Applications of a Theory of Permutations in Circular Procession to the Theory of Numbers. By Major P. A. MacMahon.

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1. In the *Comptes Rendus* of the French Academy, 11th April, 1892, there appeared a note by M. E. Jablonshi, presented by M. C. Jordan, on the subject of permutations in circular procession.

Also, in the Appendix to his "Théorie des Nombres," t. 1, M. E. Lucas gives a short investigation, which he attributes to M. Moreau.

Both investigators reach the same result, which, in a slightly different notation, may be stated as follows:—

Let there be n things, of which α are of one kind, β of a second, γ of a third, and so forth,

$$\alpha+\beta+\gamma+\ldots=n;$$

let N be the greatest common factor of the numbers $\alpha, \beta, \gamma, \ldots$, and write

$$N=\frac{\alpha}{\alpha'}=\frac{\beta}{\beta'}=\frac{\gamma}{\gamma'}=\ldots;$$

further, let $$LP(\alpha, \beta, \gamma, \ldots)$$

and $$CP(\alpha, \beta, \gamma, \ldots)$$

denote respectively the numbers of permutations of the n things in linear and in circular procession, so that

$$LP(\alpha, \beta, \gamma, \ldots)=\frac{n!}{\alpha!\,\beta!\,\gamma!\ldots},$$

and $$CP(\alpha, \beta, \gamma\ldots)$$

is the number whose expression is sought.

If d be any divisor of N, including N itself and unity, and $\phi(d)$ the *totient* of d (Sylvester's nomenclature for the numbers of integers prime to and not superior to d), the result is

$$nCP(N\alpha', N\beta', N\gamma', \ldots)=\Sigma\phi(d)\,LP\left(\frac{N}{d}\alpha', \frac{N}{d}\beta', \frac{N}{d}\gamma', \ldots\right),$$

otherwise written

$$nCP(N\alpha', N\beta', N\gamma', \ldots) = \Sigma\phi(d)\frac{\left(\frac{n}{d}\right)!}{\left(\frac{N}{d}\alpha'\right)!\left(\frac{N}{d}\beta'\right)!\left(\frac{N}{d}\gamma'\right)!\ldots}.$$

This theorem involves results of interest and importance in the pure theory of numbers.

If n be prime, N is unity, and

$$nCP(\alpha, \beta, \gamma, \ldots) = \frac{n!}{\alpha!\,\beta!\,\gamma!\ldots},$$

and this is also true whenever, n being composite, $\alpha, \beta, \gamma, \ldots$ possess no common factor greater than unity. Hence

Theorem.—The multinomial coefficient

$$\frac{n!}{\alpha!\,\beta!\,\gamma!\ldots}$$

is divisible by n, provided that the numbers $\alpha, \beta, \gamma, \ldots$ constitute a prime assemblage.

The usual statement and proof in treatises is in reference to the divisibility by n, when n is prime, of every coefficient of the multinomial expansion except those attached to powers of single letters. The theorem above is more general and shows the nature of the quotient.

Next suppose $\quad N = n = \alpha = N\alpha'$;

we find $\quad nCP(n) = \Sigma\phi(d)$,

and manifestly $\quad CP(n) = 1$;

therefore $\quad \Sigma\phi(d) = n$,

where d is any divisor of n, including n itself and unity; the well-known theorem due to Gauss.

The complete theorem may be viewed as a generalization of that of Gauss. The expression

$$\Sigma\phi(d)\frac{\left(\frac{n}{d}\right)!}{\left(\frac{N}{d}\alpha'\right)!\left(\frac{N}{d}\beta'\right)!\left(\frac{N}{d}\gamma'\right)!\ldots}$$

gives, for every partition of n, a linear function of the totients of the divisors of n which is divisible by n. Moreover the quotient is

$$CP\,(N\alpha', N\beta', N\gamma', \ldots).$$

It may be noted that, when N is a prime,

$$nCP(N\alpha', N\beta', N\gamma' \ldots) = \frac{n!}{(N\alpha')!\,(N\beta')!\,(N\gamma')!\ldots} + (N-1)\frac{\left(\frac{n}{N}\right)!}{\alpha'!\,\beta'!\,\gamma'!\ldots},$$

and then $(\alpha'\beta'\gamma' \ldots)$ is any partition of $\frac{n}{N}$.

2. The next problem is with reference to the permutations of p different objects, n at a time, when repetitions of objects are permissible.

The number of line permutations may be denoted by $RLP\,(p, n)$, and the number of circular permutations by $RCP\,(p, n)$.

The value of $RLP\,(p, n)$ is known to be p^n.

In a circular permutation there are n objects which may be all similar, or they may be of one, two, &c. ... p different kinds.

A permutation which involves a_1 objects of one kind, a_2 of a second, a_3 of a third, &c., may be said to be of type

$$(a_1 a_2 \ldots a_p),$$

where $$a_1 + a_2 + \ldots + a_p = n,$$

and 1, 2, ... or $p-1$ of the quantities a may be zero.

In order to place in evidence the existence of equalities amongst the quantities a, it is convenient to consider the general type

$$(a_1^{\kappa_1}\, a_2^{\kappa_2}\, a_3^{\kappa_3} \ldots),$$

where $$\kappa_1 + \kappa_2 + \kappa_3 + \ldots = p,$$

and $$\kappa_1 a_1 + \kappa_2 a_2 + \kappa_3 a_3 + \ldots = n.$$

We have to find the number of different combinations of objects which come under a given type.

Were the type $(a_1, a_2, \ldots a_p)$, with the quantities a all different, the number of different combinations would be clearly $p!$.

Hence, for the general type, the number of different combinations of the objects must be

$$\frac{p!}{\kappa_1!\,\kappa_2!\,\kappa_3!\ldots}$$

Ex. gr., if a permutation involves $n-2$ objects of one kind, one of a second, and one of a third, the type is

$$(n-2,\ 1^2,\ 0^{p-3}),$$

and the number of different combinations of this type is

$$\frac{p!}{1!\,2!\,(p-3)!}.$$

Denote by $CP\,(a_1^{\kappa_1},\ a_2^{\kappa_2},\ a_3^{\kappa_3},\ \ldots)$ the number of circular permutations of a set of objects of the type

$$(a_1^{\kappa_1}\,a_2^{\kappa_2}\,a_3^{\kappa_3}\ \ldots)\,;$$

then $$RCP\,(p,\ n) = \Sigma\,\frac{p!}{\kappa_1!\,\kappa_2!\,\kappa_3!\ \ldots}\,CP\,(a_1^{\kappa_1}\,a_2^{\kappa_2}\,a_3^{\kappa_3}\ \ldots).$$

But $$nCP\,(a_1^{\kappa_1}\,a_2^{\kappa_2}\,a_3^{\kappa_3}\ \ldots) = \Sigma\phi\,(d)\,LP\left\{\left(\frac{a_1}{d}\right)^{\kappa_1}\left(\frac{a_2}{d}\right)^{\kappa_2}\left(\frac{a_3}{d}\right)^{\kappa_3}\ldots\right\},$$

by the previous theorem, d denoting a divisor of the highest common factor of the numbers a.

We can now express $RCP\,(p,\ n)$ as a linear function of the expressions LP. We find that the coefficient of $\frac{1}{n}\phi\,(d)$ is

$$\Sigma\,\frac{p!}{\kappa_1!\,\kappa_2!\,\kappa_3!\ \ldots}\,LP\left\{\left(\frac{a_1}{d}\right)^{\kappa_1}\left(\frac{a_2}{d}\right)^{\kappa_2}\left(\frac{a_3}{d}\right)^{\kappa_3}\ldots\right\},$$

and this series is manifestly the value of

$$RLP\left(p,\ \frac{n}{d}\right),$$

arrived at by summation in regard to types. Hence

$$RCP\,(p,\ n) = \frac{1}{n}\,\Sigma\phi\,(d)\,RLP\left(p,\ \frac{n}{d}\right),$$

or $$RCP\,(p, n) = \frac{1}{n}\,\Sigma\phi\,(d)\,p^{n/d},$$

the summation having reference to every divisor d of the number n.

Theorem.—The number of circular permutations of p different objects n together, repetitions permissible, is equal to

$$\frac{1}{n}\,\Sigma\phi\,(d)\,p^{n/d},$$

where $\phi(d)$ represents the totient of d, and the summation is for all divisors d of the number n.

We have here another extension of Gauss's theorem concerning totients. Corresponding to every integer p, we have a linear function of the number $\phi(d)$, d a divisor of n, which is divisible by n, and the theorem shows the nature of the quotient.

Theorem.—If n and p be any positive integers, and d a divisor of n, the sum

$$\Sigma\,\phi(d)\,p^{n/d} \equiv 0 \mod n,$$

and the quotient is equal to the number of circular permutations of p different things n together, repetitions permitted.

Gauss's theorem is given by p equal to unity.

The theorem is also a generalization of Fermat's theorem concerning the divisibility of $p^{n-1}-1$ by n, when n is a prime and p prime to n.

For, since $\qquad \Sigma\,\phi(d)\,p^{n/d} \equiv 0 \mod n,$

if n be prime, $\qquad p^n + (n-1)\,p \equiv 0 \mod n,$

or $\qquad p^{n-1} - 1 \equiv 0 \mod n,$

if p be prime to n, and

$$\frac{1}{n}\left(p^{n-1}-1\right) = \frac{1}{p}\,RCP\,(p, n) - 1,$$

showing the connexion between the quotient in Fermat's theorem and the number $RCP\,(p, n)$.

The usual extension of Fermat's theorem asserts the congruence

$$p^{\phi(N)} - 1 \equiv 0 \mod N,$$

when p is prime to N.

To deduce this from the general formula, suppose N to involve the prime m to the power μ, and put

$$N = N'm^{\mu}.$$

Consider the permutations in circular procession of $p^{\phi(N')}$ objects m^{μ} together, repetitions allowed.

By the formula established above

$$m^{\mu}\,RCP\,\left\{p^{\phi(N')}, m^{\mu}\right\}$$

$$= p^{\phi(N')\,m^{\mu}} + (m-1)\,p^{\phi(N')\,m^{\mu-1}} + \ldots + (m^{\mu} - m^{\mu-1})\,p^{\phi(N')}$$

$$= p^{\phi(N')\,m^{\mu-1}}\left\{p^{\phi(N'm^{\mu})} - 1\right\} + m p^{\phi(N')\,m^{\mu-2}}\left\{p^{\phi(N'm^{\mu-1})} - 1\right\} + \ldots + m^{\mu} p^{\phi(N')};$$

or
$$RCP\,\{p^{\phi(N')},\, m^{\mu}\}$$
$$= p^{\phi(N')} + \frac{p^{\phi(N')}}{m}\{p^{\phi(N'm)}-1\} + \frac{p^{\phi(N')m}}{m^2}\{p^{\phi(N'm^2)}-1\}$$
$$+ \ldots + \frac{p^{\phi(N')m^{\mu-1}}}{m^{\mu}}\{p^{\phi(N)}-1\},$$

where the number of terms on the dexter is $\mu+1$.

Giving μ successive integer values, we establish the congruence
$$p^{\phi(N)}-1 \equiv 0 \mod m^{\mu},$$
and thence
$$p^{\phi(N)}-1 \equiv 0 \mod N.$$
To find the quotient of $p^{\phi(N)}-1$ by m^{μ}, write
$$RCP\,\{p^{\phi(N')},\, m^{\mu}\} = R_{\mu},$$
$$\frac{1}{m^{\mu}}\{p^{\phi(N'm^{\mu})}-1\} = \vartheta_{\mu},$$
$$p^{\phi(N')} = P,$$
so that
$$R_{\mu} = P + P\vartheta_1 + P^{m}\vartheta_2 + \ldots + P^{m^{\mu-1}}\vartheta_{\mu},$$
and hence
$$\vartheta_{\mu} = P^{-m^{\mu-1}}(R_{\mu}-R_{\mu-1}),$$
or
$$\frac{1}{m^{\mu}}\{p^{\phi(N)}-1\}$$
$$= p^{-\phi(N/m^{\mu})m^{\mu-1}}\left[RCP\,\{p^{\phi(N/m^{\mu})},\, m^{\mu}\} - RCP\,\{p^{\phi(N/m^{\mu})},\, m^{\mu-1}\}\right]$$
$$= \vartheta_{\mu}.$$

Let now
$$N = m_1^{\mu_1} m_2^{\mu_2} \ldots m_s^{\mu_s},$$
then
$$\frac{\{p^{\phi(N)}-1\}^s}{N} = \vartheta_{\mu_1}\vartheta_{\mu_2}\ldots\vartheta_{\mu_s},$$
leading to
$$\frac{p^{\phi(N)}-1}{N} = (\vartheta_{\mu_1}\vartheta_{\mu_2}\ldots\vartheta_{\mu_s})^{1/s}\, N^{-(s-1)/s},$$
giving the complete quotient.

Ex. gr., $N = 15,\quad m_1 = 3,\quad m_2 = 5,\quad \mu_1 = \mu_2 = 1,\quad s = 2,$
$$\vartheta_{\mu_1} = \tfrac{1}{4} RCP\,(4, 5) - 1 = \tfrac{1}{20}(4^5+4^2) - 1 = 51,$$
$$\vartheta_{\mu_2} = \tfrac{1}{16} RCP\,(16, 3) - 1 = \tfrac{1}{48}(16^3+2\,.\,16) - 1 = 85;$$

therefore $$(\vartheta_{\mu_1}\vartheta_{\mu_2})^{\frac{1}{2}} N^{-\frac{1}{2}} = \sqrt{51}\,\sqrt{85}\,\frac{1}{\sqrt{15}} = 17,$$
which is right.

3. To extend Fermat's theorem in another direction, expand the right-hand side of the formula

$$nRCP(p, n) = \Sigma\phi(d)\,p^{n/d},$$

according to powers of the prime factors of n.

It is convenient to adopt a symbolic notation by which

$$p^{[a]} = p^{a-1},$$
$$p^{[a]}p^{[b]}p^{[c]}\ldots = p^{[abc\ldots]} = p^{abc\ldots-1}.$$

As a simple case, put $n = m_1m_2m_3$, and we have

$$p^{m_1m_2m_3} + (m_1-1)\,p^{m_2m_3} + (m_2-1)\,p^{m_3m_1} + (m_3-1)\,p^{m_1m_2}$$
$$+ (m_1-1)(m_2-1)\,p^{m_3} + (m_2-1)(m_3-1)\,p^{m_1} + (m_3-1)(m_1-1)\,p^{m_2}$$
$$+ (m_1-1)(m_2-1)(m_3-1)\,p \equiv 0 \bmod m_1m_2m_3;$$

p being prime to n, a rearrangement gives

$$(p^{[m_1]}-1)(p^{[m_2]}-1)(p^{[m_3]}-1)$$
$$+ m_1(p^{[m_2]}-1)(p^{[m_3]}-1) + m_2(p^{[m_3]}-1)(p^{[m_1]}-1)$$
$$+ m_3(p^{[m_1]}-1)(p^{[m_2]}-1)$$
$$+ m_2m_3(p^{[m_1]}-1) + m_3m_1(p^{[m_2]}-1) + m_1m_2(p^{[m_3]}-1)$$
$$+ m_1m_2m_3 \equiv 0 \bmod m_1m_2m_3.$$

Now, by Fermat's theorem,

$$p^{[m_1]} - 1 \equiv 0 \bmod m_1,$$

and, taking n the product of two primes, a formula similar to the above shows the congruence

$$(p^{[m_2]}-1)(p^{[m_3]}-1) \equiv 0 \bmod m_2m_3.$$

Hence the congruence

$$(p^{[m_1]}-1)(p^{[m_2]}-1)(p^{[m_3]}-1) \equiv 0 \bmod m_1m_2m_3,$$

which is equivalent to

$$p^{m_1m_2m_3-1} - p^{m_2m_3-1} - p^{m_3m_1-1} - p^{m_1m_2-1}$$
$$+ p^{m_1-1} + p^{m_2-1} + p^{m_3-1} - 1 \equiv 0 \bmod m_1m_2m_3.$$

To generalize this congruence write

$$\Phi_p(m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}) = p^{\left[m_1^{\mu_1-1}\right]}(p^{[m_1]}-1) \dots p^{\left[m_s^{\mu_s-1}\right]}(p^{[m_s]}-1),$$

according to the law of the totient formula

$$\phi(m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}) = m_1^{\mu_1-1}(m_1-1)\, m_2^{\mu_2-1}(m_2-1) \dots m_s^{\mu_s-1}(m_s-1).$$

The above-established congruence is then written

$$\Phi_p(m_1 m_2 m_3) \equiv 0 \bmod m_1 m_2 m_3.$$

We can establish the congruence

$$\Phi_p(m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}) \equiv 0 \bmod m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s},$$

p being prime to $m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}$.

If $m_1^{a_1} m_2^{a_2} \dots m_s^{a_s}$ and $m_1^{b_1} m_2^{b_2} \dots m_s^{b_s}$ be conjugate divisors of $m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}$, we may write

$$n\Sigma\phi(d)\, p^{n/d} = \Sigma\phi(m_1^{a_1} m_2^{a_2} \dots m_s^{a_s})\, p^{m_1^{b_1} m_2^{b_2} \dots m_s^{b_s}},$$

where

$$\phi(m_1^{a_1} m_2^{a_2} \dots m_s^{a_s}) = (m_1^{a_1} - m_1^{a_1-1})(m_2^{a_2} - m_2^{a_2-1}) \dots (m_s^{a_s} - m_s^{a_s-1});$$

therefore $\Sigma\phi(m_1^{a_1} m_2^{a_2} \dots m_s^{a_s})\, p^{\left[m_1^{b_1} m_2^{b_2} \dots m_s^{b_s}\right]} \equiv 0 \bmod n.$

On the left-hand side, expansion of

$$\phi(m_1^{a_1} m_2^{a_2} \dots m_s^{a_s})$$

shows that the whole coefficient of $m_1^{a_1} m_2^{a_2} \dots m_s^{a_s}$ is

$$p^{\left[m_1^{b_1} m_2^{b_2} \dots m_s^{b_s}\right]} - p^{\left[m_1^{b_1-1} m_2^{b_2} \dots m_s^{b_s}\right]} - p^{\left[m_1^{b_1} m_2^{b_2-1} \dots m_s^{b_s}\right]} - \dots$$
$$+ p^{\left[m_1^{b_1-1} m_2^{b_2-1} \dots m_s^{b_s}\right]} + p^{\left[m_1^{b_1} m_2^{b_2-1} m_3^{b_3-1} \dots m_s^{b_s}\right]} + \dots$$
$$- \dots,$$

which is

$$\left(p^{\left[m_1^{b_1}\right]} - p^{\left[m_1^{b_1-1}\right]}\right)\left(p^{\left[m_2^{b_2}\right]} - p^{\left[m_1^{b_2-1}\right]}\right) \dots \left(p^{\left[m_s^{b_s}\right]} - p^{\left[m_s^{b_s-1}\right]}\right),$$

that is, $\Phi_p(m_1^{b_1} m_2^{b_2} \dots m_s^{b_s}).$

Hence $\Sigma m_1^{a_1} m_2^{a_2} \dots m_s^{a_s} \Phi_p (m_1^{b_1} m_2^{b_2} \dots m_s^{b_s}) \equiv 0 \bmod n$;

or, more briefly,

$$\Sigma\, d\, \Phi_p \left(\frac{n}{d}\right) \equiv 0 \bmod n,$$

which is simply another form of the congruence

$$\Sigma\, \phi(d)\, p^{n/d} \equiv 0 \bmod n.$$

The congruence $\Sigma\, d\, \Phi_p \left(\frac{n}{d}\right) \equiv 0 \bmod n$

may be written $\Phi_p(n) + \Sigma' d\, \Phi_p \left(\frac{n}{d}\right) \equiv 0 \bmod n,$

wherein the summation sign Σ' has reference to every divisor of excluding unity.

Assuming the truth of the congruence

$$\Phi_p \left(\frac{n}{d}\right) \equiv 0 \bmod \frac{n}{d},$$

for every divisor d of n which exceeds unity, manifestly

$$\Sigma' d\, \Phi_p \left(\frac{n}{d}\right) \equiv 0 \bmod n,$$

and hence $\Phi_p(n) \equiv 0 \bmod n$;

therefore by induction universally

$$\Phi_p(n) \equiv 0 \bmod n;$$

or, otherwise, $\Phi_p (m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}) \equiv 0 \bmod m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}$;

and, since $\Phi_p (m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s})$ contains a factor $p^{N/(m_1 m_2 \dots m_s) - 1}$, we have finally

$$p^{1 - N/(m_1 m_2 \dots m_s)} \Phi_p (m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}) \equiv 0 \bmod m_1^{\mu_1} m_2^{\mu_2} \dots m_s^{\mu_s}.$$

Ex. gr., Take $n = m_1^3 m_2,$

$$p^{1-m_1^2} (p^{[m_1^3]} - p^{[m_1^2]}) (p^{[m_2]} - 1) \equiv 0 \bmod m_1^3 m_2,$$

or $$p^{1-m_1^2} (p^{m_1^3 m_2 - 1} - p^{m_1^3 - 1} - p^{m_1^2 m_2 - 1} + p^{m_1^2 - 1}) \equiv 0 \bmod m_1^3 m_2,$$

or $$p^{m_1^3 m_2 - m_1^2} - p^{m_1^3 - m_1^2} - p^{m_1^2 m_2 - m_1^2} + 1 \equiv 0 \bmod m_1^3 m_2.$$

The following presents to the Library were received during the recess:—

"Imperial University of Japan," the Calendars for 1890-91, and 1891-92.

"Proceedings of the Royal Society," Vol. L., No. 307; Vol. LI., 308-313.

"Journal of the Institute of Actuaries," Vol. XXX., Pt. II., No. 166; July, 1892.

"Beiblätter zu den Annalen der Physik und Chemie," Band XVI., Stücke 6, 7, 8, 9.

"Sitzungsberichte der Physikalisch-Medicinischen Societät in Erlangen," 24 Heft; 1892.

"Royal Irish Academy—Proceedings," Vol. II., No. 2; "Transactions," Vol. XXIX., Parts 18, 19.

"Atti del Reale Instituto Veneto di Scienze, Lettere, ed Arti," Tomo XXXVIII., Serie 7, Tomo II., Dispensa 10, and Tomo L., Serie 7, Tomo III., Dispense 1, 2, and 3.

"Jahrbuch über die Fortschritte der Mathematik," Band XXI., Heft 2; 1892.

"Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig," 1892, I. and II.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXV., 5me Livraison, et Tome XXVI., 1er and 2me Livraisons.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. X., No. 6.

"Rendiconti del Circolo Matematico di Palermo," Tomo VI., Fasc. 3, 4.

"Bulletin des Sciences Mathématiques," Tome XVI., Mai-Sep., 1892; and "Table des Matières et Noms d'Auteurs."

"Bulletin de la Société Mathématique de France," Tome XX., Nos. 3, 4.

"New York Mathematical Society," Vol. I., Nos. 9, 10; 1892.

"New York Mathematical Society—List of Members, Constitution, and Bye-Laws;" June, 1892.

"Transactions of the Canadian Institute," Vol. II., Part II., No. 4; Toronto.

"Annual Archæological Report of Canadian Institute" (Session 1891); Toronto.

"Journal of the College of Science, Imperial University, Japan," Vol. V., Parts 1 and 2; 1892.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," 1892, 1-25.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. I., Fasc. 9-12, 1 Semestre; Fasc. 1-5, 2 Semestre.

"Rendiconto dell' Adunanza Solenne del 5 Giugno, 1892, onorata della Presenza di S. M. il Re," Roma, 1892.

"Acta Mathematica," XVI., 1-3.

"Annals of Mathematics," Vol. VI., No. 6; May, 1892.

"Annales de la Faculté des Sciences de Toulouse," Tome VI., Fasc. 2; Paris, 1892.

"Annali di Matematica," Tomo XX., Fasc. 1; Milano, 1892.

"Annales de l'Ecole Polytechnique de Delft," Tome VII., Livr. 2, 3; Leide, 1892.

"Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche di Napoli," Vol. VI., Serie 6, Fasc. 6; 1892.

"Journal für die reine und angewandte Mathematik," Band cx., Heft 1, 2.

"Educational Times," July–October, 1892.

"Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich," xxxvii., Heft 1, 2; 1892.

D'Ocagne (M.).—"Sur la Construction de la Parabole Osculatrice en un point d'une Courbe donnée;" "Sur la Corrélation entre les Systèmes de Coordonnées ponctuelles et les Systèmes de Coordonnées tangentielles;" "Sur une Courbe définie par la Loi de sa Rectification;" "Détermination du Rayon de Courbure en Coordonnées parallèles ponctuelles:" 8vo pamphlets. "Sur l'Application des Coordonnées Parallèles à la Démonstration d'un Théorème de Chasles relatif aux Surfaces Algébriques;" "Sur la Liaison entre les Expressions du Rayon de Courbure en Coordonnées Ponctuelles et en Coordonnées Tangentielles:" pamphlets R. 8vo.

Lemoine (Emile).—"Trois Théorèmes sur la Géométrie du Triangle;" 4to pamphlet. "Sur une Transformation relative à la Géométrie du Triangle;" "Sur les Transformations systématiques des Formules relatives au Triangle—Transformation Continue;" "Étude sur une nouvelle Transformation dite Transformation Continue:" 8vo pamphlets.

"Papers of the Mathematical Tripos," May, 1892.

Marchand (Joseph).—"Sur la Rectification des Arcs des Courbes dites Limaçons de Pascal," 8vo pamphlet.

Graf (Dr. J. H.)—"Das Leben und Wirken des Physikers und Astronomen Johann Jakob Huber (1733–1798)," 8vo; Bern, 1892.

Fleming (S.).—"An Appeal to the Canadian Institute on the Rectification of Parliament," R. 8vo; Toronto, 1892.

"Report of the Superintendent of the U. S. Naval Observatory, for the year ending June 30, 1891;" Washington, 1891.

Merriman (M.).—"Final Formulas for the Algebraic Solution of Quartic Equations," pamphlet, 8vo; New York, 1892.

"Bollettino delle Pubblicazione Italiano," Nos. 156–162.

Hankel (W. G.), and Lindenberg (H.).—"Elektrische Untersuchungen;" pamphlet.

Böcher (Maxime).—"On some Applications of Bessel's Functions with Pure Imaginary Index," 4to pamphlet.

"Temi di Premio, proclamati dal Reale Instituto Veneto, nella solenne adunanza del 29 Maggio, 1892;" Venezia, 1892.

"The Benefactors of the Toronto University," 8vo; Toronto, 1892.

"Smithsonian Report," 1887–1890, 8vo; Washington.

Greenhill (A. G.).—"Applications of Elliptic Functions," 8vo; London, 1892.

Nixon (R. C. J.).—"Elementary Plane Trigonometry," 8vo; Oxford, 1892.

Miller (W. J. C.).—"Mathematical Questions and Solutions," Vol. lvii., 8vo; London, 1892.

Schwarz (H. A.).—"Mathematische Abhandlungen," 2 Vols., R. 8vo; Berlin, 1890.

The following is the list of mathematical books bequeathed by the late Dr. Thomas Archer Hirst, F.R.S., &c., to the President and Council, for the time being, of the Society* :—

Steiner, J.—"Geometrische Gestalten." Berlin, 1832.
Zeuthen, H. G. — "Grundriss einer Elementar-Geometrischen Kegelschnittslehre." Leipzig, 1882.
Geiser, C. F.—"Synthetische Geometrie." Leipzig, 1869.
Weber, H.—"Bernhard Riemann's Mathematische Werke." Leipzig, 1876.
Burg, A.—"Compendium der höheren Mathematik." Vienna, 1836.
Lobatschewsky, N. I.—"Théorie des Parallèles." Paris, 1866.
Staudt, K. G. C.—"Geometrie der Lage." Nürnberg, 1847.
Möbius, A. F.—"Barycentrische Calcul." Leipzig, 1827.
Cremona, Dr. L.—"Geometrische Theorie der Ebenen Curven, von Maximilian Curtze." Greisswald, 1865.
Minding, F.—"Integraltafeln." Berlin, 1849.
Simpson, T.—"Essays on Mixed Mathematics." London, 1740.
Simpson, T.—"Treatise on Fluxions." London, 1737.
Todhunter, I.—"Spherical Trigonometry." Cambridge, 1859.
Joachimsthal, F.—"Géométrie Élémentaire." Berlin, 1852.
Boole, G.—"Differential Equations." London, 1865.
Todhunter, I.—"Mensuration for Beginners." London, 1869.
Todhunter, I.—"Algebra." Second edition. London, 1860.
De Morgan, A.—"Arithmetic." Fifth edition. London, 1846.
Hayward, R. B.—"Solid Geometry." London, 1890.
Boole, G.—"Calculus of Finite Differences." London, 1860.
Boole, G.—"Differential Equations." Second edition. London, 1865.
Todhunter, I.—"Differential Calculus." Fourth edition. London, 1864.
Todhunter, I.—"Integral Calculus." Third edition. London, 1868.
Todhunter, I.—"Theory of Equations." Second edition. London, 1867.
Salmon, Dr. G.—"Conic Sections." Fifth edition. London, 1869.
Wright, R. P.—"Plane Geometry." London, 1868.
Poinsot, L.—"Élemens de Statique." Ninth edition. Paris, 1848.
Cremona.—"Geometrical Memoirs." 1861–1870.
Poinsot, M.—"Théorie des Nombres." Paris, 1845.
Newton, I.—"Principia." Vols. I., II., III. 1760.
Hülsse, Dr. J. A.—"Mathematische Tafeln." Leipzig, 1849.
Schröter, Dr. H.—"Theorie der Ebenen Kurven." Leipzig, 1888.
Hesse, Dr. O.—"Analytische Geometrie des Raumes." Leipzig, 1861.
Caporali, E.—"Memorie di Geometria." Naples, 1888.
Steiner, J.—"Synthetische Geometrie." Leipzig, 1867. } One volume.
Geiser, Dr. C. F.—"Theorie der Kegelschnitte." Leipzig, 1867. }
Jonquières, E. de.—"Mélanges de Géométrie Pure." Paris, 1856.
Poinsot, L.—"Rotation des Corps." Paris, 1852.

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* The list has been verified and the volumes placed in the Society's Library by Mr. Ralph Holmes, B.A.

Cremona, L.—"Corso di Statica grafica." 1868.
Cremona, L.—"Elementi di Geometria Projettiva." Rome, 1873.
Cayley.—"Geometrical Memoirs." 1859–1871.
Cayley.—"Quantics." 1854–1874.
Cremona and others.—"Geometrical Memoirs." Vols. I. and II.
Mannheim and others.—"Geometrical Memoirs."
Poinsot and others.—"Mathematical Memoirs." Vols. I., II., III., IV., V., VI., VII.
Gauss, C. F.—"Mathematical Treatises."
Monge, G.—"Géométrie Descriptive." Par M. Brisson. Edition IV. Paris, 1820.
Halphen, G. H.—"Travaux Mathématiques." Paris, 1885.
Gauss, C. F.—"Recherches Arithmétiques." Traduites par Poullet-Delisle. Paris, 1807.
Kummer, E. E.—"Strahlensysteme." Berlin, 1867.
Jacobi, C. G. J.—"Functionum Ellipticarum." 1829.
Cremona, L., and Beltrami, E.—"Collectanea Mathematica." Naples, 1881.
Carnot, L. N. M.—"Mémoire sur la théorie des Transversales." Paris, 1806.
Chasles, M.—"Histoire des Sciences." 1867–1869.
Chasles, M.—"Traité des Sections Coniques." Paris, 1865.
Chasles, M.—"Traité de Géométrie supérieure." Paris, 1852.
Chasles, M.—"Traité de Géométrie supérieure." Second edition. Paris, 1880.
Möbius, A. F.—"Analytischen Sphärik."
Chasles, M.—"Porismes d'Euclide." Paris, 1860.
Chasles, M.—"Geometrical Memoirs." 1853–1871.
Chasles, M.—"Géométrie Supérieure." Paris, 1846.
Chasles, M.—"Cours d'Astronomie et de Géodésie." École Polytechnique, 1850.
Chasles, M.—"Cours de Machines." École Polytechnique, 1848.
Fiedler, Dr. W.—"Geometrie der Lage." Leipzig, 1875.
Steiner, J.—"Gesammelte Werke." Vols. I., II. Berlin, 1881.
Plücker, Dr. J.—"Analytisch-Geometrische Entwicklungen. 1828.
Plücker, Dr. J.—"Geometrie des Raumes." Vols. I., II. Leipzig, 1868.
Plücker, Dr. J.—"Geometrie des Raumes." Düsseldorf, 1846.
Plücker, Dr. J.—"Analytische Geometrie." Berlin, 1835.
Plücker, Dr. J.—"Theorie der Algebraischen Curven." Bonn, 1839.
Schröter, Dr. H.—"Theorie der Oberflächen zweiter Ordnung." Leipzig, 1880.
Möbius.—"Über die Symmetrie der Krystalle." Leipzig, 1846–1865.
Sturm, Dr. R.—"Flächen dritter Ordnung." Leipzig, 1867.
Joachimsthal, F.—"Elemente der Analytischen Geometrie." Berlin, 1863.
Stegmann, Dr. F. L.—"Lehrbuch des Variationsrechnung." 1854.
Hesse, Dr. Otto.—"Analytischen Geometrie." Leipzig, 1865.
Reye, Dr. T.—"Geometrie der Lage." Vols. I., II., III. Hanover, 1866, 1877, 1880.
Bruhns, Dr.—"Manual of Logarithms." Leipzig, 1870.
"Proceedings of the London Mathematical Society." Vols. I.–XVIII.
"Mathematische Annalen." Vols. I.–XXXVII.
"Crelle's Journal." Vols. XIX., XXX., XXXII., XXXIX., XLIV.–CVII.
Monge, G.—"Application de l'Analyse à la Géométrie." Fifth edition. By M. Liouville. Paris, 1850.

Jacobi, C. G. J.—"Mathematical Treatises."
Baltzer, Dr. R.—"Theorie der Determinanten." Leipzig, 1857.
Poinsot, M.—"Recherches sur l'Analyse des Sections Angulaires." Paris, 1825.
Cayley.—"Collected Mathematical Papers." Vols. I., II., III., IV. Cambridge.
Lagrange.—"Leçons sur le Calcul des Fonctions." Paris, 1806.
De Morgan.—"Trigonometry and Double Algebra." London, 1849.
Salmon.—"Higher Plane Curves." Second edition.
Salmon.—"Lessons on Higher Algebra." Third edition.
Salmon.—"Geometry of Three Dimensions." Third edition.
Carnot, L. N. M.—"Géométrie de Position." Paris, 1803.
Schubert, Dr. H.—"Kalkul der Abzählenden Geometrie." Leipzig, 1879.
Poncelet, J. V.—"Projections des Figures." Paris, 1822.
Cremona, Dr. L.—"Teoria Geometrica." Bologna, 1862.
Dupin, Ch.—"Développements de Géométrie." Paris, 1813.
MacLaurin, C.—"Geometria Organica." London, 1720.
Lagrange, J. L.—"Mécanique Analytique." Vols. I., II. Paris, 1815.
Euler, L.—"Lineas Curvas." Geneva, 1744.
Cramer, G.—"Lignes Courbes." Geneva, 1750.
Chasles, M.—"Aperçu Historique des Méthodes en Géométrie." Paris, 1875.
Chasles, M.—"Rapport sur les Progrès de la Géométrie." Paris, 1870.