Page 259, line 9, the following expression should be added to the right-hand side, viz.:

$\dot{a}^2/a^2-(w-w')\left\{(a-b)\,w-(a+b)\,w'\right\}/a+(v-v')\left\{(c-a)\,v+(c+a)\,v'\right\}/a.$

Page 261, line 6, *read* $w=w'=\tfrac{1}{2}\zeta$.

,, ,, equation (23), *read* $3\dot{\alpha}^2/\alpha^4$ *instead of* $3\dot{\alpha}^2/4\dot{\alpha}^4$.

Geometry of the Quartic. *By* R. Russell, M.A.

[*Read Nov.* 10*th*, 1887.]

The system of points, the properties of which I intend to discuss, arose from an attempt to interpret geometrically the sextic covariant of a quartic.

I consider a quartic whose coefficients may be any whatever, real or imaginary. Its roots are of the form a_1+ib_1, a_2+ib_2, a_3+ib_3, a_4+ib_4. These are represented as follows:—Assume any two rectangular axes and take the point whose coordinates are a_1, b_1; that point may be considered to represent the complex quantity a_1+ib_1. We see, therefore, that the four roots of a quartic may be represented by four points in a plane.

I. If α, β, γ, δ be the four roots of the quartic, the factors of the sextic covariant are the numerators of

$$\frac{1}{z-\beta}+\frac{1}{z-\gamma}-\frac{1}{z-\alpha}-\frac{1}{z-\delta};\quad \frac{1}{z-\gamma}+\frac{1}{z-\alpha}-\frac{1}{z-\beta}-\frac{1}{z-\delta};$$

$$\frac{1}{z-\alpha}+\frac{1}{z-\beta}-\frac{1}{z-\gamma}-\frac{1}{z-\delta};$$

z of course denoting a quantity $x+iy$.

Let us consider the roots of the quadratic

$$\frac{1}{z-\beta}+\frac{1}{z-\gamma}-\frac{1}{z-\alpha}-\frac{1}{z-\delta}=0;$$

and let z represent a root of it. Now $z-\alpha$ defines the length and direction of the line joining z and α, and therefore $\dfrac{1}{z-\alpha}$ defines a line whose length is the reciprocal of that line, and whose direction is the re-

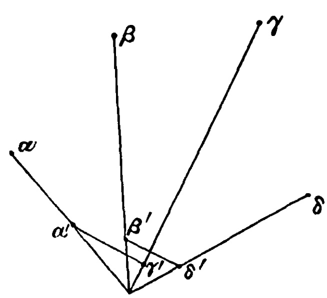

Fig. 1.

flexion of it with respect to axis of x. Consequently, any property which holds for the points

$$\frac{1}{z-\alpha}\cdot\frac{1}{z-\beta}\cdot\frac{1}{z-\gamma}\cdot\frac{1}{z-\delta},$$

will equally hold if these points are taken in the directions $z\alpha$, $z\beta$, $z\gamma$, $z\delta$ respectively. We may, therefore, represent the four quantities

$$\frac{1}{z-\alpha},\ \frac{1}{z-\beta},\ \frac{1}{z-\gamma},\ \frac{1}{z-\delta}$$

by the points α', β', γ', δ'. The above quadratic reduces to $\gamma'-\alpha'=\delta'-\beta'$, showing that the lines γ', α' and δ', β' are equal and parallel. Hence z is such a point that the quadrilateral has inverted into a parallelogram. *If therefore the roots of a quartic be represented by four points in a plane, the roots of the sextic covariant are those six points (all real) from which as origin the quadrilateral inverts into a parallelogram.*

The six points are arranged as follows:—I_1 and J_1 are the points from which the quadrilateral inverts into a parallelogram, the extremities of whose diagonals are the inverses of β, γ and α, δ respectively; I_2, J_2 those from which the extremities of the diagonals are the inverses of γ, α and β, δ; and I_3, J_3 those from which the extremities of the diagonals are the inverses of α, β and γ, δ.

II. If two quadratics $a_1z^2+2b_1z+c_1$, $a_2z^2+2b_2z+c_2$ be connected harmonically (*i.e.*, $a_1c_2+a_2c_1-2b_1b_2=0$), then the two pairs of points representing their roots are concyclic and harmonic. The proof is obvious.

The following properties of the quadratic factors of the sextic covariant are well known.

$$\frac{1}{z-\beta}+\frac{1}{z-\gamma}-\frac{1}{z-\alpha}-\frac{1}{z-\delta}$$

is connected harmonically with $z-\beta\,.\,z-\gamma$, and $z-\alpha\,.\,z-\delta$. Similarly with respect to the other two quadratic factors. Besides, the three factors themselves are, two and two, harmonic. Hence $\beta\gamma I_1J_1$, $\alpha\delta I_1J_1$ are both concyclic and harmonic, and similar properties hold with regard to I_2J_2 and I_3J_3. Also $I_1J_1I_2J_2$, $I_2J_2I_3J_3$, and $I_3J_3I_1J_1$ are concyclic and harmonic. Three circles so related are of course orthogonal, two and two. *Hence the six points "I J" are the points of intersection of three mutually orthogonal circles*, which we shall call the "*IJ*" circles.

Before giving a geometrical construction for these points, I shall establish the following proposition, and give the interpretation of it.

III. There are three homographic transformations which leave the roots of a quartic unchanged.

The most general homographic relation connecting z and ζ is

$$lz\zeta+mz+n\zeta+p=0.$$

If, when $z=\beta,\ \gamma,\ \alpha,\ \delta,\ \zeta=\gamma,\ \beta,\ \delta,\ \alpha$, respectively, then we have

$m=n$, and $\qquad l\beta\gamma+m(\beta+\gamma)+p=0,$

$$l\alpha\delta+m(\alpha+\delta)+p=0,$$

therefore $$\frac{l}{\beta+\gamma-\alpha-\delta}=\frac{m \text{ or } n}{\alpha\delta-\beta\gamma}=\frac{p}{\beta\gamma(\alpha+\delta)-\alpha\delta(\beta+\gamma)},$$

thus determining the homographic relation.

If in the above we put $z=\zeta$, we get a quadratic $lz^2+2mz+p=0$, which gives two quantities absolutely unaltered by the transformation. This quadratic is equivalent to

$$\frac{1}{z-\beta}+\frac{1}{z-\gamma}-\frac{1}{z-\alpha}-\frac{1}{z-\delta}=0.$$

IV. What is the geometrical significance of this?

Since $m=n$, the homographic relation is obviously equivalent to $(z-\theta)(\zeta-\theta)=\phi^2$, where θ and ϕ are constants. Now, denoting z, ζ, and θ by three points in the plane, it is obvious that the product of the distances $z\theta$ and $\zeta\theta$ is equal to the modulus of ϕ^2, and that their directions are equally inclined to direction of ϕ, and therefore the double points IJ are situated on a line through θ in the direction of ϕ, and such that $\theta I^2=\theta J^2=z\theta\,.\,\zeta\theta$.

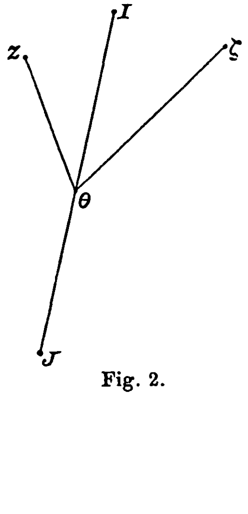

Fig. 2.

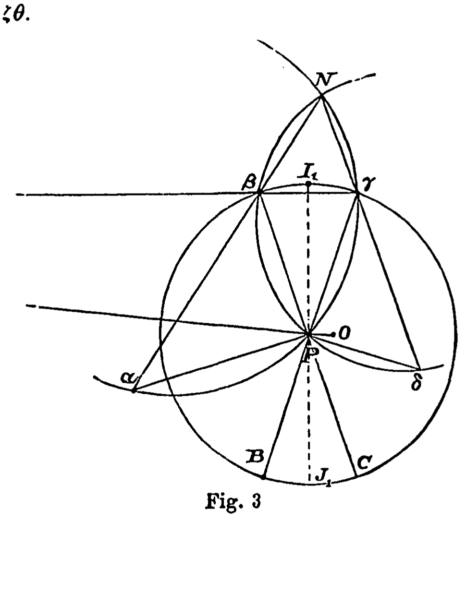

Fig. 3

We are now in a position to give a geometrical construction for the "IJ" points. Let P be the point denoting the quantity θ in the last section, and I_1J_1 the direction of ϕ. When $z = \beta . \gamma . \alpha . \delta$, then $\zeta = \gamma . \beta . \delta . \alpha$; therefore, from the last section, the lines $P\beta$ and $P\gamma$ are equally inclined to I_1J_1, and $P\beta . P\gamma = PI^2$. Exactly the same statement holds with regard to α and δ. We see, therefore, that the triangles $\alpha P\beta$ and $\gamma P\delta$ are similar, and therefore the angle $P\gamma\delta = \beta\alpha P$, and $\alpha\beta P = \gamma\delta P$. *Hence P is determined as follows*:—Produce $\alpha\beta$ and $\gamma\delta$ to meet at N, and describe circles round $\alpha\gamma N$ and $\delta\beta N$; then P is the intersection of these circles.

I_1, J_1 lie on the internal bisector of the angle $\beta P\gamma$, and PI_1 or PJ_1 is a mean proportional between $P\beta$ and $P\gamma$ or between $P\alpha$ and $P\delta$.

I shall in what follows discuss the problem from a geometrical point of view.

V. It is obvious, from Fig. 3, that if we invert with respect to P as origin, and a circle whose radius is PI_1 or PJ_1, we obtain a quadrilateral similar and equal to the given one, and they are reflexions with respect to the line I_1J_1. There are, of course, two other points, Q and R, which possess the same property, and PQ and PR are equally inclined to I_1J_1, and such that $PQ . PR = PI_1^2$.

We shall now see that I_1, J_1 possess the property of inverting the quadrilateral into a parallelogram.

Produce βP and γP so that $PB = P\beta$ and $PC = P\gamma$; then, since

$$PI^2 = PJ_1^2 = P\beta . P\gamma = P\beta . PC = P\gamma . PB,$$

a circle passes through β, γ, I_1, J_1, B, C. Draw PO perpendicular to I_1J_1. Since PI_1 and PO are the internal and external bisectors of the angle $\beta P\gamma$, therefore they meet the line $\beta\gamma$ in two points harmonic conjugates with respect to β and γ, and therefore the lines I_1J_1 and $\beta\gamma$ are conjugates; *i.e.*, each passes through the pole of the other, and therefore the four points β, γ, I_1, J_1 are concyclic and harmonic Similarly α, δ, I_1, J_1 are concyclic and harmonic. Now invert with respect to J_1.

The two circles invert into two lines, and in each line I_1' is the harmonic conjugate of a point at infinity (inverse of J_1) with respect to $\beta'\gamma'$ and $\alpha'\delta'$ respectively, that is, I_1' is simultaneously the middle point of $\beta'\gamma'$ and $\alpha'\delta'$. We thus have a parallelogram.

VI. The triangle PQR is a new *fundamental triangle* related to the quadrilateral. The "IJ" points lie on the internal bisectors of the vertical angles and at distances from the vertices in each case, which are mean proportionals between the conterminous sides.

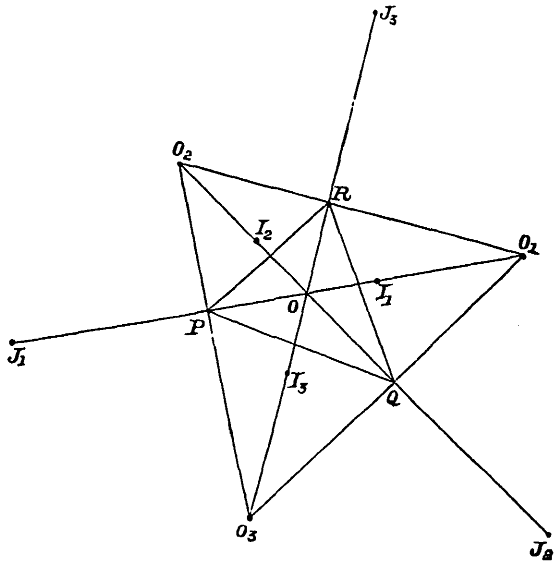

Fig. 4.

Now, having constructed the triangle PQR, let O_1, O_2, O_3, O be the centres of circles touching the sides; then, if we describe circles on O_2O_3, O_3O_1, O_1O_2 as diameters, these circles meet the internal bisectors in the "IJ" points.

VII. The circles $I_2J_2I_3J_3$, $I_3J_3I_1J_1$, $I_1J_1I_2J_2$ are mutually orthogonal, and their centres are at O_1, O_2, O_3,

$$I_1O \,.\, J_1O = PI_1^2 - PO^2 = PQ \,.\, PR - PO^2$$

$$= PO_2 \,.\, PO_3 - PO^2 = PO_1 \,.\, PO - PO^2 = PO \,.\, OO_1.$$

Hence $$I_1O \,.\, J_1O = I_2O \,.\, J_2O = I_3O \,.\, J_3O,$$

therefore $I_2J_2I_3J_3$ lie on a circle, and, since I_3J_3 and I_2J_2 are bisected at right angles by RO_1 and QO_1 respectively, therefore O_1 is the centre of that circle.

The "IJ" circles having O_1 and O_2 as centres are orthogonal, for their radii are O_1J_3 and O_2J_3, and the angle $O_1J_3O_2$ is right; therefore, etc.

VIII. If we invert the four original points α, β, γ, δ and the "IJ" points with respect to any circle, the points still retain their property.

For, since $\beta\gamma I_1J_1$ and $\alpha\delta I_1J_1$ are concyclic and harmonic, their inverses will be so also, and therefore the inverses of the "IJ" points will be "IJ" points of the inverses of the original points.

IX. Given the "IJ" and any one of the four original points—δ, suppose—find the remaining points.

$\alpha\delta I_1J_1$, $\beta\delta I_2J_2$, $\gamma\delta I_3J_3$ are concyclic and harmonic, and therefore, when δ is given, α, β, and γ are *singly determinate.*

I may remark that this shows that, if we are given a sextic which is the sextic covariant of some quartic unknown, then that quartic is of the form $lU+mV$, where U and V are any two quartics satisfying the condition.

X. We shall now consider the properties of another system of *four points* related to the original system in a most remarkable manner. *These are the points from which if we invert, the original four invert into a triangle and its orthocentre.*

The three "IJ" circles have a common orthogonal circle whose centre is at O, and the negative square of whose radius is the value of

$$OP \, . \, OO_1 = OQ \, . \, OO_2 = OR \, . \, OO_3.$$

Let D be the inverse of one of the original points δ with respect to this circle, and let us invert the whole figure with respect to a circle having D as its centre.

Let α', β', γ', δ' be the inverses of α, β, γ, δ with respect to D; then, since D and δ are inverse points with respect to the circle, orthogonal to the "IJ" circles, therefore, from (VIII.), in the inverted figure, δ' will be the centre of the common orthogonal circle; and since $\alpha'\delta' I_1'J_1'$, $\beta'\delta' I_2'J_2'$, $\gamma'\delta' I_3'J_3'$ are collinear and harmonic, therefore α', β', γ' are situated at the centres of the "$I'J'$" circles. This will be obvious if we consider Figure 4. The quadrilateral has therefore inverted into a triangle and its orthocentre.

If therefore, we take the inverses of the four original points with respect to the circle, orthogonal to the "IJ" circles, we obtain four new points from which as origins the quadrilateral inverts into a triangle and its ortho-centre. Denote these four points by the letters A, B, C, D.

The four points A, B, C, D have the same "IJ" points as α, β, γ, δ. This needs no proof.

XI. I shall next prove that A, B, C, D are the inverses of δ, γ, β, α respectively with respect to circle going through $I_2J_2I_3J_3$. I use the same letters as in Fig. 4.

α, δ, I_1, J_1 are concyclic and harmonic. Invert with respect to circle $I_2J_2I_3J_3$ (centre O_1), then A', D', I_1, J_1 are harmonic. Hence I_1J_1 is common segment of harmonic section of $\alpha\delta$, $A'D'$; therefore $A'\alpha$, $D'\delta$ must intersect on that line at a point O which is the harmonic conjugate of O_1 with respect to I_1J_1. But that is exactly the point O in Fig. 4. Hence the points A', D', in Fig. 5, are the same as the points A, D in (X.).

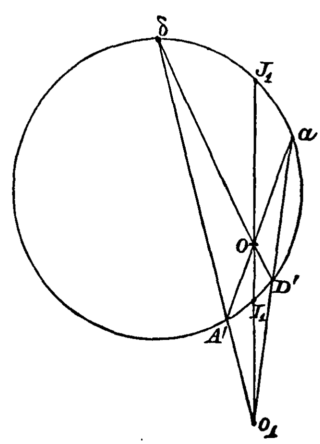

Fig. 5.

Thus we see that A, B, C, D are the inverses of

δ, γ, β, α with respect to circle $I_2J_2I_3J_3$,

γ, δ, α, β with respect to circle $I_3J_3I_1J_1$,

β, α, δ, γ with respect to circle $I_1J_1I_2J_2$,

α, β, γ, δ with respect to common orthogonal circle;

and thus they form two quadrilaterals simultaneously in perspective from four different centres.

I never recollect having come across two quadrilaterals so related before.

XII. If the roots of a cubic be represented by three points in a plane, the roots of its Hessian may be represented by the two points from which the triangle by inversion becomes equilateral.

Let α, β, γ be the roots of the cubic, then

$$\frac{1}{z-\alpha}+\frac{\omega}{z-\beta}+\frac{\omega^2}{z-\gamma} \quad \text{and} \quad \frac{1}{z-\alpha}+\frac{\omega^2}{z-\beta}+\frac{\omega}{z-\gamma}$$

are the factors of its Hessian. We have, therefore, to determine the property of a point in the plane satisfying the condition

$$\frac{1}{z-\alpha}+\frac{\omega}{z-\beta}+\frac{\omega^2}{z-\gamma}=0, \text{ where } \omega^3=1.$$

This may be written

$$\frac{1}{z-\alpha}-\frac{1}{z-\gamma}+\omega\left(\frac{1}{z-\beta}-\frac{1}{z-\gamma}\right)=0,$$

showing that the lines joining $\frac{1}{z-a}$, $\frac{1}{z-\beta}$, $\frac{1}{z-\gamma}$ are inclined at angles of 60°

XIII. If the PQR triangle be equilateral, then, P, Q, R denoting the vertices (considered as complex quantities), we have

$$\overline{Q-R}^2+\overline{R-P}^2+\overline{P-Q}^2=0.$$

But $$P=\frac{\beta\gamma-a\delta}{\beta+\gamma-a-\delta},\quad Q=\frac{\gamma a-\beta\delta}{\gamma+a-\beta-\delta},\quad R=\frac{a\beta-\gamma\delta}{a+\beta-\gamma-\delta},$$

and therefore the above condition amounts to

$$(\beta-\gamma)^2(a-\delta)^2(\beta+\gamma-a-\delta)^4+(\gamma-a)^2(\beta-\delta)^2(\gamma+a-\beta-\delta)^4$$
$$+(a-\beta)^2(\gamma-\delta)^2(a+\beta-\gamma-\delta)^4=0$$

or denoting the roots of $4a^3t^3-Iat+J=0$, by λ, μ, ν,

$$\left(\lambda-\frac{H}{a^2}\right)^2(\mu-\nu)^2+\left(\mu-\frac{H}{a^2}\right)^2(\nu-\lambda)^2+\left(\nu-\frac{H}{a^2}\right)^2(\lambda-\mu)^2=0,$$

or $$a^2I^2+36aJH+12H^2I=0,$$

where $$H\equiv ac-b^2.$$

We see therefore that, if we invert from any of the eight-point roots of $12H^2I-36HUJ+U^2I^2=0$, the "PQR triangle" of the new quartic is equilateral. We see also, from the way in which the "IJ" points are obtained from the "PQR triangle," that $I_1I_2I_3$ and $J_1J_2J_3$ form two equilateral triangles symmetrically arranged, the centre of perspective being their common centre. The arrangement is as in Fig. 6. We may obtain these eight points as follows:—Taking one root or point from each factor of the sextic covariant, we obtain two triangles or cubics. These two cubics have a common Hessian whose roots are two roots of the above equation

$$12H^2I-36HUJ+U^2I^2=0.$$

There are obviously four ways of thus choosing the cubics or triangles, and in each case they will have a common Hessian, thus giving rise to the eight points mentioned. Or, again:—

In Fig. 6 the circles round $I_1I_2I_3$ and $J_1J_2J_3$ are concentric. We must therefore, in the original figure, have inverted from one of the limiting points of the circles round $I_1I_2I_3$ and $J_1J_2J_3$. But there are four ways of thus describing circles, and we arrive, as before, at these same eight points.

The two quartics represented by the expression

$$12H^2I-36HUJ+U^2I^2$$

are the quartics of the system $aU+\beta H$ whose "I" vanishes. In Fig. 6 they are represented by the points P, Q, R, ∞, and $O_1O_2O_3O$.

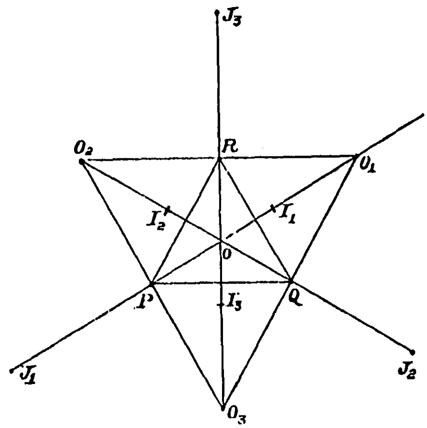

Fig. 6.

There is no difficulty in proving that O_1 and P are the limiting points of the circles $I_2I_3J_1$ and $J_2J_3I_1$.

In the general figure, since the "I" of each of the above quartics vanishes, the four points denoting the roots of either quartic are such that, if from one point we invert, the remaining three become vertices of an equilateral triangle.

An inspection of Fig. 6 will also show that the four points denoting the roots of either quartic are the inverses with respect to the "IJ circles" of the points denoting roots of the other.

XIV. This again leads to a *new canonical form for the quartic and sextic covariant*, which we proceed to find.

Taking origin at O and axis of x along I_1J_1 we have, on denoting the length of the side of triangle PQR by A,

$$OI_1 = a\left(1-\frac{1}{\sqrt{3}}\right), \quad OJ_1 = a\left(1+\frac{1}{\sqrt{3}}\right);$$

therefore
$$I_1 = a\left(1-\frac{1}{\sqrt{3}}\right), \qquad J_1 = -a\left(1+\frac{1}{\sqrt{3}}\right),$$

$$I_2 = a\left(1-\frac{1}{\sqrt{3}}\right)e^{\frac{2}{3}i\pi}, \quad J_2 = -a\left(1+\frac{1}{\sqrt{3}}\right)e^{\frac{2}{3}i\pi},$$

$$I_3 = a\left(1-\frac{1}{\sqrt{3}}\right)e^{\frac{4}{3}i\pi}, \quad J_3 = -a\left(1+\frac{1}{\sqrt{3}}\right)e^{\frac{4}{3}i\pi};$$

therefore
$$(z-I_1)(z-J_1) = z^2+\frac{2az}{\sqrt{3}}-\frac{2a^2}{3},$$

$$(z-I_2)(z-J_2) = z^2+\frac{2az}{\sqrt{3}}\,\omega-\frac{2a^2}{3}\,\omega^2, \text{ where } \omega^3=1,$$

$$(z-I_3)(z-J_3) = z^2+\frac{2az}{\sqrt{3}}\,\omega^2-\frac{2a^2}{3}\,\omega.$$

These are the factors of the sextic covariant. We may obviously put $a = \frac{\sqrt{3}}{\sqrt{2}}$, when the factors become

$$z^2+z\sqrt{2}-1, \quad z^2+z\sqrt{2}\,\omega-\omega^2, \quad z^2+z\sqrt{2}\,\omega^2-\omega;$$

and therefore the sextic covariant is

$$z^6+5\sqrt{2}\,z^3-1=0.$$

The most general form for the quartic corresponding to this form is

$$\lambda\omega\,(z^2+z\sqrt{2}\,\omega-\omega^2)^2+\mu\omega^2\,(z^2+z\sqrt{2}\,\omega^2-\omega)^2,$$

or

$$U \equiv (\lambda\omega+\mu\omega^2)\,z^4+2\sqrt{2}\,(\lambda\omega^2+\mu\omega)\,z^3-2\sqrt{2}\,(\lambda\omega+\mu\omega^2)\,z+(\lambda\omega^2+\mu\omega)$$

$$-2H \equiv (\lambda\omega^2+\mu\omega)^2z^4+2\sqrt{2}\,(\lambda\omega+\mu\omega^2)^2z^3-2\sqrt{2}\,(\lambda\omega^2+\mu\omega)^2z+(\lambda\omega+\mu\omega^2)^2.$$

XV. I have not, as yet, succeeded in obtaining as neat a geometrical interpretation of the Hessian as might be desired. The following is the simplest method I can think of for determining the points denoting the Hessian, being given those which represent the quartic.

1. Given the "*PQR* triangle" and the centre of gravity of the four points representing roots of a quartic, determine the quartic.

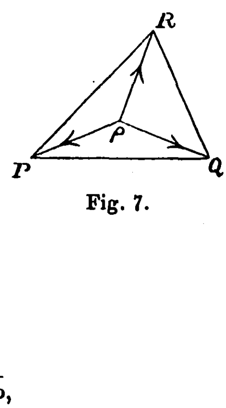

Fig. 7.

$$P = \frac{\beta\gamma-a\delta}{\beta+\gamma-a-\delta}, \quad Q = \frac{\gamma a-\beta\delta}{\gamma+a-\beta-\delta},$$

$$R = \frac{a\beta-\gamma\delta}{a+\beta-\gamma-\delta},$$

and

$$\rho = \tfrac{1}{4}\,(a+\beta+\gamma+\delta).$$

There is no difficulty in proving that

$$\frac{\beta+\gamma-a-\delta}{4} = \sqrt{Q-\rho\,.\,R-\rho},$$

$$\frac{\gamma+a-\beta-\delta}{4} = \sqrt{R-\rho\,.\,P-\rho},$$

$$\frac{a+\beta-\gamma-\delta}{4} = \sqrt{P-\rho\,.\,Q-\rho}.$$

But $\quad \frac{\beta+\gamma-a-\delta}{4}, \quad \frac{\gamma+a-\beta-\delta}{4}, \quad \frac{a+\beta-\gamma-\delta}{4}$

represent the directions and half the lengths of the lines joining the middle points of the opposite connectors of the points α, β, γ, δ. These pass through the centre of gravity.

Hence, if from ρ (Fig. 7) we draw lines bisecting internally the angles $Q\rho R$, $R\rho P$, $P\rho Q$ and on each side of ρ take off mean proportionals between the conterminous sides, we have the six middle points of the opposite connectors of the points α, β, γ, δ. The points α, β, γ, δ are then found by completing the parallelograms.

This leads at once to a construction for the Hessian. For, if ρ' denote centre of gravity of points representing the Hessian, we have

$$\rho'-\rho = -\frac{ad-bc}{2(ac-b^2)}+\frac{2b}{a} = -\frac{a^2d-3abc+2b^3}{2a(ac-b^2)}.$$

Expressing the quantity on right-hand side in terms of $P-\rho$, $Q-\rho$, $R-\rho$, this easily reduces to

$$\frac{3}{\rho'-\rho} = \frac{1}{P-\rho}+\frac{1}{Q-\rho}+\frac{1}{R-\rho}, \quad \text{or} \quad \frac{P-\rho'}{P-\rho}+\frac{Q-\rho'}{Q-\rho}+\frac{R-\rho'}{R-\rho} = 0,$$

showing that, if we invert from ρ (centre of gravity of original quartic), then ρ' inverts into centre of gravity of the triangle $P'Q'R'$. Hence, if we invert round ρ, the centre of gravity of the Hessian is the inverse of the centre of gravity of the triangle $P'Q'R'$. The Hessian is then constructed as in the preceding.

XVI. I shall finish the present paper by proving another property of the "IJ" points. It is (denoting $z-\alpha$, $z-\beta$, $z-\gamma$, $z-\delta$ by U_z) that

$$\int_\beta^{I_1 \text{ or } J_1} \frac{\partial z}{\sqrt{U_z}} = \int_{I_1 \text{ or } J_1}^{\gamma} \frac{\partial z}{\sqrt{U_z}}, \qquad \int_\alpha^{I_1 \text{ or } J_1} \frac{\partial z}{\sqrt{U_z}} = \int_{I_1 \text{ or } J_1}^{\delta} \frac{\partial z}{\sqrt{U_z}},$$

so that I or J is the place at which the result of integration between two of the singular points (roots of quartic) is bisected.

Transforming the element $\frac{\partial z}{\sqrt{U_z}}$ by means of the homographic relation in (III.), we obtain without any difficulty

$$\int_\beta^{I_1 \text{ or } J_1} \frac{\partial z}{\sqrt{U_z}} = -\int_\gamma^{I_1 \text{ or } J_1} \frac{\partial \zeta}{\sqrt{U_\zeta}},$$

and

$$\int_\alpha^{I_1 \text{ or } J_1} \frac{\partial z}{\sqrt{U_z}} = -\int_\delta^{I_1 \text{ or } J_1} \frac{\partial \zeta}{\sqrt{U_\zeta}};$$

therefore

$$\int_\beta^{I_1 \text{ or } J_1} \frac{\partial z}{\sqrt{U_z}} = \int_{I_1 \text{ or } J_1}^{\gamma} \frac{\partial \zeta}{\sqrt{U_\zeta}},$$

and $$\int_{0}^{I_1 \text{ or } J_1} \frac{\partial z}{\sqrt{U_z}} = \int_{I_1 \text{ or } J_1}^{\delta} \frac{\partial \zeta}{\sqrt{U_\zeta}},$$ proving the proposition.

Of course, similar properties hold with regard to I_2, J_2, I_3, J_3.

On the Stability or Instability of certain Fluid Motions, II. *By* Lord RAYLEIGH, Professor of Natural Philosophy in the Royal Institution.

[*Read Nov. 10th*, 1887.]

As the question of the stability, or otherwise, of fluid motions is attracting attention in consequence of Sir W. Thomson's recent work, I think it advisable to point out an error in the solution which I gave some years ago* of one of the problems relating to this subject; and I will take the opportunity to treat the problem with greater generality.

In the steady laminated motion, the velocity (U) is a function of y only. In the disturbed motion $U+u$, v, the small quantities u, v are supposed to be periodic functions of x, proportional to e^{ikx}, and, as dependent upon the time, to be proportional to e^{int}, where n is a constant, real or imaginary. Under these circumstances the equation determining v (51) is

$$\left(\frac{n}{k}+U\right)\left(\frac{d^2v}{dy^2}-k^2v\right) - \frac{d^2U}{dy^2}v = 0 \quad \text{.................. (1).}$$

The vorticity (Z) of the steady motion is $\frac{1}{2}\frac{dU}{dy}$. If throughout any layer Z be constant, d^2U/dy^2 vanishes, and wherever $n+kU$ does not also vanish

$$\frac{d^2v}{dy^2}-k^2v = 0 \quad \text{.................................. (2),}$$

or

$$v = Ae^{ky}+Be^{-ky} \quad \text{............................ (3).}$$

If there are several layers in each of which Z is constant, the various solutions of the form (3) are to be fitted together, the arbitrary constants being so chosen as to satisfy certain boundary conditions. The first of these conditions is evidently

$$\Delta v = 0 \quad \text{.................................. (4).}$$

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* *Math. Soc. Proc.* XI., p. 57, 1880.