if it proved not to do so, you would doubt, not the basis of your conviction, but the workmanship of the builders.

Considerations of this kind suggest that in the interests of clear thinking we should give up the argument from superposition, and place the logical treatment of congruence on its real foundations. Those, I submit, are: (1) the (assumed) possibility that a figure, occurring anywhere, may be exactly repeated anywhere else, and (2) the (assumed) fact that certain elementary constructions, such as drawing a line through two given points, measuring off a given length along a ray, or setting off an angle of given magnitude, can be carried out in only one way. Fusing these assumptions into one, we have the Principle of Congruence: namely, that figures produced by combining the aforesaid elementary constructions in any given (unambiguous) manner are all geometrically equivalent—or, in simpler language, that if a given geometrical construction can be carried out in only one way it produces equivalent figures whether carried out *here* or *there*. For instance, on one side of a given line AB, of length c, it is clearly possible to construct only one triangle ABC, such that AC is of given length b, and the angle between AB and AC of a given magnitude A. It follows from the principle of congruence that triangles drawn to this specification must be equivalent or congruent wherever they may be.

D. Hilbert, in his well-known book,* follows this principle, but limits it to the following axiom: If in two triangles $A'B'=AB$, $A'C'=AC$ and $\angle A'=\angle A$, then $\angle B'=\angle B$ and $\angle C'=\angle C$. From this assumption the equality of BC and $B'C'$ follows and (eventually) all the other congruence-theorems which do not involve the parallel postulate.† I venture to think Hilbert's limitation too drastic for school use. A boy will gain what I have called a more philosophical view of geometry if he is taught to apply the principle in the broader form. For example, it should be used to prove not only the whole of Euclid I. 4, but also the first case of Euclid I. 26, though it

* *The Foundations of Geometry.* A translation is published by the Open Court Publishing Company.

† He might equally well have started with the assumption that if $AB=A'B'$, $\angle A=\angle A'$, and $\angle B=\angle B'$, then $AC=A'C'$ and $BC=B'C'$. From this it follows that $\angle C=\angle C'$, and Euclid I. 4 can also be deduced.

is a valuable exercise to show subsequently that *either* of these theorems can be deduced from the other. For convenience of reference let the former be called Con. I., the latter Con. II. Then Con. III. (Euclid, I. 5) can at once be deduced from Con. I., and Con. V. (Euclid, I. 8) from Con. I. and III. together. For symmetry's sake I leave a space for Con. IV. (Euclid, I. 6). Personally, I like to deduce this from Con. II. as Con. III. is deduced from Con. I.; the proof used being in each case suggested by Hilbert's proof of Euclid, I. 5. With these five theorems in our hands, we have all the tools needed to develop geometry as far as the principle of congruence alone can support it.

It is, I submit, of great importance to bring out clearly that though the principle of congruence accounts for many striking properties of figures, it does not account for all. For instance, it does not explain the properties of parallelograms, nor Pythagoras's theorem, nor why the angles of a triangle always add up to 180°, nor even the perfectly obvious fact that all the angles of a square must be right angles. For more than two thousand years mathematicians sought to bring these properties within its purview, but were always baffled. Finally, at the beginning of the nineteenth century, it became clear that the problem is insoluble, and that the intractable truths to which I have referred depend upon a second great property of space, additional to and in that sense independent of the property which makes congruent figures possible.

Now what I specially wish to discuss to-night is the question how this second great property should be formulated. Euclid, of course, expressed it in his parallel postulate ("Postulate 5" or "Axiom 12"), and so started a tradition which has been almost universally followed. To challenge a policy laid down by so great a man and hallowed by centuries of acceptance is, I admit, an audacious act. I venture, however, humbly to suggest that Euclid might have served the world better if he had followed another line, and I am about to urge that we should take that line now. I shall argue the question from the standpoint of one whose main interest is in the *teaching* of geometry, but I believe there is a great deal to be said for the proposal from the purely scientific point of view.

I will begin by stressing again the importance of making boys understand clearly the architecture of geometry. From the teaching standpoint it is a serious weakness of Euclid's procedure that he introduces the parallel postulate only to prove the *converse* of a theorem, and thereafter never (I think) mentions it again. It is proved (I. 27) that if a transversal cuts two lines at the same angle the lines cannot meet; but, to get on, we must also know that, if they do not meet, any transversal cuts them both at the same angle. It is to guarantee this conclusion (I. 29) that Axiom 12 (or Playfair's equivalent) makes its solitary appearance. It needs little psychology to see that this procedure does not give the axiom a fair chance. Everyone knows how slow boys are to be convinced that, although a primary proposition has been proved, the truth of its converse remains an open question. It follows that the entrance of the new axiom into the geometrical scheme is psychologically inconspicuous, and that the momentous consequences of admitting it are not clearly realised. And, as I have said, the case is made worse by the fact that, having once admitted it, we never have occasion to notice its presence explicitly again.

Now, from the list of truths not deducible from the principle of congruence, we have omitted by far the most important instance: namely, the existence and properties of similar figures. That figures of very different sizes may yet have exactly the same shape is a fact borne in upon a child from his earliest hours. His mother, as she approaches his cradle, is presented (to use the psychological term) as a series of such figures; it needs no argument or persuasion to make him recognise the cat and the dog in his picture-book; and at a later stage he accepts maps and plans as the most natural things in the world.* In short, the main facts about similar figures are so familiar, so interesting and so useful in their applications that most good teachers now give them a conspicuous place in the heuristic stage of geometry.† I cannot see how it can be denied that they are most clearly entitled to it and that it is a serious error to ignore them. If teachers were wholly free to obey their teaching-sense, they would probably give similarity a still more important position in their schemes of work. What chiefly deters them is the unhappy tradition which postponed the *logical* treatment of the subject until so late a point that the majority of boys never reached it. Though the Euclidean proofs are gone, the tradition still operates.‡ Its evil effects will not wholly disappear unless we give to the principle of similarity a place in logical geometry corresponding to its psychological importance and its value as an instrument of investigation. That is what I wish to do. Instead of deducing the existence of similar figures from the (assumed) existence of parallels, I propose that we shall deduce the existence and properties of parallels from the (assumed) existence of similar figures. I make the proposal for the reasons explained in this paragraph; and I shall proceed to defend it by arguing (i) that the change would avoid the weakness pointed out in the preceding paragraph, *i.e.* would make the architecture of elementary geometry much clearer, and (ii) that the proofs needed are at least as easy as the proofs they replace, and are sometimes much simpler.

With regard to the first point. The best way to show that one property of space is independent of another is to invent a kind of space in which the latter exists but the former is absent. That is, in effect, what Lobatchevsky did to prove the independence of the parallel postulate. His argument is far too difficult for ordinary school-boys,§ but it is easy to show that congruence may exist where similarity is absent. Take a sphere of such a size that a degree along its equator measures one inch, and draw on it a triangle composed of great circular arcs measuring respectively 6, 8 and 10 inches. (Great circular arcs are not, of course, straight lines, but they correspond to them inasmuch as they mark out the shortest path between two points on a sphere.) If a triangle with sides of these lengths were drawn on a flat sheet of paper, the angles would be 36° 52′, 53° 8′ and 90°, of which the sum is exactly 180°. On the globe the corresponding angles are 36° 54′, 53° 10′ and 90° 2′, making a total of 180° 6′. Thus the triangle on the sphere is slightly different in form from the triangle on the flat; but, like the latter, it would have exactly the same shape wherever drawn. That is to say, the surface of a sphere resembles a plane in admitting endless repetition of the same figure.

The important difference comes into view when we attempt to reproduce the triangle on a different scale—making the sides, for instance, four times as long. On the flat sheet this can be done without disturbing the shape of the triangle, but on the sphere it is otherwise. For if the sides were lengthened to 24, 32 and 40 inches, the angles would be increased to 39° 13′, 55° 28′ and

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* The difficulty with regard to maps is, indeed, to persuade him that they are not merely reduced diagrams of the areas they represent.

† The work should include the enlargement and reduction of drawings and a simple treatment of perspective and should incidentally teach the correct technical use of the term "similar." Mr. Fawdry pointed out in the discussion that boys will call all ellipses (for instance) "similar." This, I suggest, is because, according to the ordinary meaning of the word, they *are* similar—just as all triangles are. A new technical term, without misleading associations, would be acceptable. Would *homomorphic* or *identiform* be too alarming?

‡ Some of the Universities now permit candidates for matriculation to refer proofs to the principle of similarity. This is a great advance. It tends to give similarity the place here claimed for it, and it enables teachers to substitute for Euclid's cumbrous proofs of I. 47, III. 35, 36, the simple arguments recommended long ago in Mr. W. C. Fletcher's text-books —from which many of us, in our early days, learnt a great deal. One must also refer gratefully to the help given to the cause by authorities in the service of the Board of Education whose position compels them to be anonymous.

§ I concur with Mr. Carson's opinion (*Mathematical Education*, p. 104) that everyone who proceeds to a University "should gain some slight idea of the nature of non-Euclidean geometry," and I submit that what follows here is not a bad introduction to the subject.

92° 19′, making a total of 187°. Thus the surface of a sphere, being uniform, allows of congruent figures yet does not possess the property which makes similar figures possible. It is clear, therefore, that there is no necessary connexion between the properties of congruence and of similarity. Space as we know it appears to possess both, but it might conceivably have had the first without also having the second.

In conformity with this conclusion it is, then, proposed to organise the whole of geometry on the basis of two (assumed) properties of space. Expressed in popular language they are:

(1) A given figure can be exactly reproduced anywhere.

(2) A given figure can be reproduced anywhere on any (enlarged or diminished) scale.

No further property of space need be assumed, for no property of figures has ever been discovered which cannot be derived from one or both of these.

It cannot, I submit, be denied that logical geometry, built up in the way proposed, would gain greatly in clearness and symmetry. There is no visible kinship between the postulate of congruence and the parallel postulate; but the postulate of similarity resembles and supplements the postulate of congruence in a way that is both obvious and gratifying to the aesthetic sense.

We turn now to the proofs, which are to be based on the axiom that, given a rectilinear figure and any straight line, it is always possible to construct on the given straight line a figure similar to the given figure. Armed with this axiom we can easily show that there are three sets of conditions for similarity between triangles, corresponding, one by one, to the conditions for congruence. The proofs all follow the same lines, so that it will suffice to give the first.

Let $A'B'$ and $A'C'$ have the same ratio to AB and AC respectively, and let $\angle A' = \angle A$. Then we are to prove that $\angle B' = \angle B$, $\angle C' = \angle C$, and $B'C' : BC = A'B' : AB$. Take any line $A''B'' = A'B'$. By the axiom there can be drawn on it a triangle $A''B''C''$ similar to ABC. In that triangle $\angle A'' = \angle A = \angle A'$, and $A''C'' : AC = A''B'' : AB = A'B' : AB = A'C' : AC$; whence $A''C'' = A'C'$. It follows that the triangles $A''B''C''$ and $A'B'C'$ are congruent; so that $\angle B' = \angle B'' = \angle B$, $\angle C' = \angle C'' = \angle C$, and

$$B'C' : BC = B''C'' : BC = A''B'' : AB = A'B' : AB.$$

You will observe that I have said nothing about the commensurability of the magnitudes. As a matter of fact, I think Legendre was right in maintaining that the measurement of ratios is a question for arithmetic, and that we are not necessarily called upon to discuss it in geometry. If we give the proofs of similarity now current in text-books we *must* say something about it; for those proofs deliberately make the false assumption that all magnitudes of the same kind are commensurable. But I count it one of the merits of the proof given above that it does not require us to deal with the question at all. Even if a "rigorous" argument is insisted on, it is easy to meet the requirement by prefacing the above proof with a few axioms embodying the properties of ratios without any reference to their measurement. Nevertheless something must be said somewhere about the measurement of ratios, and it may be conveniently said here and, perhaps, take the following form. If two quantities of the same kind, P and Q, contain respectively p and q units exactly, the ratio of their magnitudes is measured by the fraction p/q. But it is a rare thing to find a quantity which contains the unit an exact number of times, however small the unit may be. What we actually find is, as a rule, that P's magnitude is between p and $p+1$ units, Q's between q and $q+1$. In this case (which, I repeat, is the usual one) we cannot measure the ratio exactly; we can only say that it is between $(p+1)/q$ and $p/(q+1)$. If R and S are two other quantities, we can similarly determine that their ratio lies between $(r+1)/s$ and $r/(s+1)$. Now if there are any fractions which lie between the members both of the first pair and of the second, it is clearly possible that the two ratios *may* be the same ;* and we shall conclude that they *are* the same if such fractions can always be found however small the unit is taken. This is, in fact, the only definition of equal ratios which can be practically applied. Having explained it, you may, if you please, go on to show that there are quantities whose magnitudes cannot, even conceivably, be both measured exactly in terms of the same unit ; but the discussion would be a luxury which might well be postponed.

When the fundamental congruence theorems have been proved there is much to be said for going straight on to the corresponding similarity theorems, so that the whole substructure may be laid down before we proceed to build on it. On the other hand, it is important to separate clearly the properties which can be deduced by the congruence principle alone from those which are deducible only with the aid of the principle of similarity. Thus there is also something to be said for postponing the study of similarity for a while. But this is purely a question of expediency, which might affect the numbering of theorems (if it should be decided to give them official numbers), but could not affect their logical order.

The standard theorems deducible from the congruence theorems alone fall naturally into two groups. I should enunciate those of the first group in the following order. (i) If the rays AP and BQ, on the same side of AB, make the angles PAB and QBA together equal to two right angles, then the rays do not meet. (ii) The same may be proved if the angles are together greater than two right angles. (iii) If the lines do meet the angles are together less than two right angles. (*N.B.*—The converse cannot be proved.) (iv) The exterior angle of a triangle is greater than either of the interior opposite angles.

I choose this order because, as Hilbert suggests, (i) can be proved by direct application of the first congruence theorem and because the remaining proofs then become simpler than Euclid's. Moreover, the order seems "prettier" than his. It will, however, be shown below that the whole group can be deduced still more easily from the second similarity theorem, so that the only ground for taking it at this stage is to show how far the authority of the principle of congruence extends.

The second of the two groups consists of Euclid, I. 18-20 and 24. These all depend on (iv) above and therefore could be postponed until similarity has been treated.

It is much more important for my purpose to formulate the standard theorems immediately derived from the fundamental similarity theorems; for, as I have already said, they are to include the doctrine of parallels. I suggest the following order. (i) If any transversal cuts two lines at the same angle those lines do not meet. (This is, of course, equivalent to (i) above.) (ii) On the same supposition, any transversal that intersects the former one also cuts the lines at the same angle. (iii) The angle-sum of a triangle is two right angles. (iv) A triangle can be constructed with angles equal to any three whose sum is two right angles. (v) If two lines in a plane do not meet, any transversal must cut them both at the same angle. (vi) Through a given point there can be drawn only one line which will not meet a given line in the same plane with it. (Playfair's Axiom.)

I will give the proofs of (i) and (v) (Euclid, I. 27, 29). You may find it amusing to work out the others for yourselves.

(i) Let the transversal ABC cut the lines L and M in B and C, and suppose L and M to meet in N. Then the triangles ABN, ACN have two equal angles and are therefore similar. Hence $\angle ANB = \angle ANC$, which is impossible; so the lines cannot meet.

(v) Let $\angle ABL$ be greater than $\angle ACM$; then the interior angles CBL and BCM are together less than two right angles. Let the defect from two right

* This would, for example, be the case if the first pair were 3·463 ... and 3·482 ... while the second pair were 3·471 ... and 3·502 ..., but not if the second pair were 3·483 ... and 3·496 ....

angles be $\angle R$. Then by (iv) a triangle can be constructed whose angles are respectively equal to $\angle B$, $\angle C$ and $\angle R$. Moreover, by the postulate of similarity, a triangle similar to this one can be constructed on BC. But this means that BL and CM must meet, which is contrary to the hypothesis. Hence $\angle ABL$ cannot be greater than $\angle ACM$, etc.

The substitution of a postulate of similarity for the parallel postulate is, as I have admitted, an audacious step. I am anxious, therefore, to plead for it the authority of great names. The first is John Wallis who, in seeking to "demonstrate the fifth postulate of Euclid," followed a method very different from mine but based upon the same assumption.* A much greater authority, Laplace, definitely held that a postulate of similarity is more natural than Euclid's parallel postulate, and considered this view to be confirmed by the remarkable fact that Nature appears to take no account of absolute size, but applies her mechanical laws indifferently to systems almost infinitely big and little. This argument has gained still more force since Laplace's day. Lastly, since this paper was read, Prof. Coulichère of Petrograd has kindly pointed out to me that W. K. Clifford, in his fragmentary but brilliant *Common Sense of the Exact Sciences*, has adumbrated a treatment of geometry similar in principle to the one here advocated. I find (I cannot candidly say with entire pleasure !) that he anticipated the proof I have just given of Euclid, I. 29.

At the end of so long an address only a brief reference can be made to the "foundations of geometry." My view about this part of the subject is that it should be dealt with after matriculation. At that stage it is possible to treat it in accordance with the modern spirit—which has progressed, though many seem not to know it, far beyond Euclid. In short, I recommend a course of the utmost "rigour" based on the work of Pasch, Hilbert, Peano, Veblen and our own Russell and Whitehead. No boy who finds the work unattractive should be compelled to take it, but my experience is that to lads of 17-19 it is often extraordinarily stimulating and interesting. T. P. NUNN.

GLEANINGS FAR AND NEAR.

121. Licensing of the Press. One of these gentlemen (who have never printed their names but to their licenses), said to a geometrician : "I cannot permit the publication of your book : you dare to say, that, between two given points, the shortest line is the straight line. Do you think me such an idiot as not to perceive your allusion ? If your work appeared, I should make enemies of all those who find, by crooked ways, an easier admittance into court, than by a straight line. Consider their number !" At this moment the censors in Austria appear singularly inept ; for, not long ago, they condemned as heretical, two books, one of which, entitled *Principes de la Trigonométrie*, the censor would not allow to be printed, because the *Trinity*, which he imagined to be included in trigonometry, was not permitted to be discussed ; and the other, on the *Destruction of Insects*, he insisted had a covert allusion to the Jesuits. . . . Malebranche could not get a license for his *Recherches après la Verité* until Mezeray approved of it as a work on Geometry. . . .—Disraeli, *Curiosities of Literature*, p. 254.

122. Mr. Shirley (alias Dr. Shirley) . . . subsists, as other authors must expect, by a sort of Geometry.—Dunton's *Life and Errors*, i. 185.

* It wil be of interest to quote his actual words : "Praesumo tandem . . . ut communem notionem
Datae unicunque Figurae, Similem aliam cujuscunque magnitudinis possibilem esse.
Hoc enim (propter quantitates continuas in infinitum divisibiles, pariter atque in infinitum augibiles) videtur ipsa Quantitatis natura fluere ; figuram scilicet quamlibet continue posse (retenta figurae specie) tam minui, tam augeri in infinitum." From *Opera* (1693), vol. ii. p. 674. I have followed up the references in Bonola's invaluable *Non-Euclidean Geometry* (Open Court Series).