$$\frac{(-ik)^p}{1.2\ldots\ldots p}\,b^{p-1}\,\frac{\frac{p-1}{2}\,\frac{p-3}{2}\ldots\ldots\frac{p-2n+1}{2}}{1.2\ldots\ldots n}\left(-\frac{2r}{b}\mu\right)^n$$

$$=\frac{(-ikb)^p}{1.2\ldots\ldots p}\,\frac{(p-1)(p-3)\ldots\ldots(p-2n+1)}{1.2\ldots\ldots n}\,(-1)^n\frac{r^n\mu^n}{b^{n+1}},$$

so that

$$\mathrm{H}=(-1)^n\frac{r^n}{1.2\ldots n.b^{n+1}}\sum_{p=-1}^{p=\infty}\frac{(p-1)(p-3)\ldots(p-2n+1)}{1.2.3\ldots\ldots p}(-ikb);$$

and from (104)

$$\mathrm{A}_n''=(-1)^n\frac{2n+1}{b\,(ikb)^n}\sum_{p=-1}^{p=\infty}\frac{(p-1)(p-3)\ldots\ldots(p-2n+1)}{1.2.3\ldots p}(-ikb)^p \quad\ldots\ldots\ldots (139).$$

The value of ϕ due to a source at a distance b from the pole is thus determined by (134), (139). The remainder of the investigation proceeds as before.*

On the Summation of Certain Series. By Prof. WOLSTENHOLME.

(*Read January 9th*, 1873.)

Suppose a person to draw successively from a number n of different vessels, each containing white and black balls, and let p_r be the chance that a drawing from the rth vessel gives a white ball, then his whole expectation of drawing a white ball is

$$1-(1-p_1)(1-p_2)(1-p_3)\ldots\ldots(1-p_n);$$

for he must either draw a white ball sometime or other, or fail in every attempt. But the sum of his several expectations at the first, second, ... nth drawings, is

$$p_1+(1-p_1)\,p_2+(1-p_1)(1-p_2)\,p_3+\ldots+(1-p_1)(1-p_2)\ldots(1-p_{n-1})\,p_n;$$

whence we have the algebraical identity

$$p_1+(1-p_1)\,p_2+(1-p_1)(1-p_2)\,p_3+\ldots+(1-p_1)(1-p_2)\ldots(1-p_{n-1})\,p_n$$
$$=1-(1-p_1)(1-p_2)\ldots(1-p_n).$$

This method of proof assumes that $p_1, p_2, \ldots$ are all less than 1, but it is obvious that such a restriction is altogether unnecessary. In fact, this identity, when obtained, is obviously true, but it is one which includes several series of interest, some examples of which are appended.

$$(1.)\ \frac{n}{m+n}+\frac{m}{m+n}\,\frac{n}{m+n-1}+\frac{m\,(m-1)\,n}{(m+n)(m+n-1)(m+n-2)}+\ldots$$
$$\ldots\text{ to }(m+1)\text{ terms}\equiv 1,$$

* This Paper appears as it was originally read before the Society, without alterations other than purely verbal.

or

$$\frac{1}{m+n}+\frac{m}{(m+n)(m+n-1)}+\frac{m(m-1)}{(m+n)(m+n-1)(m+n-2)}+\ldots\equiv\frac{1}{n}.$$

(2.) $$\frac{n}{m+n}+\frac{m}{m+n}\,\frac{n}{m+n+1}+\frac{m(m+1)}{(m+n)(m+n+1)}\,\frac{n}{m+n+2}+\ldots$$

$$\ldots \text{ to } r \text{ terms} \equiv 1-\frac{m(m+1)\ldots(m+r-1)}{(m+n)(m+n+1)\ldots(m+n+r-1)};$$

or $$1+\frac{m}{m+n+1}+\frac{m(m+1)}{(m+n+1)(m+n+2)}+\ldots\ldots \text{ to } r \text{ terms}$$

$$\equiv\frac{m+n}{n}\left\{1-\frac{m(m+1)\ldots(m+r-1)}{(m+n)(m+n+1)\ldots(m+n+r-1)}\right\};$$

or $$1+\frac{m}{m+n}+\frac{m(m+1)}{(m+n)(m+n+1)}+\ldots\ldots \text{ to } r \text{ terms}$$

$$\equiv\frac{m+n-1}{n-1}\left\{1-\frac{m(m+1)\ldots(m+r-1)}{(m+n-1)(m+n)\ldots(m+n+r-2)}\right\}.$$

(3.) $$\frac{p-q}{p}+\frac{q}{p}\,\frac{p-q}{p+r}+\frac{q(q+r)}{p(p+r)}\,\frac{p-q}{p+2r}+\frac{q(q+r)(q+2r)}{p(p+r)(p+2r)}\,\frac{p-q}{p+3r}$$

$$+\ldots\ldots \text{ to } n \text{ terms} \equiv 1-\frac{q(q+r)(q+2r)\ldots(q+\overline{n-1}r)}{p(p+r)(p+2r)\ldots(p+\overline{n-1}r)};$$

or $$1+\frac{q}{p+r}+\frac{q(q+r)}{(p+r)(p+2r)}+\frac{q(q+r)(q+2r)}{(p+r)(p+2r)(p+3r)}+\ldots\ldots$$

$$=\frac{p}{p-q}\left\{1-\frac{q(q+r)(q+2r)\ldots(q+\overline{n-1}r)}{p(p+r)\ldots(p+\overline{n-1}r)}\right\};$$

or $$1+\frac{q}{p}+\frac{q(q+r)}{p(p+r)}+\frac{q(q+r)(q+2r)}{p(p+r)(p+2r)}+\ldots\ldots \text{ to } n \text{ terms}$$

$$=\frac{p-r}{p-q-r}\left\{1-\frac{q(q+r)\ldots(q+\overline{n-1}r)}{(p-r)p(p+r)\ldots(p+\overline{n-2}r)}\right\};$$

whence we see that this well known series is convergent when $p>q+r$, and divergent when $p<q+r$; also when $p=q+r$ it becomes the common harmonical series, and is divergent.

In the case when this series is convergent, we get

$$1+\frac{q}{p}+\frac{q(q+r)}{p(p+r)}+\ldots\ldots \text{ to } \infty=\frac{p-r}{p-q-r},$$

$$\frac{q}{p}\left(1+\frac{q+r}{p+r}+\frac{(q+r)(q+2r)}{(p+r)(p+2r)}+\ldots\ldots\right)=\frac{q}{p}\,\frac{p}{p-q-r},$$

$$\frac{q(q+r)}{p(p+r)}\left(1+\frac{q+2r}{p+2r}+\ldots\ldots\right)=\frac{q(q+r)}{p(p+r)}\,\frac{p+r}{p-q-r},$$

&c. = &c.;

therefore $1+2\frac{q}{p}+3\frac{q(q+r)}{p(p+r)}+4\frac{q(q+r)(q+2r)}{p(p+r)(p+2r)}+\dots$ to ∞

$$=\frac{1}{p-q-r}\left\{p-r+q\left(1+\frac{q+r}{p}+\frac{(q+r)(q+2r)}{p(p+r)}+\dots\right)\right\}$$

$$=\frac{1}{p-q-r}\left\{p-r+q\frac{p-r}{p-q-2r}\right\}=\frac{(p-r)(p-2r)}{(p-q-r)(p-q-2r)},$$

provided that $p > q+2r$. These results suggest the general law.

Assuming that, when m is an integer, and $p > q+mr$,

$$1+m\frac{q}{p}+\frac{m(m+1)}{2}\frac{q(q+r)}{p(p+r)}+\frac{m(m+1)(m+2)}{\underline{|3}}\frac{q(q+r)(q+2r)}{p(p+r)(p+2r)}+\dots \text{ to } \infty$$

$$=\frac{(p-r)(p-2r)\dots(p-mr)}{(p-q-r)(p-q-2r)\dots(p-q-mr)},$$

we shall then have

$$\frac{q}{p}\left(1+m\frac{q+r}{p+r}+\frac{m(m+1)}{\underline{|2}}\frac{(q+r)(q+2r)}{(p+r)(p+2r)}+\dots \text{ to } \infty\right)$$

$$=\frac{q}{p}\cdot\frac{p(p-r)\dots p-(m-1)r}{(p-q-r)\dots(p-q-mr)},$$

$$\frac{q(q+r)}{p(p+r)}\left(1+m\frac{q+2r}{p+2r}+\dots \text{ to } \infty\right)$$

$$=\frac{q(q+r)}{p(p+r)}\cdot\frac{(p+r)p\dots p-(m-2)r}{(p-q-r)\dots(p-q-mr)},$$

&c. = &c.;

or adding,

$$1+(m+1)\frac{q}{p}+\frac{(m+1)(m+2)}{\underline{|2}}\frac{q(q+r)}{p(p+r)}$$

$$+\frac{(m+1)(m+2)(m+3)}{\underline{|3}}\frac{q(q+r)(q+2r)}{p(p+r)(p+2r)}+\dots$$

$$=\frac{(p-r)(p-2r)\dots(p-mr)}{(p-q-r)(p-q-2r)\dots(p-q-mr)}$$

$$\times\left\{1+\frac{q}{p-mr}+\frac{q(q+r)}{(p-mr)\{p-(m-1)r\}}+\dots \text{ to } \infty\right\};$$

or if $p-mr > q+r$, and therefore $p > q+(m+1)r$,

$$=\frac{(p-r)(p-2r)\dots(p-mr)}{(p-q-r)(p-q-2r)\dots(p-q-mr)}\cdot\frac{p-mr-r}{p-mr-q-r}$$

$$=\frac{(p-r)(p-2r)\dots\{p-(m+1)r\}}{(p-q-r)(p-q-2r)\dots\{p-q-(m+1)r\}}.$$

Hence, when $p > q+mr$, and m is a whole number, we have the limit of the sum of the series

$$1+m\frac{q}{p}+\frac{m(m+1)}{2}\frac{q(q+r)}{p(p+r)}+\frac{m(m+1)(m+2)}{\lfloor 3}\frac{q(q+r)(q+2r)}{p(p+r)(p+2r)}+\dots \quad \dots \text{ to } \infty$$

equal to

$$\frac{(p-r)(p-2r)\dots(p-mr)}{(p-q-r)(p-q-2r)\dots(p-q-mr)}.$$

Hence, putting $r=1$, we get the result

$$\frac{\Gamma(q)}{\Gamma(p)}+m\frac{\Gamma(q+1)}{\Gamma(p+1)}+\frac{m(m+1)}{\lfloor 2}\frac{\Gamma(q+2)}{\Gamma(p+2)}+\dots \text{ to } \infty$$
$$=\frac{\Gamma(q)}{\Gamma(p)}\frac{\Gamma p}{\Gamma(p-m)}\cdot\frac{\Gamma(p-q-m)}{\Gamma(p-q)};$$

and putting $-m$ for m in this result, and striking out corresponding factors, we have, when m is a whole number,

$$1-m\frac{q}{p}+\frac{m(m-1)}{\lfloor 2}\frac{q(q+1)}{p(p+1)}+\frac{m(m-1)(m-2)}{\lfloor 3}\frac{q(q+1)(q+2)}{p(p+1)(p+2)}-\dots$$
$$\equiv\frac{(p-q)(p-q+1)\dots(p-q+m-1)}{p(p+1)\dots(p+m-1)}$$
$$\equiv\left(1-\frac{q}{p}\right)\left(1-\frac{q}{p+1}\right)\left(1-\frac{q}{p+2}\right)\dots\dots\left(1-\frac{q}{p+m-1}\right).$$

Strictly this is not a proof of the result; but, as used to be continually said of the use of $\sqrt{-1}$, only serves to suggest a result which should be proved independently.

This equality may in this case be very readily proved, either by the method of so-called "induction," or by proving that $q-p$, $q-p-1$, ... are factors of the sinister, since the coefficients of q^m are manifestly the same.

It has, however, been pointed out to me that the simplest proof is by equating coefficients of x^m in the identity

$$(1+x)^{p+q}=(1+x)^q\left(1-\frac{x}{1+x}\right)^{-p}.$$

It may of course be written equally in the form

$$1-m\frac{q}{p}+\frac{m(m-1)}{\lfloor 2}\frac{q(q-1)}{p(p-1)}-\dots \quad \text{(where } m \text{ is a whole number)}$$
$$\equiv\left(1-\frac{q}{p}\right)\left(1-\frac{q}{p-1}\right)\left(1-\frac{q}{p-2}\right)\dots \text{ to } m \text{ factors},$$

in which it appeared as a Senate House Problem in 1869. It is really only another form of Vandermonde's Theorem, which is that, if a_n denote $a(a-1)\dots(a-n+1)$, then

$$(a+b)_n\equiv a_n+n\,a_{n-1}\,b_1+\frac{n(n-1)}{\lfloor 2}a_{n-2}\,b_2+\dots;$$

whence, dividing by a_n,

$$\left(1+\frac{b}{a}\right)\left(1+\frac{b}{a-1}\right)\cdots\cdots\left(1+\frac{b}{a-n+1}\right)$$
$$\equiv 1+n\frac{b}{a-n+1}+\frac{n(n-1)}{\lfloor 2}\frac{b(b-1)}{(a-n+1)(a-n+2)}+\cdots\cdots;$$

or putting $a-n+1=-c$, and reversing the order of the factors,

$$\left(1-\frac{b}{c}\right)\left(1-\frac{b}{c-1}\right)\left(1-\frac{b}{c-2}\right)\cdots\cdots\left(1-\frac{b}{c-n+1}\right)$$
$$=1-n\frac{b}{c}+\frac{n(n-1)}{\lfloor 2}\frac{b(b-1)}{c(c-1)}-\cdots\cdots.$$

I believe I was the first to discover that my question was really only a very well-known theorem, slightly disguised; and I did penance accordingly by appending the proper title to the question when reprinted in the Calendar.

The series which can be brought under the general formula may be written as follows:

$$1+\frac{x}{y+a}+\frac{x(x+a)}{(y+a)(y+b)}+\frac{x(x+a)(x+b)}{(y+a)(y+b)(y+c)}+\ldots \text{ to } r \text{ terms},$$

the sum of which is

$$\frac{y}{y-x}\left\{1-\frac{x(x+a)(x+b)\ldots \text{ to } r \text{ factors}}{y(y+a)(y+b)\ldots \text{ to } r \text{ factors}}\right\}.$$

They are all included in the formulæ given by Gauss in his Memoir "Disquisitiones generales circa seriem infinitam

$$1+\frac{\alpha\,.\,\beta}{1\,.\,\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\,.\,2\,.\,\gamma(\gamma+1)}x^2+\cdots\cdots,"$$

Göttingen Comm. 1812, and Werke, Band iii. See Todhunter's "Algebra," fifth edition, p. 511. The above reference was kindly supplied by Mr. J. W. L. Glaisher.

March 13th, 1873.

Dr. HIRST, F.R.S., President, in the Chair.

Prof. A. G. Greenhill, M.A., of Cooper's Hill College, was elected a Member.

The following papers were read:—

Mr. R. B. Hayward, M.A., "On an extension of the term *Area* to any closed circuit in space."

Mr. J. W. L. Glaisher "On the Evaluation of a class of Definite Integrals involving circular functions in the Numerator and powers of the variable only in the Denominator."

Mr. S. Roberts, Vice-President, "Note on Normals and the Surface of Centres of an Algebraical Surface."

Mr. M. Jenkins (Hon. Sec.), "Proof of the proposition that a number which divides the product of two numbers, and is prime to one of them, will divide the others."

In the course of a discussion on Mr. Hayward's paper, Mr. Merrifield remarked that if a cone of revolution has its vertex on the surface of a sphere, and passes through the centre of a sphere, one of the projections of the intersection is a common parabola; and that the projection by lines parallel to the axis of the cone on a plane at right angles to it is Pascal's Limaçon, reducing to the cardioid when the angle at the vertex of the cone is a right angle.

The following presents were received:—

"On the Contact of Surfaces" (read before the Royal Society, Feb. 22nd, 1872); and "On some Recent Generalizations of Algebra," by Mr. W. Spottiswoode, F.R.S., Vice-President: from the Author.

"A Complete Practical Treatise on the Nature and Use of Logarithms, and on Plane Trigonometry, with Logarithmic and Trigonometrical Tables," by James Elliot (5th edition), with Key (3rd edition): from the Publishers.

"Monatsbericht," Nov. and Dec. 1872.

"Annali di Matematica," serie ii[a]., tom. v., fasc. 3°., Nov. 1872.

"Bulletin de la Société Mathématique de France," publié par les Sécrétaires, No. 1: from the Society.

"Journal of London Institution," No. 19.

"Bulletin des Sciences Mathématiques et Astronomiques," Feb. 1873.

"Papers relating to the Transit of Venus in 1874, prepared under the direction of the Commission authorised by Congress, and published by authority of the Hon. Sec. of the Navy;" Part ii.; New York, 1872. In duplicate.

"Reports on Observations of Encke's Comet during its return in 1871," by Asaph Hall and Wm. Harkness; Washington, 1872.

"On the Right Ascensions of the Equatorial Fundamental Stars, and the corrections necessary to reduce the Right Ascensions of different Catalogues to a mean homogeneous system," by Simon Newcomb: from U. S. Royal Observatory, Washington.

"Bulletins de l'Académie Royale des Sciences, des Lettres et des Beaux Arts de Belgique," 2[e] série, tomes 31—34.

Ditto, Annuaires de 1872 et 1873: Centième Anniversaire de Fondation, tomes i°. and ii°.; also, Tables de Mortalité, Notices extraites de l'Annuaire de l'Observatoire pour 1873; Notice sur Babbage: from M. A. Quetelet.

Salmon's "Higher Plane Curves" (2nd edition): from the Author.

"Crelle," 75 Band, 4[es] Heft, Feb. 1873.

"Journal of the Institute of Actuaries and Assurance Magazine," No. xc., Jan. 1873.

"Bulletin des Sciences Mathématiques et Astronomiques," tome troisième, Dec. 1872; tome quatrième, Mars 1873.

"Extrait du Bulletin des Sciences Mathématiques et Astronomiques" (from tom. iv. 1873).

Prof. J. Clerk Maxwell on "Electricity and Magnetism" (2 vols.): from the Author.

On an Extension of the term "Area" to any closed Circuit in Space. *By* R. B. Hayward, M.A.

[*Read March 13th*, 1873.]

De Morgan has shown (*Cam. and Dub. Math. Journ.*, vol. v.) that the term "Area" may be applied in a perfectly definite sense to any closed circuit in one plane, however *autotomic* its character, and that it may be regarded as having an algebraical sign determined by the direction in which the circuit is supposed to be described.

I propose to show that there is, for any closed circuit in space, a plane area determined by the circuit itself without any extrinsic elements, which is definite as well in the *aspect* of its plane as in magnitude, and which, since it reduces to the area of the circuit in De Morgan's sense when the circuit is plane, may properly be termed the area of the circuit in the general case.

In this sense, area is no longer a mere magnitude, or a magnitude affected only with the positive or negative sign, but a magnitude affected with direction; in other words, it is a *vector*, not simply a *scalar*.

Conceive a point to describe continuously a given closed circuit in space of three dimensions. Then its projection on any given plane will describe a closed circuit in that plane, the area of which will have a definite sign depending on the convention which is made as to the positive direction of revolution on the positive aspect of the plane. Let the positive aspects of the rectangular coordinate planes of yz, zx, xy be those which are turned towards the positive sides of the axes of x, y, z respectively, then the projections of the given circuit on the coordinate planes will have areas determinate in sign as well as in magnitude. Let them be denoted by A_x, A_y, A_z respectively. Then, if the direction, cosines of a given plane, that is, the cosines of the angles between its