

## 17. The Theory of Determinants in the Historical Order of its Development. By Thomas Muir, M.A., LL.D.

PART I. (continued). *Determinants in General* (1779–1812).

Now it is at once manifest that the successive developments here obtained of the determinant  $[xyzt]$  are letter by letter identical with the successive "*lignes*" obtained by Bézout from the unreal product  $xyzt$ ; but that instead of having one arbitrary step succeeding another, as in the application of Bézout's rule, there is here a fluent reasonableness characterising the whole process.\* As for the peculiarities requiring elucidation in the series of special examples above referred to, they are seen, when looked at in this light, to be but matters of course.

Not only so, but it will be found that the translation of  $xy$  into  $[xy]$ , &c., is an unfailing key to much that follows in Bézout in connection with the subject. For example, let us take the wide extension of the rule which is expounded later on in the treatise, in a section headed

\* If the fact at the basis of the process were made use of nowadays, it would be advantageous, of course, in the first instance to simplify the determinant as much as possible. For example, the equations being (Bézout, p. 178)

$$\left. \begin{aligned} 2x + 4y + 5z &= 22 \\ 3x + 5y + 2z &= 30 \\ 5x + 6y + 4z &= 43 \end{aligned} \right\},$$

we might proceed as follows:—

$$\begin{aligned} & \begin{vmatrix} 2 & 4 & 5 & -22 \\ 3 & 5 & 2 & -30 \\ 5 & 6 & 4 & -43 \\ x & y & z & t \end{vmatrix} = \begin{vmatrix} 0 & 2 & 11 & -6 \\ 1 & 1 & -3 & -8 \\ 0 & -3 & -3 & 9 \\ x & y & z & t \end{vmatrix} \\ & = 3 \begin{vmatrix} 0 & 0 & 9 & 0 \\ 1 & 0 & -4 & -5 \\ 0 & -1 & -1 & 3 \\ x & y & z & t \end{vmatrix} = 27 \begin{vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -5 \\ 0 & -1 & 0 & 3 \\ x & y & z & t \end{vmatrix} \\ & = 27 \{ -t + 0z - 3y - 5x \}; \end{aligned}$$

whence  $x=5$ ,  $y=3$ ,  $z=0$ .

*Considérations utiles pour abréger considérablement  
le calcul des coefficients qui servent à l'élimination.*

There are in all fifteen pages (pp. 208–223, §§ 252–270) devoted to the subject. The contents of three paragraphs will give a sufficiently clear idea of the nature of the whole. The notation used is identical with that of Laplace, *e.g.*,

$$\begin{aligned}(ab') &= ab' - a'b, \\ (ab'c'') &= (ab' - a'b)c'' - (ab'' - a''b)c' + (a'b'' - a''b')c, \\ &\dots\dots\dots\end{aligned}$$

Two of the three selected paragraphs stand as follows:—

“(264.) Cette manière de procéder au calcul des inconnues, en les groupant, n'est pas applicable seulement à notre objet; elle peut en général être appliquée dans toutes les équations du premier degré.

“Si l'on avoit, par exemple, les quatre équations suivantes

$$\begin{aligned}ax + by + cz + dt + e &= 0, \\ a'x + b'y + c'z + d't + e' &= 0, \\ a''x + b''y + c''z + d''t + e'' &= 0, \\ a'''x + b'''y + c'''z + d'''t + e''' &= 0.\end{aligned}$$

En se rappelant que chaque inconnue a pour valeur le coefficient qu'elle se trouve avoir dans la dernière *ligne*, divisé constamment par celui que l'inconnue introduite aura dans cette même *ligne*, on verra bientôt qu'on peut réduire le calcul à chercher le coefficient de l'une quelconque des inconnues dans la dernière ligne; parce que de la même manière qu'on en aura calculé un, on calculera de même tous les autres: ou même, lorsqu'on en aura calculé un, on pourra en déduire tous les autres, lorsque les équations auront toute la généralité possible. Or pour avoir la valeur du coefficient d'une des inconnues dans la dernière ligne, la question se réduit à calculer la valeur du produit des autres inconnues. Mais pour ne pas se tromper sur les signes, il faudra toujours ne pas perdre de vue, la place que cette inconnue est censée occuper dans le produit de toutes les inconnues. Ainsi, dans le cas présent, au lieu de calculer généralement la dernière *ligne* pour avoir *xyztu*, je calcule

seulement cette dernière ligne pour  $yztu$  : et pour l'avoir de la manière la plus commode, je groupe en cette manière  $yz.tu$ , et je procède comme il suit, au calcul des lignes, observant que  $y$  est censé à la seconde place.

Première ligne.  $- bz.tu - yz.du$ ,

Seconde ligne.  $+(bc').tu - bz.d'u + b'z.du + yz.(de')$ ,

Troisième ligne.  $-(bc').d'u + (bc'').d'u - bz.(d'e''') - (b'c'').du + b'z.(de'') - b''z.(de')$ ,

Quatrième ligne.  $+(bc').(d'e''') - (bc'').(d'e''') + (bc''').(d'e'') + (b'c'').(de''') - (b'c''').(de'') + (b''c'').(de')$ ;

c'est le coefficient de  $x$  dans la dernière ligne.

“Pour avoir celui de  $u$ , je calculerois de même la valeur de  $xyzt$ , en le groupant ainsi,  $xy.zt$ , et je trouverois pour valeur du coefficient de  $u$  dans la dernière ligne, la quantité

$$(ab').(c'd''') - (ab'').(c'd''') + (ab''').(c'd'') + (a'b'').(cd''') - (a'b'').(cd'') + (a''b''').(cd')$$

“D'où je conclus

$$x = \frac{+(bc').(d'e''') - (bc'').(d'e''') + (bc''').(d'e'') + (b'c'').(de''') - (b'c''').(de'') + (b''c'').(de')}{(ab').(c'd''') - (ab'').(c'd''') + (ab''').(c'd'') + (a'b'').(cd''') - (a'b'').(cd'') + (a''b''').(cd')}$$

et ainsi de suite.

“(265.) Si j'avois les cinq équations suivantes—

$$ax + by + cz + dr + et + f = 0,$$

$$a'x + b'y + c'z + d'r + e't + f' = 0,$$

$$a''x + b''y + c''z + d''r + e''t + f'' = 0,$$

$$a'''x + b'''y + c'''z + d'''r + e'''t + f''' = 0,$$

$$a^{iv}x + b^{iv}y + c^{iv}z + d^{iv}r + e^{iv}t + f^{iv} = 0.$$

Je calculerois, par exemple, le coefficient de  $x$  dans la dernière ligne, en calculant  $yzr.tu$ , ou  $yz.rtu$ , ou  $yz.rt.u$ .

“Si j'avois six équations dont les inconnues fussent  $x, y, z, r, s$  et  $t$ , je calculerois, par exemple, le coefficient de  $x$ , en calculant ou  $yz.rs.tu$ , ou  $yzrs.tu$ , ou  $yzr.stu$ , et ainsi de suite.”

The next paragraph deals with an illustrative example. The twelve equations—

$$\begin{array}{rcl}
Aa + A'a' + A''a'' & & = 0 \\
Ab + A'b' + A''b'' & & = 0 \\
Ac + A'c' + A''c'' + Ba + B'a' + B''a'' & & = 0 \\
& + Bb + B'b' + B''b'' & = 0 \\
& + Bc + B'c' + B''c'' & = 0 \\
& + Bd + B'd' + B''d'' + Ca + C'a' + C''a'' & = 0 \\
& & + Cb + C'b' + C''b'' & = 0 \\
& & + Cc + C'c' + C''c'' & = 0 \\
& & + Cd + C'd' + C''d'' + Da + D'a' + D''a'' & = 0 \\
& & & + Db + D'b' + D''b'' & = 0 \\
& & & + Dc + D'c' + D''c'' & = 0 \\
Ad + A'd' + A''d'' & & & + Da + D'a' + D''a'' & = 0
\end{array}$$

are given, and what is required is the result of the elimination (*équation de condition*) of the twelve quantities— $a, a', a'', b, b', b'', c, c', c'', d, d', d''$ . This is found to be—

$$(ab'c'').[(bc'd'')^3 - (ab'c'')^2(ab'd'')] = 0.$$

The two paragraphs quoted (§§ 264, 265) show that Bézout could obtain with considerably increased ease and certitude any one of Laplace's expansions of numerator and denominator. What it accomplished in the illustrative example is virtually, in modern symbolism, the reduction of

$$\begin{vmatrix}
a & a' & a'' & . & . & . & . & . & . & . & . & . \\
b & b' & b'' & . & . & . & . & . & . & . & . & . \\
c & c' & c'' & a & a' & a'' & . & . & . & . & . & . \\
. & . & . & b & b' & b'' & . & . & . & . & . & . \\
. & . & . & c & c' & c'' & . & . & . & . & . & . \\
. & . & . & d & d' & d'' & a & a' & a'' & . & . & . \\
. & . & . & . & . & . & b & b' & b'' & . & . & . \\
. & . & . & . & . & . & c & c' & c'' & . & . & . \\
. & . & . & . & . & . & d & d' & d'' & a & a' & a'' \\
. & . & . & . & . & . & . & . & . & b & b' & b'' \\
. & . & . & . & . & . & . & . & . & c & c' & c'' \\
d & d' & d'' & . & . & . & . & . & . & a & a' & a''
\end{vmatrix}$$

to the form

$$|ab'c''|. |bc'd''|^3 - |ab'c''|^2. |ab'd''|.$$

Although this can be done nowadays with ease by means of Laplace's expansion-theorem in its modern garb, it may be safely affirmed that Laplace himself, using his own process, would not have succeeded in making the reduction. Considerable importance thus attaches from more than one point of view to Bézout's curious "rule."

The only other section with which we are concerned bears the heading

*Méthode pour trouver des fonctions d'un nombre quelconque de quantités, qui soient zéro par elles-mêmes.*

In the second paragraph of the section the principle is explained as follows:—

"(216) Concevons un nombre  $n$  d'équations du premier degré renfermant un nombre  $n + 1$  d'inconnues, et sans aucun terme absolument connu.

"Imaginons que l'on augmente le nombre de ces équations, de l'une d'entr'elles; alors il est clair que ce que nous appelons la dernière ligne, sera non seulement l'équation de condition nécessaire pour que ce nombre  $n + 1$  d'équations ait lieu; mais encore que cette équation de condition aura lieu; en sorte qu'elle sera une fonction des coefficients de ces équations, laquelle sera zéro par elle-même.

"Voilà donc un moyen très-simple pour trouver un nombre  $n + 1$ \* de fonctions d'un nombre  $n + 1$  de quantités, lesquelles fonctions soient zéro par elles-mêmes."

For example, the pair of equations

$$\left. \begin{aligned} ax + by + cz &= 0 \\ a'x + b'y + c'z &= 0 \end{aligned} \right\}$$

is taken, the first equation is repeated, and for this set of three equations the *équation de condition* is found to be

$$(ab' - a'b)c - (ac' - a'c)b + (bc' - b'c)a = 0.$$

"Or il est clair que la troisième équation n'exprimant rien de différent de la première, cette dernière quantité doit être zéro par elle-même: donc si on a ces deux suites de quantités

\* Should be  $n$ .

$$\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \end{array}$$

on peut être assuré qu'on aura toujours

$$(ab' - a'b)c - (ac' - a'c)b + (bc' - b'c)a = 0.$$

“ Et si au lieu de joindre la première équation, c'eût été la seconde, nous aurions trouvé de même

$$(ab' - a'b)c' - (ac' - a'c)b' + (bc' - b'c)a' = 0.”$$

Similarly in regard to the quantities

$$\begin{array}{cccc} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{array}$$

the identity

$$\begin{aligned} & [(ab' - a'b)c'' - (ac' - a'c)b'' + (bc' - b'c)a'']d \\ & - [(ab' - a'b)d'' - (ad' - a'd)b'' + (bd' - b'd)a'']c \\ & + [(ac' - a'c)d'' - (ad' - a'd)c'' + (cd' - c'd)a'']b \\ & - [(bc' - b'c)d'' - (bd' - b'd)c'' + (cd' - c'd)b'']a = 0 \end{aligned}$$

and two others are established, the general theorem of course being merely referred to as easily obtainable.

Thus far there is in substance nothing new. What we have obtained is simply a different aspect of Vandermonde's theorem, that *when two indices of either set are alike the function vanishes*, or, as we should now say, *a determinant with two rows identical is equal to zero*. Indeed the identities are used by Vandermonde in Bézout's form when solving a set of simultaneous equations. But what follows is important.

By taking two of these identities

$$\begin{aligned} (ab' - a'b)c - (ac' - a'c)b + (bc' - b'c)a &= 0 \\ (ab' - a'b)c' - (ac' - a'c)b' + (bc' - b'c)a' &= 0, \end{aligned}$$

multiplying both sides of the first by  $d'$ , both sides of the second by  $d$ , and subtracting, there is obtained in regard to the quantities

$$\begin{array}{cccc} a, & b, & c, & d \\ a', & b', & c', & d' \end{array}$$

the identity

$$(ab' - a'b)(cd' - c'd) - (ac' - a'c)(bd' - b'd) + (bc' - b'c)(ad' - a'd) = 0.$$

Similarly by taking the three next identities before obtained, which for shortness we may write in modern notation,

$$\begin{aligned} |ab'c''|d - |ab'd''|c + |ac'd''|b - |bc'd''|a &= 0, \\ |ab'c''|d' - |ab'd''|c' + |ac'd''|b' - |bc'd''|a' &= 0, \\ |ab'c''|d'' - |ab'd''|c'' + |ac'd''|b'' - |bc'd''|a'' &= 0, \end{aligned}$$

there is deduced in regard to the quantities

$$\begin{array}{cccccc} a, & b, & c, & d, & e \\ a', & b', & c', & d', & e' \\ a'', & b'', & c'', & d'', & e'' \end{array}$$

the identities

$$\begin{aligned} |ab'c''|. |d'e'| - |ab'd''|. |ce'| + |ac'd''|. |be'| - |bc'd''|. |ae'| &= 0, \\ |ab'c''|. |d'e''| - |ab'd''|. |ce''| + |ac'd''|. |be''| - |bc'd''|. |ae''| &= 0, \\ |ab'c''|. |d'e'''| - |ab'd''|. |c'e'''| + |ac'd''|. |b'e'''| - |bc'd''|. |a'e'''| &= 0. \end{aligned}$$

Finally these last three identities are taken, both sides of the first multiplied by  $f''$ , both sides of the second by  $-f'$ , both sides of the third by  $f$ , and then by addition there is obtained in regard to the quantities

$$\begin{array}{cccccc} a, & b, & c, & d, & e, & f \\ a', & b', & c', & d', & e', & f' \\ a'', & b'', & c'', & d'', & e'', & f'' \end{array}$$

the identity

$$|ab'c''|. |d'e'f''| - |ab'd''|. |c'e'f''| + |ac'd''|. |b'e'f''| - |bc'd''|. |a'e'f''| = 0.$$

The subject of what may appropriately be called *vanishing aggregates of determinant-products* is not pursued farther, the concluding paragraph being

“(223) En voilà assez pour faire connoître la route qu'on doit tenir, pour trouver ces sortes des théorèmes. On voit qu'il y a une infinité d'autres combinaisons à faire, et qui donneront chacune de nouvelles fonctions, qui seront zéro par elles-mêmes : mais cela est facile à trouver actuellement.”\*

\* It is very curious to observe, in passing, that although Bézout does not obtain all his vanishing aggregates directly by means of the principle which he so carefully states at the commencement, nevertheless every one of them can be so obtained. He does not extend the principle beyond the case where only *one* of the original equations is repeated. If, however, we take the equations

$$\begin{aligned} ax + by + cz + dw &= 0, \\ a'x + b'y + c'z + d'w &= 0, \end{aligned}$$

Our second list of Bézout's contributions thus is:—

(1) An unexplained artificial process for finding the numerators and denominators of fractions which express the values of the unknowns in a set of linear equations, or for finding the resultant of the elimination of  $n$  quantities from  $n+1$  linear equations,—a process especially useful when the coefficients have particular values. (II. 3 + III. 4 + IV. 2.)

(2) An improved mode of finding Laplace's expansions, especially (but not exclusively) useful when the coefficients have particular values. (XIV. 3.)

(3) A proof of Vandermonde's theorem regarding the effect of the equality of two indices belonging to the same set. (XII. 3.)

(4) A series of identities regarding vanishing aggregates of products. (XXIII.)

### HINDENBURG, C. F. (1784).

[*Specimen analyticum de lineis curvis secundi ordinis, in delucidationem Analyseos Finitorum Kaestnerianæ. Auctore Christiano Friderico Rüdiger. Cum præfatione Caroli Friderici Hindenburgii, professoris Lipsiensis.* (xlviii + 74 pp.) pp. xiv-xlviii. *Lipsiæ.*]\*

One of the problems dealt with by Rüdiger being the finding of the equation of the conic passing through five given points ("coefficientium determinatio Trajectoriæ secundi ordinis per data quinque puncta"), Hindenburg, in his preface, takes occasion to show how the generalised problem for  $\frac{1}{2}n(n+3)$  points has been treated, pointing out that it is, of course, immediately dependent on the solution of a set of simultaneous linear equations. He directs attention to the labours of Cramer and Bézout, specially lauding the method of the latter, given in the treatise of 1779. Then he repeat both of them so as to have a set of four, and then proceed by the *méthode pour abrégé* to find the *équation de condition*, we obtain

$$|ab'| \cdot |cd'| - |ac'| \cdot |bd'| + |ad'| \cdot |bc'| + |bc'| \cdot |ad'| - |bd'| \cdot |ac'| + |cd'| \cdot |ab'| = 0,$$

$$\text{i.e. } 2\{|ab'| \cdot |cd'| - |ac'| \cdot |bd'| + |ad'| \cdot |bc'|\} = 0.$$

This is the identity at foot of p. 457, and all the others are readily seen to be obtainable in the same way.

\* My best thanks are due the Committee of Management of University College, London, for the loan of a copy of Hindenburg's tract from the Graves Library.



says—“*Hæc de Opere Bezoldino in universam, quod plurimis adhuc Lectoribus nostris ignotum erit, dicta sufficiant. Nunc Regulam ipsam proponam.*” . . . The seventeen pages which follow, contain a tolerably close Latin translation of the *Règle générale pour calculer* . . . , and the *Méthode pour trouver* . . . , pp. 172–187, §§ 198–223, which have been expounded above. Cramer’s rule is next given, the second mode of putting it being in words, and the first as follows:—

“Sint plures Incognitæ  $z, y, x, w$ , &c. totidemque Aequationes simplices indeterminatæ

$$A^1 = Z^1z + Y^1y + X^1x + W^1w + \&c.$$

$$A^2 = Z^2z + Y^2y + X^2x + W^2w + \&c.$$

$$A^3 = Z^3z + Y^3y + X^3x + W^3w + \&c.$$

$$A^4 = Z^4z + Y^4y + X^4x + W^4w + \&c.$$

$$\&c. \ \&c. \ \&c. \ \&c. \ \&c. \ \&c.$$

Erit, . . . . ., positis terminorum signis, ut præcipitur in fine Tabulæ, pag. seq.

$$z = \frac{\begin{array}{cccccc} A & Y & X & W & V & U & T & \dots \end{array}}{\begin{array}{cccccc} \text{Permut} (1, 2, 3, 4, 5, 6, 7, \dots) \\ \hline \text{Permut} (1, 2, 3, 4, 5, 6, 7, \dots) \\ Z & Y & X & W & V & U & T & \dots \end{array}} \quad (\text{VII. 3.})$$

The similar expressions for  $y, x, w, v, u, t$ , are given, and then the “*regula signorum.*” After an illustrative example, the question of the *sequence* of the signs is taken up.

“Quod si itaque  $+sg(1, 2, 3, \dots, n)$  denotet signorum vicissitudines, quibus hic afficiuntur Permutationum a numeris 1, 2, 3, . . .  $n$  singulæ species, et  $-sg(1, 2, 3, \dots, n)$  signa *contraria* vel *opposita*: appatet fore

$$\begin{aligned} sg(1, 2) &= +sg(1) && -sg(1) \\ sg(1, 2, 3) &= +sg(1, 2) && -sg(1, 2) && +sg(1, 2) \\ sg(1, 2, 3, 4) &= +sg(1, 2, 3) && -sg(1, 2, 3) && +sg(1, 2, 3) && -sg(1, 2, 3) \end{aligned}$$

. . . . .

unde, quia  $sg(1)$  est +, facile eruitur

$$\begin{aligned} sg(1, 2) &\text{ esse } + - \\ sg(1, 2, 3) &\dots + - - + + - \\ sg(1, 2, 3, 4) &\dots + - - + + - - + + - - + \\ &\dots + - - + + - - + + - - + \end{aligned}$$

and it is pointed out that the first sign is always +, and the last + or - according as the number  $1 + 2 + 3 + \dots + (n-1)$  is even or odd.

Bearing in mind that Hindenburg wrote his permutations in a definite order, this remark regarding the sequence of signs entitles us to view him as the author of a combined rule of term-formation and rule of signs, which may be formulated as follows:—

*Write the permutations of 1, 2, 3, . . . , n in ascending order of magnitude as if they were numbers; make the first sign +, the second -, the next pair contrary in sign to the first pair, the third pair contrary in sign to the second pair, the next six (1.2.3) contrary in sign to the first six, the third six contrary in sign to the second six, the fourth six contrary in sign to the third six, the next twenty-four (1.2.3.4) contrary in sign to the first twenty-four, and so on.* (II. 4 + III. 5.)

ROTHE, H. A. (1800).

[Ueber Permutationen, in Beziehung auf die Stellen ihrer Elemente.

Anwendung der daraus abgeleiteten Sätze auf das Eliminationsproblem. *Sammlung combinatorisch-analytischer Abhandlungen, herausg. v. C. F. Hindenburg*, ii. pp. 263–305.]

Rothe was a follower of Hindenburg, knew Hindenburg's preface to Rüdiger's Specimen Analyticum, and was familiar with what had been done by Cramer and Bézout (see his words at p. 305). His memoir is very explicit and formal, proposition following definition, and corollary following proposition, in the most methodical manner.

The idea which is made the basis of it, that of *place-index* ("Stellenexponent"), is an ill-advised and purposeless modification of Cramer's idea of a "dérangement." The definition is as follows:—In any permutation of the first  $n$  integers, the *place-index* of any integer is got by counting the integer itself, and all the elements after it which are less than it. For example, in the permutation

6, 4, 3, 9, 8, 10, 1, 7, 2, 5

of the first ten integers, the place-index of 9 is 6, and that of 7 is 3. The counting of the integer itself makes the place-index always *one more* than the number of "dérangements" connected with the

integer. This necessitates the introduction of a corresponding modification of Cramer's "rule of signs," viz.

"3. Willkürlicher Satz. Jede Permutation der Elemente 1, 2, 3, . . . ,  $r$ , werde mit dem Zeichen + versehen, wenn entweder gar keine, oder eine gerade Menge gerader Zahlen, unter ihren Stellenexponenten vorkommt; mit dem Zeichen - hingegen, wenn die Menge der geraden Zahlen, unter den Stellenexponenten ungerade ist." (III. 6.)

It is difficult to suggest any justification for the changes here introduced. The author himself refers to none. Indeed, in the very next paragraph he points out that to ascertain whether there be an even number of even integers among the place-indices is the same as to diminish each of the place-indices by 1, and ascertain whether there be an even number of odd integers, that is, whether the *sum* of the odd integers be even. He then concludes—

"Man kann also auch die Regel so ausdrücken: Jede Permutation bekommt das Zeichen + wenn die Summe der um 1 verminderten Stellenexponenten gerade, - hingegen, wenn sie ungerade ist."

This is simply Cramer's rule, and it is the only rule of signs employed henceforward in the memoir, the expression "die Summe der um 1 verminderten Stellenexponenten," occurring over and over again as a periphrasis for "the number of *dérangements*."

The next four pages are occupied with a very lengthy but thorough investigation of the theorem that *two permutations differ in sign, if they be so related that either is got from the other by the interchange of two of the elements of the latter*. Strictly speaking, however, the proposition proved is something more definite than this, viz.—

*If in a permutation of the integers 1, 2, . . . ,  $r$  there be  $d$  integers intermediate in place and value between any two,  $A$  and  $B$ , of the integers, the interchanging of the said two would increase or diminish the number of inversions of order by  $2d + 1$ .* (III. 7.)

The proof consists in finding the sum of the place-indices for the given permutation in terms of  $d$  as just defined,  $c$  the number of elements less than both  $A$  and  $B$  and situated between them,  $f$  the number of such elements situated to the right of  $B$ , and  $e$  the

number of elements between A and B in value and situated to the right of B; then finding in like manner the sum of the place-indices for the new permutation; and finally comparing the two sums. The concluding sentence is as follows:—

“Denn da . . . . ., so ist die Summe der Stellenexponenten der zweyten Permutation um  $d+e+1-e+d$  oder um  $2d+1$  grösser, als bey der ersten Permutation; folglich gilt das auch bey der Summe der um 1 verminderten Stellenexponenten, da bey beyden Permutationen  $r$  einerley ist. Also ist die eine Summe gerade, die andere ungerade, folglich haben nach (4) beyde Permutationen verschiedene Zeichen.”

As immediate deductions from this, it is pointed out that

*The sign of any one permutation may be determined when the sign of any other is known, by counting the number of interchanges necessary to transform the one permutation into the other; (III. 8.) and that*

*If one element of a permutation be made to take up a new place, by being, as it were, passed over  $m$  other elements, the sign of the new permutation is the same as, or different from, that of the original according as  $m$  is even or odd. (III. 9.)*

A third corollary is given, but it is, strictly speaking, a self-evident corollary to the second corollary, and is quite unimportant.

Rothe's next theorem is—

*The permutations of 1, 2, 3, . . . . ,  $n$  being arranged after the manner in which numbers are arranged in ascending order of magnitude, any two consecutive permutations will have the same sign, if the first place in which they differ be the  $(4n+3)^{\text{th}}$  or  $(4n+4)^{\text{th}}$  from the end, and will be of opposite sign if the said place be the  $(4n+1)^{\text{th}}$  or  $(4n+2)^{\text{th}}$  from the end. (III. 10.)*

Thus if the permutations of 1, 2, 3, . . . . , 10 be taken, and arranged as specified, two which will occur consecutively are

8, 4, 9, 3, 10, 7, 6, 5, 2, 1  
8, 4, 9, 5, 1, 2, 3, 6, 7, 10;

and as the first place in which these differ is the 7<sup>th</sup> from the end, it is affirmed that the signs preceding them must be alike. The

mode of proving the theorem will be readily understood by seeing it applied to this illustrative example. Taking the permutation

$$8, 4, 9, 3, 10, 7, 6, 5, 2, 1,$$

and interchanging 3 and 5 we have the permutation

$$8, 4, 9, 5, 10, 7, 6, 3, 2, 1,$$

and thence by cyclical changes the permutation

$$8, 4, 9, 5, 1, 2, 3, 6, 7, 10,$$

the number of alterations of sign thus being

$$1 + (5 + 4 + 3 + 2 + 1)$$

$$\text{i.e. } 1 + \frac{1}{2}(5 \times 6),$$

—an even number.

Annexed to the theorem is the following corollary, which is not essentially sufficient from Hindenburg's proposition regarding the sequence of signs,—

*If the permutations of 1, 2, 3, . . . , n - 1 be arranged after the manner in which numbers are arranged in ascending order of magnitude, and also in like manner the permutations of 1, 2, 3, . . . , n - 1, n, then those permutations of the latter arranged set which begin with r, say, have in order the same signs as the permutations of the former arranged set, or different signs, according as r is odd or even.* (III. 11.)

For example, arranging the permutations of 1, 2, 3, each with its proper sign in front, we have

$$\begin{array}{l} +1, 2, 3 \\ -1, 3, 2 \\ -2, 1, 3 \\ +2, 3, 1 \\ +3, 1, 2 \\ -3, 2, 1; \end{array} \quad (\text{A})$$

then arranging those permutations of 1, 2, 3, 4 which begin with 3 say, each with its proper sign, we have

$$\begin{array}{l} +3, 1, 2, 4 \\ -3, 1, 4, 2 \\ -3, 2, 1, 4 \\ +3, 2, 4, 1 \\ +3, 4, 1, 2 \\ -3, 4, 2, 1; \end{array} \quad (\text{B})$$

and the two series of signs are seen to be identical, 3 being an odd number. Viewing this quite independently of the theorem to which it is annexed, it is evident that a change of sign at any point in the series (A) implies a change at the corresponding point in the other series, and consequently attention need only be paid to the first sign of (B) as compared with the first sign of (A). Now the first sign of (A) must necessarily be always plus, there being no inversions; and the first sign of (B) depends on the changes necessary for the transformation of the natural order 1, 2, 3, 4, into 3, 1, 2, 4. The truth of the corollary is thus apparent.

A second corollary is given, but it is of still less consequence, the difference between it and the first being that in the arranged set (B) the place whose occupant remains unchanged may be any one of the  $n$  places. (III. 12.)

The next few paragraphs concern the subject of "conjugate permutations" (*verwandte Permutationen*),—apparently a fresh conception. The definition is—

*Two permutations of the numbers 1, 2, 3, . . . , n are called CONJUGATE when each number and the number of the place which it occupies in the one permutation are interchanged in the case of the other permutation.* (xxiv.)

For example, the permutations

$$\begin{array}{ll} 3, & 8, 5, 10, 9, 4, 6, 1, 7, 2 & (A) \\ 8, & 10, 1, 6, 3, 7, 9, 2, 5, 4 & (B) \end{array}$$

are conjugate, because 3 is in the 1<sup>st</sup> place of (A) and 1 is in the 3<sup>rd</sup> place of (B), 8 is in the 2<sup>nd</sup> place of (A), and 2 is in the 8<sup>th</sup> place of B, and so on in every case.

The first theorem obtained is—

*Conjugate permutations have the same sign.* (III. 13.)

This is proved in a curious and interesting way, a special conjugate pair being considered, viz., the pair just given as an example. To commence with, a square divided into  $10 \times 10$  equal squares is drawn, the vertical rows of small squares being numbered 1, 2, 3, &c. from left to right, and the horizontal rows 1, 2, 3, &c. from the top downwards. The permutation

$$3, 8, 5, 10, 9, 4, 6, 1, 7, 2$$

is then represented by putting a dot in each of the horizontal rows, in the first under 3, in the second under 8, and so on; so that if the rows be taken in order, and the number above each dot read, the given permutation is obtained. For the representation of the conjugate permutation nothing further is necessary: we obtain it at once if we only turn the paper round clockwise until the vertical rows are horizontal, and read off in order the numbers above the dots. In the next place the number of "dérangements" belonging to the permutation 3, 8, 5, . . . is indicated by inserting a cross in every small square which is to the left of one dot and above another; thus the two crosses in the first horizontal row correspond to the two "dérangements" 32, 31; the six crosses in the second horizontal row to the six "dérangements" 85, 84, 86, 81, 87, 82; and so on. Then it is observed that if we turn the paper and try to indicate the "dérangements" of the conjugate permutation by inserting a cross in every small square which is to the right of one dot and above another, we obtain exactly the same crosses as before. The signs of the two permutations must thus be alike.

	1	2	3	4	5	6	7	8	9	10
1	×	×	.							
2	×	×		×	×	×	×	.		
3	×	×		×	.					
4	×	×		×		×	×		×	.
5	×	×		×		×	×		.	
6	×	×		.						
7	×	×				.				
8	.									
9		×					.			
10	.									

Immediately following this, the 24 permutations of 1, 2, 3, 4 are given in a column, each one having opposite it, in a parallel column, its conjugate permutation. The existence of *self-conjugate* permutations, *e.g.*, the permutation 3, 4, 1, 2 is thus brought to notice, and the substance of the following theorem in regard to them is given:—

*If  $U_n$  be the number of self-conjugate permutations of the first  $n$  integers, then*

$$U_n = U_{n-1} + (n-1)U_{n-2} . . . . . \text{ (xxv.)}$$

*where  $U_1 = 1$  and  $U_2 = 2$ .*

This, however, is the only one of his results which Rothe does not attempt to prove.

In the second part of the memoir, which contains the application of the theorems of the first part to the solution of a set of linear

equations, there is not so much that is noteworthy. Methods previously known are followed, the new features being formality and rigour of demonstration.

The coefficients of the equations being

$$\begin{array}{ccccccc} 11, & 12, & 13, & \dots, & 1r \\ 21, & 22, & 23, & \dots, & 2r \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r1, & r2, & r3, & \dots, & rr \end{array}$$

it is noted, as Vandermonde had remarked, that the common denominator of the values of the unknown may be got in two ways, viz., by permuting either all the second integers of the couples, 11, 22, 33, . . . , rr, or all the first integers: but this is supplemented by a proof, that *if any term be taken, e.g.,*

$$16.24.33.47.51.68.79.82.95$$

*with the couples so arranged that the first integers are in ascending order, and the sign be determined from the number of inversions in the series of second integers, then the sign obtained will be the same as would be got by arranging the couples so as to have the second integers in ascending order, and determining the sign from the inversions in the series of first integers.* The proof rests entirely on the previous theorem, that conjugate permutations have the same sign; indeed the new proposition is little else than another form of this theorem. (III. 14.)

The desirability of an appropriate notation for the cofactor, which any one of the coefficients has in the common denominator is recognised,\* and the want supplied by prefixing f to the coefficient in question; for example, the cofactor of 32 is denoted by

$$f32.$$

It is thus at once seen that the denominator itself is equal to

$$1n.f1n + 2n.f2n + \dots + rn.frn,$$

$$\text{or} \quad n1.fn1 + n2.fn2 + \dots + nr.fnr. \quad (\text{VI. 2.})$$

Also by this means one of Bézout's (or Vandermonde's) general theorems becomes easily expressible in symbols, viz.,

$$1n.f1m + 2n.f2m + \dots + rn.frm = 0, \quad (\text{XII. 4.})$$

\* Lagrange's use of a corresponding letter from a different alphabet must not be forgotten.



the proof of which is given as follows. In all the terms of  $f1m$ , every one of the integers except one occurs as the first integer of a couple, and every one of the integers except  $m$  occurs as the second integer of a couple: consequently, in every term of  $1n.f1m$ , the first places of the couples are occupied by the integers from 1 to  $r$  inclusive, while in the second places,  $m$  is still the only integer awaiting, and  $n$  occurs twice. Suppose then all the terms of

$$1n.f1m + 2n.f2m + \dots + rn.frm$$

so written, that the first integers of the couples are in ascending order of magnitude, and let us attend to a single term

$$\dots \cdot pn \cdot \dots \cdot qn \cdot \dots$$

in which the two couples, having  $n$  for second integer, are the  $p^{\text{th}}$  and  $q^{\text{th}}$ . If we inquire from which of the expressions  $1n.f1m$ ,  $2n.f2m$ ,  $\dots$  this term comes, we see that it is a term of both  $pn.fpm$  and  $qn.fqm$ , and must, therefore, occur twice. Further, we see that in  $pn.fqm$  it has the sign of the term

$$\dots \cdot pm \cdot \dots \cdot qn \cdot \dots$$

of the common denominator, and that in  $qn.fpm$ , it has the sign of the term

$$\dots \cdot pn \cdot \dots \cdot qm \cdot \dots$$

of the common denominator. But these two terms of the common denominator have different signs: consequently

$$1n.f1m + 2n.f2m + \dots + rn.frm$$

consists of pairs of equal terms with unlike signs, and thus vanishes identically. (XII. 4.)

These preparations having been attended to, the set of  $r$  equations with  $r$  unknowns is solved by Laplace's method; and a verification made after the manner of Vandermonde. It is also pointed out, that if the solution of a set of equations, say the four

$$\left. \begin{aligned} ax_1 + bx_2 + cx_3 + dx_4 &= s_1 \\ ex_1 + fx_2 + gx_3 + hx_4 &= s_2 \\ ix_1 + kx_2 + lx_3 + mx_4 &= s_3 \\ nx_1 + ox_2 + px_3 + qx_4 &= s_4 \end{aligned} \right\}$$

be

$$\left. \begin{aligned} x_1 &= As_1 + Bs_2 + Cs_3 + Ds_4 \\ x_2 &= Es_1 + Fs_2 + Gs_3 + Hs_4 \\ x_3 &= Is_1 + Ks_2 + Ls_3 + Ms_4 \\ x_4 &= Ns_1 + Os_2 + Ps_3 + Qs_4 \end{aligned} \right\},$$

then the solution of the set

$$\left. \begin{aligned} ay_1 + ey_2 + iy_3 + ny_4 &= v_1 \\ by_1 + fy_2 + ky_3 + oy_4 &= v_2 \\ cy_1 + gy_2 + ly_3 + py_4 &= v_3 \\ dy_1 + hy_2 + my_3 + qy_4 &= v_4 \end{aligned} \right\},$$

which has the same coefficients differently disposed, will be

$$\left. \begin{aligned} y_1 &= Av_1 + Ev_2 + Iv_3 + Nv_4 \\ y_2 &= Bv_1 + Fv_2 + Kv_3 + Ov_4 \\ y_3 &= Cv_1 + Gv_2 + Lv_3 + Pv_4 \\ y_4 &= Dv_1 + Hv_2 + Mv_3 + Qv_4 \end{aligned} \right\}; \quad \dots \quad (\text{xxvi.})$$

and hence, that the solution of a set having the special form

$$\left. \begin{aligned} ax_1 + bx_2 + cx_3 + dx_4 &= s_1 \\ bx_1 + ex_2 + fx_3 + gx_4 &= s_2 \\ cx_1 + fx_2 + hx_3 + ix_4 &= s_3 \\ dx_1 + gx_2 + ix_3 + jx_4 &= s_4 \end{aligned} \right\}$$

will itself take the same form, viz.

$$\left. \begin{aligned} As_1 + Bs_2 + Cs_3 + Ds_4 &= x_1 \\ Bs_1 + Es_2 + Fs_3 + Gs_4 &= x_2 \\ Cs_1 + Fs_2 + Hs_3 + Is_4 &= x_3 \\ Ds_1 + Gs_2 + Is_3 + Js_4 &= x_4 \end{aligned} \right\} \quad \dots \quad (\text{xxvi. 2.})$$

GAUSS (1801).

[*Disquisitiones Arithmeticae*. Auctore D. Carolo Friderico Gauss.

167 pp. Lips.]

The connection of Gauss with our theory was very similar to that of Lagrange, and doubtless was due to the fact that Lagrange had preceded him. The fifth chapter of his famous work, which is the only chapter we are concerned with, bears the title "*De formis æquationibusque indeterminatis secundi gradus*," and its subject may be described in exactly the same words as Lagrange used in regard

to his memoir *Recherches d'Arithmétique* (1773: see above), viz. "les nombres qui peuvent être représentées par la formule  $Bt^2 + Ctu + Du^2$ ."

Gauss writes his form of the second degree thus—

$$axx + 2bxy + cyy;$$

and for shortness speaks of it as the form  $(a, b, c)$ . The function of the coefficients  $a, b, c$ , which was found by Lagrange to be of notable importance in the discussion of the form, Gauss calls the "*determinant* of the form," the exact words of his definition being

"Numerum  $bb - ac$ , a cuius indole proprietates formæ  $(a, b, c)$  imprimis pendere in sequentibus docebimus, *determinantem* huius formæ uocabimus." (xv. 2.)

Here then we have the first use of the term which with an extended signification has in our day come to be so familiar. It must be carefully noted that the more general functions, to which the name came afterwards to be given, also repeatedly occur in the course of Gauss' work, e.g. the function  $\alpha\delta - \beta\gamma$  in his statement of Lagrange's theorem (xxii.)

$$b'b' - a'c' = (bb - ac)(\alpha\delta - \beta\gamma)^2.$$

But such functions are not spoken of as belonging to the same category as  $bb - ac$ . In fact the new term introduced by Gauss was not "determinant" but "determinant of a form," being thus perfectly identical in meaning and usage with the modern term "discriminant."

Notwithstanding the title of the chapter Gauss did not confine himself to forms of two variables. A digression is made for the purpose of considering the ternary quadratic form ("formam ternariam secundi gradus"),

$$axx + a'x'x' + a''x''x'' + 2bx'x'' + 2b'xx'' + 2b''xx',$$

or as he shortly denotes it

$$\begin{pmatrix} a, & a', & a'' \\ b, & b', & b'' \end{pmatrix}.$$

In the matter of nomenclature the following paragraph of this digression is interesting

$$\begin{aligned} \text{"Ponendo } bb - a'a'' = A, \quad b'b' - aa'' = A', \quad b''b'' - aa' = A'', \\ ab - b'b'' = B, \quad a'b' - bb'' = B', \quad a''b'' - bb' = B'', \end{aligned}$$

oritur alia forma

$$\begin{pmatrix} A & A' & A'' \\ B & B' & B'' \end{pmatrix} \dots F$$

quam formæ

$$\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \end{pmatrix} \dots f$$

*adjunctam* dicemus. Hinc rursus inuenitur, denotando breuitatis caussa numerum

$$abb + a'b'b' + a''b''b'' - aa'a'' - 2bb'b'' \text{ per } D,$$

$$\begin{aligned} BB - A'A'' &= aD, & B'B' - AA'' &= a'D, & B''B'' - AA' &= a''D, \\ AB - B'B'' &= bD, & A'B' - BB'' &= b'D, & A''B'' - BB' &= b''D, \end{aligned}$$

unde patet, formæ  $F$  adjunctam esse formam

$$\begin{pmatrix} aD & a'D & a''D \\ bD & b'D & b''D \end{pmatrix}.$$

Numerum  $D$ , a cuius indole proprietates formæ ternariæ  $f$  imprimis pendent, *determinantem* huius formæ uocabimus (xv. 2); hoc modo determinans formæ  $F$  sit  $= DD$ , sive æqualis quadrato determinantis formæ  $f$ , cui adjuncta est."

In this there is no advance so far as the theory of modern determinants is concerned, the identities given being those numbered (xx) and (xxi) under Lagrange. On the same page, however, an extension is given of Lagrange's theorem (xxii), regarding the determinant of the new form obtained by effecting a linear substitution on a given form. Gauss' words in regard to this are—

"Si forma aliqua ternaria  $f$  determinantis  $D$ , cuius indeterminatæ sunt  $x, x', x''$  (puta prima  $= x$ , &c.) in formam ternariam  $g$  determinantis  $E$ , cuius indeterminatæ sunt  $y, y', y''$ , transmutatur per substitutionem talem

$$\begin{aligned} x &= \alpha y + \beta y' + \gamma y'', \\ x' &= \alpha' y + \beta' y' + \gamma' y'', \\ x'' &= \alpha'' y + \beta'' y' + \gamma'' y'', \end{aligned}$$

ubi nouem coefficientes  $\alpha, \beta$ , &c. omnes supponuntur esse numeri integri, breuitatis caussa neglectis indeterminatis simpliciter dicemus,  $f$  transire in  $g$  per substitutionem ( $S$ )



$f$  transmutatum iri per substitutionem

$$\begin{array}{lll} \alpha\delta + \beta\delta' + \gamma\delta'', & \alpha\epsilon + \beta\epsilon' + \gamma\epsilon'' & \alpha\zeta + \beta\zeta' + \gamma\zeta'' \\ \alpha'\delta + \beta'\delta' + \gamma'\delta'' & \alpha'\epsilon + \beta'\epsilon' + \gamma'\epsilon'' & \alpha'\zeta + \beta'\zeta' + \gamma'\zeta'' \\ \alpha''\delta + \beta''\delta' + \gamma''\delta'' & \alpha''\epsilon + \beta''\epsilon' + \gamma''\epsilon'' & \alpha''\zeta + \beta''\zeta' + \gamma''\zeta''. \end{array} \quad (\text{xxii. 3.})$$

MONGE (1809).

[Essai d'application de l'analyse a quelques parties de la géométrie élémentaire. *Journ. de l'Ec. Polyt.*, viii. pp. 107–109.]

Lagrange, as we have already seen, was led to certain identities regarding the expression

$$xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x''$$

in the course of investigations on the subject of triangular pyramids. The position of Monge is that of Lagrange reversed. From the theory of equations he derives identities connecting such expressions, and translates them into geometrical theorems.

The simpler of these identities, as being already chronicled, we pass over. At p. 107 he takes the three equations

$$\begin{aligned} a_1u + b_1x + c_1y + d_1z + e_1 &= 0 \\ a_2u + b_2x + c_2y + d_2z + e_2 &= 0 \\ a_3u + b_3x + c_3y + d_3z + e_3 &= 0, \end{aligned}$$

and eliminating every pair of the letters  $u, x, y, z$ , obtains the six equations

$$\begin{aligned} \beta u + \alpha x + P &= 0 & (1) \\ \gamma x + \beta y + Q &= 0 & (2) \\ \delta y + \gamma z + M &= 0 & (3) \\ \alpha z + \delta u + N &= 0 & (4) \\ \gamma u - \alpha y + S &= 0 & (5) \\ \beta z - \delta x + R &= 0 & (6); \end{aligned}$$

the ten letters

$$\alpha, \beta, \gamma, \delta, M, N, P, Q, R, S$$

being used to stand for the lengthy expressions which we now-days denote by

$$\begin{aligned} |b_1c_2d_3|, |a_1c_2d_3|, |a_1b_2d_3|, |a_1b_2c_3|, \\ |a_1b_2e_3|, |b_1c_2e_3|, |c_1d_2e_3|, -|a_1d_2e_3|, |a_1c_2e_3|, |b_1d_2e_3|. \end{aligned}$$

Then, taking triads of these six equations, *e.g.*, the triads (1), (2), (5), he derives the identities

$$\left. \begin{aligned} \alpha Q + \beta S - \gamma P &= 0 \\ \delta P + \alpha R - \beta N &= 0 \\ -\gamma N + \delta S + \alpha M &= 0 \\ -\beta M + \gamma R + \delta Q &= 0 \end{aligned} \right\},$$

or

$$\left. \begin{aligned} -|b_1 c_2 d_3| \cdot |a_1 d_2 e_3| + |a_1 c_2 d_3| \cdot |b_1 d_2 e_3| - |a_1 b_2 d_3| \cdot |c_1 d_2 e_3| &= 0 \\ |a_1 b_2 c_3| \cdot |c_1 d_2 e_3| + |b_1 c_2 d_3| \cdot |a_1 c_2 e_3| - |a_1 c_2 d_3| \cdot |b_1 c_2 e_3| &= 0 \\ -|a_1 b_2 d_3| \cdot |b_1 c_3 e_3| + |a_1 b_2 c_3| \cdot |b_1 d_2 e_3| + |b_1 c_2 d_3| \cdot |a_1 b_2 e_3| &= 0 \\ -|a_1 c_2 d_3| \cdot |a_1 b_2 e_3| + |a_1 b_2 d_3| \cdot |a_1 c_2 e_3| - |a_1 b_2 c_3| \cdot |a_1 d_2 e_3| &= 0 \end{aligned} \right\} \text{(xxiii. 2.)}$$

which in their turn, he says, by processes of elimination, may be the source of many others. For example, each of the four being linear and homogeneous in  $\alpha, \beta, \gamma, \delta$ , these letters may all be eliminated with the result

$$RS + QN - PM = 0,$$

or

$$|a_1 c_2 e_3| \cdot |b_1 d_2 e_3| - |a_1 d_2 e_3| \cdot |b_1 c_2 e_3| - |c_1 d_2 e_3| \cdot |a_1 b_2 e_3| = 0.$$

Also, eliminating  $P$  from the first and second,  $S$  from the first and third,  $Q$  from the first and fourth, and so on, we have

$$\begin{aligned} -\beta \gamma N + \delta \alpha Q + \beta \delta S + \alpha \gamma R &= 0, \\ \alpha \beta M + \gamma \delta P - \beta \gamma N - \delta \alpha Q &= 0, \\ \alpha \beta M - \gamma \delta P + \beta \delta S - \alpha \gamma R &= 0, \\ &\&c. \qquad \&c. \end{aligned}$$

*i.e.*

$$\begin{aligned} -|a_1 c_2 d_3| \cdot |a_1 b_2 d_3| \cdot |b_1 c_2 e_3| - |a_1 b_2 c_3| \cdot |b_1 c_2 d_3| \cdot |a_1 d_2 e_3| \\ + |a_1 c_2 d_3| \cdot |a_1 b_2 c_3| \cdot |b_1 d_2 e_3| + |b_1 c_2 d_3| \cdot |a_1 b_2 d_3| \cdot |a_1 c_2 e_3| \end{aligned} \Big\} = 0,$$

&c. &c. (xxviii.)

Monge does not pursue the subject further. His method, however, is seen to be quite general; and we can readily believe that he possessed numerous other identities of the same kind. This is borne out by a statement in Binet's important memoir of 1812. Binet, who was familiar with what had been done by Vandermonde, Laplace, and Gauss, says (p. 286):—"M. Monge m'a communiqué, depuis la lecture de ce mémoire, d'autres théorèmes très-remarquables sus ces résultantes; mais ils ne sont pas du genre de ceux que nous nous proposons de donner ici."

HIRSCH (1809).

[Sammlung von Aufgaben aus der Theorie der algebraischen Gleichungen, von Meier Hirsch. pp. 103–107. Berlin, 1809.]

The 4th Chapter *Von der Elimination u. s. w.*, contains five pages on the subject of the solution of simultaneous linear equations. These embrace nothing more noteworthy than a statement, without proof, of Cramer's rule, separated into three parts (iv., iii. 2, v.), and carefully worded.

BINET (May 1811).

[Mémoire sur la théorie des axes conjugués et des momens d'inertie des corps. *Journ. de l'École Polytechnique*, ix. (pp. 41–67), pp. 45, 46.]\*

In this well-known memoir, in which the conception of the *moment of inertia of a body with respect to a plane* was first made known, there repeatedly occur expressions, which at the present day would appear in the notation of determinants. There is only one paragraph, however, containing anything new in regard to these functions. It stands as follows:—

“Le moment d'inertie minimum pris par rapport au plan (C), a pour valeur

$$\Sigma mk^2 = f^2 \times$$

$$\frac{ABC - AF^2 - BE^2 - CD^2 + 2DEF}{g^2(BC - F^2) + h^2(AC - E^2) + i^2(AB - D^2) + 2gh(EF - CD) + 2gi(DF - BE) + 2hi(DE - AF)}.$$

Si, dans le numérateur,

$$ABC - AF^2 - BE^2 - CD^2 + 2DEF$$

on remplace A, B, C, &c. par  $\Sigma mx^2$ ,  $\Sigma my^2$ , &c. que ces lettres représentent, on a

$$\begin{aligned} & \Sigma mx^2 \Sigma my^2 \Sigma mz^2 - \Sigma mx^2 (\Sigma myz)^2 - \Sigma my^2 (\Sigma mxz)^2 \\ & - \Sigma mz^2 (\Sigma mxy)^2 + 2 \Sigma mxy \Sigma mxz \Sigma myz, \end{aligned}$$

et l'on peut s'assurer que cette expression est identique à

$$\Sigma mm'm''(xy'z'' + yz'x'' + zx'y'' - xz'y'' - yx'z'' - zy'x'')^2;$$

par une transformation analogue, on peut ramener la quantité

\* An abstract of this is given in the *Nouv. Bull. des Sciences par la Société Philomatique*, ii. pp. 312–316.



$$g^2(BC - F^2) + h^2(AC - E^2) + i^2(AB - D^2) \\ + 2gh(EF - CD) + 2gi(DF - BE) + 2hi(DE - AF),$$

à celle-ci

$$\Sigma mm'[g(yz' - zy') + h(zx' - xz') + i(xy' - yx')]^2."$$

Now the numerator referred to would at the present day be written.

$$\begin{vmatrix} A & D & E \\ D & B & F \\ E & F & C \end{vmatrix},$$

and since  $\Sigma mx^2$ , &c. stand for  $mx^2 + m_1x_1^2 + m_2x_2^2 + \dots$ , &c., the first identity given may be put in the form

$$\begin{vmatrix} mx^2 + m_1x_1^2 + m_2x_2^2 + \dots & mxy + m_1x_1y_1 + m_2x_2y_2 + \dots & mxz + m_1x_1z_1 + m_2x_2z_2 + \dots \\ mxy + m_1x_1y_1 + m_2x_2y_2 + \dots & my^2 + m_1y_1^2 + m_2y_2^2 + \dots & myz + m_1y_1z_1 + m_2y_2z_2 + \dots \\ mxz + m_1x_1z_1 + m_2x_2z_2 + \dots & myz + m_1y_1z_1 + m_2y_2z_2 + \dots & mz^2 + m_1z_1^2 + m_2z_2^2 + \dots \end{vmatrix} \\ = mm_1m_2 \begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ z & z_1 & z_2 \end{vmatrix}^2 + mm_1m_3 \begin{vmatrix} x & x_1 & x_3 \\ y & y_1 & y_3 \\ z & z_1 & z_3 \end{vmatrix}^2 + \dots \quad (\text{XVIII. 2.})$$

where  $x_1, y_2, \dots$  are for convenience written instead of  $x', y'', \dots$ . It will be seen that this is an important extension of a theorem of Lagrange, the latter theorem being the very special case of the present obtained by putting  $m = m_1 = m_2 = 1$ , and  $m_3 = m_4 = \dots = 0$ , —a fact which is brought still more clearly into evidence if, instead of the left-hand member of the identity, we write the modern contraction for it, viz.

$$\begin{vmatrix} mx & m_1x_1 & m_2x_2 & m_3x_3 & \dots \\ my & m_1y_1 & m_2y_2 & m_3y_3 & \dots \\ mz & m_1z_1 & m_2z_2 & m_3z_3 & \dots \end{vmatrix} \times \begin{vmatrix} x & x_1 & x_2 & x_3 & \dots \\ y & y_1 & y_2 & y_3 & \dots \\ z & z_1 & z_2 & z_3 & \dots \end{vmatrix}.$$

Again the denominator

$$g^2(BC - F^2) + h^2(AC - E^2) + i^2(AB - D^2) \\ + 2gh(EF - CD) + 2gi(DF - BE) + 2hi(DE - AF)$$

being in modern notation

$$\begin{vmatrix} . & g & h & i \\ g & A & D & E \\ h & D & B & F \\ i & E & F & C \end{vmatrix},$$

the second identity may be written

$$\begin{vmatrix} \cdot & g & h & i \\ g & mx^2 + m_1x_1^2 + \dots & mxy + m_1x_1y_1 + \dots & mxz + m_1x_1z_1 + \dots \\ h & mxy + m_1x_1y_1 + \dots & my^2 + m_1y_1^2 + \dots & myz + m_1y_1z_1 + \dots \\ i & mxz + m_1x_1z_1 + \dots & myz + m_1y_1z_1 + \dots & mz^2 + m_1z_1^2 + \dots \end{vmatrix} \\ = mm_1 \begin{vmatrix} g & x & x_1 \\ h & y & y_1 \\ i & z & z_1 \end{vmatrix}^2 + mm_2 \begin{vmatrix} g & x & x_2 \\ h & y & y_2 \\ i & z & z_2 \end{vmatrix}^2 + m_1m_2 \begin{vmatrix} g & x_1 & x_2 \\ h & y_1 & y_2 \\ i & z_1 & z_2 \end{vmatrix}^2 + \dots \text{(XXIX.)}$$

This also is an important theorem, and is not so much an extension of previous work as a breaking of fresh ground.

### BINET (November 1811).

[Sur quelques formules d'algèbre, et sur leur application à des expressions qui ont rapport aux axes conjugués des corps. *Nouv. Bull. des Sciences par la Société Philomatique*, ii. pp. 389-392.]

In this paper Binet returns to the consideration of the first of the two identities which have just been referred to, writing it now in the form

$$\Sigma(xy'z'' - xz'y'' + yz'x'' - yx'z'' + zx'y'' - zy'x'')^2 \\ = \Sigma x^2 \Sigma y^2 \Sigma z^2 - \Sigma x^2 (\Sigma yz)^2 - \Sigma y^2 (\Sigma xz)^2 - \Sigma z^2 (\Sigma xy)^2 + 2 \Sigma xy \Sigma xz \Sigma yz.$$

He puts it in the same category as the identity

$$\Sigma(y'z - zy')^2 = \Sigma y^2 \Sigma z^2 - (\Sigma yz)^2,$$

which he speaks of as being then known. Further, he says

“Ces deux formules sont du même genre que la suivante

$$\left\{ \begin{aligned} & ux'y'z''' - ux'z'y''' + uy'z'x''' - uy'x'z''' + uz'x'y''' - uz'y'x''' + xy'u'z''' - xy'z'u''' \\ & + xz'y'u''' - xz'u'y''' + xu'z'y''' - xu'z'y''' + yz'u'x''' - yz'u'x''' + yu'x'z''' - yu'x'z''' \\ & + yx'z'u''' - yx'u'z''' + zu'y'x''' - zu'y'x''' + zx'y'u''' - zx'y'u''' + zy'u'x''' - zy'u'x''' \end{aligned} \right\}^2 \\ = \Sigma u^2 \Sigma x^2 \Sigma y^2 \Sigma z^2 - \Sigma u^2 \Sigma x^2 (\Sigma yz)^2 - \Sigma u^2 \Sigma y^2 (\Sigma xz)^2 - \Sigma u^2 \Sigma z^2 (\Sigma xy)^2 \\ - \Sigma x^2 \Sigma y^2 (\Sigma uz)^2 - \Sigma x^2 \Sigma z^2 (\Sigma uy)^2 - \Sigma y^2 \Sigma z^2 (\Sigma ux)^2 \\ + 2 \Sigma u^2 \Sigma xy \Sigma xz \Sigma yz + 2 \Sigma x^2 \Sigma uy \Sigma uz \Sigma yz + 2 \Sigma y^2 \Sigma ux \Sigma uz \Sigma xz \\ + 2 \Sigma z^2 \Sigma ux \Sigma uy \Sigma xy + (\Sigma ux)^2 (\Sigma yz)^2 + (\Sigma uy)^2 (\Sigma xz)^2 + (\Sigma uz)^2 (\Sigma xy)^2 \\ - 2 \Sigma ux \Sigma xy \Sigma yz \Sigma zu - 2 \Sigma uy \Sigma yz \Sigma xz \Sigma xu - 2 \Sigma uy \Sigma yz \Sigma xz \Sigma zu,$$

—a result which in modern notation would take the form

$$\begin{vmatrix} u & u_1 & u_2 & u_3 \\ x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \end{vmatrix}^2 + \begin{vmatrix} u & u_1 & u_2 & u_4 \\ x & x_1 & x_2 & x_4 \\ y & y_1 & y_2 & y_4 \\ z & z_1 & z_2 & z_4 \end{vmatrix}^2 + \dots$$

$$= \begin{vmatrix} u^2 + u_1^2 + \dots & ux + u_1x_1 + \dots & uy + u_1y_1 + \dots & uz + u_1z_1 + \dots \\ ux + u_1x_1 + \dots & x^2 + x_1^2 + \dots & xy + x_1y_1 + \dots & xz + x_1z_1 + \dots \\ uy + u_1y_1 + \dots & xy + x_1y_1 + \dots & y^2 + y_1^2 + \dots & yz + y_1z_1 + \dots \\ uz + u_1z_1 + \dots & xz + x_1z_1 + \dots & yz + y_1z_1 + \dots & z^2 + z_1^2 + \dots \end{vmatrix} \quad (\text{XVIII. 3.})$$

It is thus clear that, in November 1811, Binet was well on the way towards a great generalisation. He even says that the three identities may be looked upon

“comme les trois premières d’une suite de formules construites d’après une même loi facile à saisir.”

He merely indicates, however, the mode of proof he would adopt for the results obtained, and refers to possible applications of them in investigations regarding the Method of Least Squares (Laplace, *Connaissance des Temps*, 1813) and the Centre of Gravity (Lagrange, *Mém. de Berlin*, 1783). The mode of proof need not be given here, as it turns up again in the far more important memoir in which the theorem in all its generality falls to be considered.

#### DE PRASSE (1811).

[*Commentationes Mathematicæ. Auctore Mauricio de Prasse. 120 pp. Lips., 1804, 1812. Pp. 89–102; Commentatio vii.\*: Demonstratio eliminationis Cramerianæ.*]

Of previous writings the one which De Prasse’s most resembles is Rothe’s. There is less of it, and it shows less freshness; but there is the same stiff formality of arrangement, and the same effort at rigour of demonstration.

\* Separate copies of the *Demonstratio eliminationis Cramerianæ* are also to be found, bearing the invitation title-page:

*Ad memoriam Kregelio-Sternbachianam in auditorio philosophorum die xviii Julii MDCCCXI. h. ix celebrandam invitavit ordinum Academicæ Lips. Decani seniores cæterique adsesores . . . Demonstratio eliminationis Cramerianæ.*

It is these copies which fix the date. See *Nature*, xxxvii. pp. 246, 247.

The definition of a permutation (*variatio*) being given, the first problem (which, however, is called a theorem) is propounded, viz., to tabulate the permutations of  $\alpha, \beta, \gamma, \delta, \dots$  ("Variationum ex elementis  $\alpha, \beta, \gamma, \dots$  constructarum et in Classes combinatorias digestarum Tabulam parare"). The result is

$\alpha$	$\beta$	$\gamma$	$\delta$
$\alpha\beta$	$\alpha\gamma$	$\alpha\delta$	}
$\beta\alpha$	$\beta\gamma$	$\beta\delta$	
$\gamma\alpha$	$\gamma\beta$	$\gamma\delta$	
$\delta\alpha$	$\delta\beta$	$\delta\gamma$	
<hr/>			
	$\alpha\beta\gamma$	$\alpha\beta\delta$	}
	$\alpha\gamma\beta$	$\alpha\gamma\delta$	
	$\alpha\delta\beta$	$\alpha\delta\gamma$	
	$\beta\alpha\gamma$	$\beta\alpha\delta$	
	$\beta\gamma\alpha$	$\beta\gamma\delta$	
	$\beta\delta\alpha$	$\beta\delta\gamma$	
	$\gamma\alpha\beta$	$\gamma\alpha\delta$	
	$\gamma\beta\alpha$	$\gamma\beta\delta$	
	$\gamma\delta\alpha$	$\gamma\delta\beta$	
	$\delta\alpha\beta$	$\delta\alpha\gamma$	
	$\delta\beta\alpha$	$\delta\beta\gamma$	
	$\delta\gamma\alpha$	$\delta\gamma\beta$	
<hr/>			
	$\alpha\beta\gamma\delta$		}
	$\alpha\beta\delta\gamma$		
	$\alpha\gamma\beta\delta$		
	$\alpha\gamma\delta\beta$		
	$\alpha\delta\beta\gamma$		
	$\alpha\delta\gamma\beta$		
	$\beta\alpha\gamma\delta$		
	$\beta\alpha\delta\gamma$		
	$\beta\gamma\alpha\delta$		
	$\beta\gamma\delta\alpha$		
	$\beta\delta\alpha\gamma$		
	$\beta\delta\gamma\alpha$		
	$\gamma\alpha\beta\delta$		
	$\gamma\alpha\delta\beta$		
	$\gamma\beta\alpha\delta$		
	$\gamma\beta\delta\alpha$		
	$\gamma\delta\alpha\beta$		
	$\gamma\delta\beta\alpha$		
	$\delta\alpha\beta\gamma$		
	$\delta\alpha\gamma\beta$		
	$\delta\beta\alpha\gamma$		
	$\delta\beta\gamma\alpha$		
	$\delta\gamma\alpha\beta$		
	$\delta\gamma\beta\alpha$		

The first row of the permutations involving two letters is got by taking the first letter of the previous row and annexing each of the others to it in succession and in the order of their occurrence; the second row is got in like manner from the second letter; and so on. Similarly the first row of permutations involving three letters is got from  $\alpha\beta$  the first obtained permutation of two letters, the second row from  $\alpha\gamma$  the next obtained permutation of two letters, and so on.\*

The second problem (and on this occasion actually so designated) is somewhat quaint in its indefiniteness, viz., to prefix to each permutation the sign + or the sign -, so that the sum of all the permutations involving the same number of letters ( $>1$ ) may vanish ("*Singulis Variationibus, omissis repetitionibus, signa + et - ita praefigere, ut summa secundae et cujuslibet classis insequentis evanescat*"). There is no indefiniteness or multiplicity about the solution, which in substance is:—Make the permutations in every row of the preceding table alternately + and -, the first sign of all being +, and the first permutation of every other row having the same sign as the permutation from which it was derived. In this way the table becomes

$+ \alpha, - \beta, + \gamma, - \delta$	}
$+ \alpha\beta, - \alpha\gamma, + \alpha\delta$	}
$- \beta\alpha, + \beta\gamma, - \beta\delta$	
$+ \gamma\alpha, - \gamma\beta, + \gamma\delta$	
$- \delta\alpha, + \delta\beta, - \delta\gamma$	
$+ \alpha\beta\gamma, - \alpha\beta\delta$	}
$- \alpha\gamma\beta, + \alpha\gamma\delta$	
$+ \alpha\delta\beta, - \alpha\delta\gamma$	
$- \beta\alpha\gamma, + \beta\alpha\delta$	
$+ \beta\gamma\alpha, - \beta\gamma\delta$	
$- \beta\delta\alpha, + \beta\delta\gamma$	
$+ \gamma\alpha\beta, - \gamma\alpha\delta$	
$- \gamma\beta\alpha, + \gamma\beta\delta$	
$+ \gamma\delta\alpha, - \gamma\delta\beta$	
$- \delta\alpha\beta, + \delta\alpha\gamma$	
$+ \delta\beta\alpha, - \delta\beta\gamma$	
$- \delta\gamma\alpha, + \delta\gamma\beta$	

\* It will be seen that the order in which the permutations come to hand in this process of tabulation is the order in which they would be arranged according to magnitude if each permutation were viewed as a number of which  $\alpha, \beta, \gamma, \delta$  were the digits,  $\alpha$  being  $< \beta < \gamma < \delta$  ("ordo lexicographicus," "lexicographische Anordnung" of Hindenburg).

$$\begin{array}{l}
 +\alpha\beta\gamma\delta \\
 -\alpha\beta\delta\gamma \\
 -\alpha\gamma\beta\delta \\
 +\alpha\gamma\delta\beta \\
 +\alpha\delta\beta\gamma \\
 -\alpha\delta\gamma\beta \\
 \\ 
 -\beta\alpha\gamma\delta \\
 +\beta\alpha\delta\gamma \\
 +\beta\gamma\alpha\delta \\
 -\beta\gamma\delta\alpha \\
 -\beta\delta\alpha\gamma \\
 +\beta\delta\gamma\alpha \\
 \\ 
 +\gamma\alpha\beta\delta \\
 -\gamma\alpha\delta\beta \\
 -\gamma\beta\alpha\delta \\
 +\gamma\beta\delta\alpha \\
 +\gamma\delta\alpha\beta \\
 -\gamma\delta\beta\alpha \\
 \\ 
 -\delta\alpha\beta\gamma \\
 +\delta\alpha\gamma\beta \\
 +\delta\beta\alpha\gamma \\
 -\delta\beta\gamma\alpha \\
 -\delta\gamma\alpha\beta \\
 +\delta\gamma\beta\alpha
 \end{array}
 .$$

A proof by the method of mathematical induction (so-called) is given that with these signs the sum of all the permutations of any group vanishes.

Up to this point the essence of what has been furnished is a combined rule of term-formation and rule of signs. (II. 5 + III. 15.) In connection with it Bézout's rule of the year 1764 may be recalled.

The third problem is to determine the sign of any single permutation from consideration of the permutation itself. The solution is:—Under each letter of the given permutation put all the letters which precede it in the natural arrangement and which are not found to precede it in the given permutation; and make the sum + or - according as the total number of such letters is even or odd.

“EXEMP. Datae complexiones sint hæ :

$$\epsilon\gamma\delta\beta, \quad \delta\alpha\epsilon\gamma, \quad \epsilon\delta\gamma\alpha, \quad \delta\beta\epsilon\gamma.$$

Literæ secundum I subjiciantur

$\alpha\alpha\alpha\alpha$	$\alpha.\beta\beta$	$\alpha\alpha\alpha.$	$\alpha\alpha\alpha\alpha$
$\beta\beta\beta$	$\beta\gamma$	$\beta\beta\beta$	$\beta.\gamma$
$\gamma$	$\gamma$	$\gamma\gamma$	$\gamma$
$\delta$		$\delta$	

quarum numeri sunt

9          6          9          7

qui complexionibus datis præfigi jubent signa

-          +          -          -."

The proof that this rule of signs, which is manifestly nothing else than Cramer's, leads to the same results as the previous rule, is quite easily understood if a particular permutation be first considered. For example, let the sign of the particular permutation  $\delta\beta\alpha\gamma$  be wanted. Following the first rule, we should require to note four different members, viz.,

- (1) the no. of the column in which  $\delta\beta\alpha\gamma$  occurs in the 4th group,
- (2)        "                "                 $\delta\beta\alpha$         "        3rd    "
- (3)        "                "                 $\delta\beta$             "        2nd    "
- (4)        "                "                 $\delta$              "        1st    " .

The first of these numbers being 1, we should infer that in fixing the sign of  $\delta\beta\alpha\gamma$  in the fourth group there had been no change from the sign of  $\delta\beta\alpha$  in the third group; the second number being also 1, we should make a like inference; the third number being 2, we should infer that in fixing the sign of  $\delta\beta$  in the second group there had been 1 change from the sign of  $\delta$  in the first group; and finally, the fourth number being 4, we should infer that in fixing the sign of  $\delta$  in the first group there had been 3 changes from the sign of  $\alpha$  in that group. The total number of changes from the sign of  $\alpha$  in the first group being thus  $3 + 1 + 0 + 0$ , i.e., 4, the sign would be made +. Now the 3 in this aggregate is simply the number of letters in the first group which precede  $\delta$ , the 1 is simply the number of letters taken along with  $\delta$  before  $\beta$  comes to be taken along with it to form  $\delta\beta$  in the second group, and the two zeros correspond to the fact that  $\delta\beta\alpha$  on the third group and  $\delta\beta\alpha\gamma$  on the fourth group have no permutation standing to the left of them. Consequently to count the number of changes ( $3 + 1 + 0 + 0$ ) from the

sign of  $\alpha$  in accordance with the first rule is the same as to count the number of letters placed under the given permutation, thus,

$$\begin{array}{c} \delta\beta\alpha\gamma \\ \alpha\alpha\dots \\ \beta \\ \gamma \end{array}$$

in accordance with the second rule.

Another point of resemblance between Rothe and De Prasse is thus made manifest, viz., that they both refused to accept Cramer's rule of signs as fundamental, preferring to base their work on a rule equally arbitrary, and then to deduce Cramer's from it.

In case it may have escaped the reader, attention may likewise be drawn to the fact that De Prasse prefixes a sign not only to permutations involving all the letters dealt with, but also to any permutation whatever involving a less number; so that in reckoning the sign of  $\alpha\delta\beta$ , say, the full number of letters from which  $\alpha$ ,  $\delta$ ,  $\beta$  are chosen must be known.

A theorem like Hindenburg's is next given, viz., *If the permutations of any group be separated into sub-groups, (1) those which begin with  $\alpha$ , (2) those which begin with  $\beta$ , and so on, then the series of signs of the 3rd, 5th, and other odd sub-groups is identical with the series of signs of the 1st sub-group, and the signs of any one of the even sub-groups is got by changing each sign of the first sub-group into the opposite sign.* (III. 16.)

It is more extensive than Hindenburg's in that it is true of permutations which involve less than all the letters, provided such permutations have had their signs fixed in accordance with De Prasse's rule. The proof depends, of course, on the first rule of signs, and consists in showing that if the theorem be true for any group it must, by the said rule, be true for the next group. It will be remembered that Hindenburg gave no proof.

Following this is Rothe's theorem regarding the interchange of two elements of a permutation, or rather an extension of the theorem to signed permutations involving less than the whole number of letters. The proof is as lengthy as Rothe's, even more unnecessary letters than Rothe's  $c$ ,  $f$ ,  $e$  being introduced. (III. 17.)

The last theorem is Vandermonde's (XII.); and this is followed by



two pages of application to the solution of simultaneous linear equations.

No reference is made by De Prasse to Hindenburg, Rothe, or Vandermonde.

### WRONSKI (1812).

[*Réfutation de la Théorie des Fonctions Analytiques de Lagrange.*

Par Höéné Wronski, pp. 14, 15, . . . , 132, 133. Paris.]

In 1810 Wronski presented to the Institute of France a memoir on the so-called *Technie de l'Algorithmie*, which with his usual sanguine enthusiasm he viewed as the essential part of a new branch of Mathematics. It contained a very general theorem, now known as "Wronski's theorem," for the expansion of functions,—a theorem requiring for its expression the use of a notation for what Wronski styled *combinatory sums*. The memoir consisted merely of a statement of results, and probably on this account, although favourably reported on by Lagrange and Lacroix, was not printed. The subject of it, however, turns up repeatedly in the *Réfutation* printed two years later; and from the indications there given we can so far form an idea of the grasp which Wronski had of the theory of the said *sums*.

At page 14 the following passage occurs:—

"Soient  $X_1, X_2, X_3$ , &c. plusieurs fonctions d'une quantité variable. Nommons *somme combinatoire*, et désignons par la lettre hébraïque *sin*, de la manière que voici

$$\psi[\Delta^a X_1 . \Delta^b X_2 . \Delta^c X_3 . . . \Delta^p X_\pi], \quad (\text{xv. 3}) \quad (\text{vii. 4})$$

la somme des produits des différences de ces fonctions, composés de la manière suivante: Formez, avec les exposans  $a, b, c, . . . , p$  des différences dont il est question, toutes les permutations possibles; donnez ces exposans, dans chaque ordre de leurs permutations, aux différences consécutives qui composent le produit

$$\Delta X_1 . \Delta X_2 . \Delta_3 . . . \Delta X_\pi;$$

donnez de plus, aux produits séparés, formés de cette manière, le signe positif lorsque le nombre de variations des exposans  $a, b, c$ , etc., considérés dans leur ordre alphabétique, est nul ou

pair, et le signe négatif lorsque ce nombre de variations est impair; enfin, prenez la somme de tous ces produits séparés.— Vous aurez ainsi, par exemple,

$$\begin{aligned}\psi[\Delta^a X_1] &= \Delta^a X_1, \\ \psi[\Delta^a_1 \cdot \Delta^b X_2] &= \Delta^a X_1 \cdot \Delta^b X_2 - \Delta^b X_1 \cdot \Delta^a X_2, \\ &\dots\dots\dots\end{aligned}$$

The new name, *combinatory sum*, and the new notation, did not originate in ignorance of the work of previous investigators, for memoirs of Vandermonde and Laplace are referred to. The only fresh and real point of interest lies in the fact that the first index of every pair of indices is not attached to the same letter as the second index, but belongs to an operational symbol preceding this letter, and is used for the purpose of denoting repetition of the operation. This and the allied fact that the elements are not all independent of each other,  $\Delta^1 X_1$  and  $\Delta^2 X_1$ , for example, being connected by the equation

$$\Delta^2 X_1 = \Delta(\Delta^1 X_1),$$

indicate that Wronski's combinatory sums form a special class with properties peculiar to themselves.

#### BINET (November 1812).

[Mémoire sur un système de formules analytiques, et leur application à des considérations géométriques. *Journ. de l'Ec. Polyt.*, ix. cah. 16, pp. 280–302, . . .]

It would seem as if the above-noted frequent recurrence of functions of the same kind had led Binet to a special study of them. In the memoir we have now come to, his standpoint towards them is changed. They are viewed as functions having a history: for information regarding them, the writings of Vandermonde, Laplace, Lagrange, and Gauss are referred to: they are spoken of by Laplace's name for them, *résultantes à deux lettres, à trois lettres, à quatre lettres, &c.*; and the first twenty-three pages of the memoir are devoted expressly to establishing new theorems regarding them.

Of these the fundamental, and by far the most notable, is the afterwards well-known *multiplication-theorem*. It is enunciated at the outset as follows:—

“Lorsqu'on a deux systèmes de  $n$  lettres chacun, et nous supposons chaque système écrit avec une seule lettre portant divers accens, qui serviront à ranger dans le même ordre les deux systèmes; on peut former avec ces lettres un nombre  $n\frac{n-1}{2}$  de résultantes à deux lettres, en ne prenant dans le second terme de chacune, que des lettres portant les mêmes accens que celles du premier. Si, avec deux autres systèmes de lettres, on forme encore des résultantes à deux lettres, et qu'on les multiplie chacune par sa correspondante obtenue des deux premiers systèmes, c'est-à-dire, par celle dont les lettres portent les mêmes accens; la somme des produits de toutes ces résultantes correspondantes sera elle-même une résultante à deux lettres, dont les termes ou lettres seront des sommes de produits des élémens des deux systèmes portant les mêmes accens. Avec deux groupes de trois systèmes de lettres chacun, on peut former semblablement deux séries de résultantes à trois lettres; faisant ensuite la somme des produits de celles qui se correspondent par les accens de leurs lettres, on aura encore une résultante à trois lettres. Pareille chose ayant lieu pour des résultantes à quatre lettres, &c., on peut conclure ce théorème: Le produit d'un nombre quelconque de sommes de produits\* de deux résultantes correspondantes de même ordre, est encore une résultante de cet ordre.”

(xvii. 4 + xviii. 4.)

The mode of proof adopted is lengthy, laborious, and not very satisfactory, except as affording a verification of the theorem for the cases of “résultantes” of low orders. It rests too on certain identities, the demonstration of which is open to similar criticism. All that Binet says regarding these absolutely essential identities is (p. 284)—

“Je représenterai par  $\Sigma a$  la somme  $a' + a'' + a''' + \&c.$ , des quantités  $a'$ ,  $a''$ ,  $a'''$ , &c.; par  $\Sigma ab$  la somme des produits  $ab + a'b' + a''b'' + \&c.$ , dans chacun desquels les lettres  $a$  et  $b$  ont le même accent; par  $\Sigma ab'$  la somme  $a'b'' + b'a'' + a'b''' + \&c.$ ,

\* There is an extension here which one is scarcely prepared for, viz., “*le produit d'un nombre quelconque de sommes de produits,*” instead of *la somme d'un nombre de produits.*

là tous les produits d'un des  $a$  par un des  $b$ , portent un accent différent de celui de  $a$ ; par  $\Sigma ab'c''$  la somme  $a'b''c''' + b'c''a''' + c'a''b''' + \&c.$ , et ainsi de suite. Cela posé, on vérifie aisément les formules suivantes:

$$\begin{aligned}\Sigma ab' &= \Sigma a \Sigma b - \Sigma ab, \\ \Sigma ab'c'' &= \Sigma a \Sigma b \Sigma c + 2 \Sigma abc - \Sigma a \Sigma bc - \Sigma b \Sigma ca - \Sigma c \Sigma ab, \\ \Sigma ab'c''d''' &= \Sigma a \Sigma b \Sigma c \Sigma d - 6 \Sigma abcd \\ &\quad - \Sigma a \Sigma b \Sigma cd - \Sigma a \Sigma c \Sigma bd - \Sigma a \Sigma d \Sigma bc \\ &\quad - \Sigma c \Sigma d \Sigma ab - \Sigma b \Sigma d \Sigma ac - \Sigma b \Sigma c \Sigma ad \\ &\quad + \Sigma ab \Sigma cd + \Sigma ac \Sigma bd + \Sigma ab^* \Sigma bc \\ &\quad + 2 \Sigma a \Sigma bcd + 2 \Sigma b \Sigma cda + 2 \Sigma c \Sigma dab \\ &\quad + 2 \Sigma d \Sigma abc, \\ \Sigma ab'c''d'''e^{iv} &= \Sigma a \Sigma b \Sigma c \Sigma d \Sigma e + \&c., \\ &\quad \&c.\end{aligned}$$

It is thus seen that not only is no general proof of the identities given, but that even the law of formation of the right-hand members of the identities themselves is left undivulged. The exact words employed in the demonstration of the first case of the multiplication-theorem are (p. 286)—

“Avec un nombre  $n$  de lettres  $y', y'', y'''$ , &c. et un même nombre de  $z', z'', z'''$ , &c. on peut former  $n \frac{n-1}{2}$  résultantes à deux lettres ( $y', z''$ ), ( $y', z'''$ ), &c. ( $y'', z'''$ ) &c.; ayant formé pareillement avec les lettres,  $v', v'', v'''$ , &c.,  $\zeta', \zeta'', \zeta'''$  &c., les résultantes ( $v', \zeta''$ ), ( $v', \zeta'''$ ), &c., ( $v'', \zeta'''$ ), &c., considérons la somme  $\Sigma(y, z')(v, \zeta')$  des produits des résultantes qui se correspondent par les accens dans les deux systèmes. On voit, en développant, par la multiplication, chacun des termes de cette somme, qu'elle revient à

$$\Sigma yv.z'\zeta' - \Sigma zv.y'\zeta'.$$

A ces deux dernières intégrales, on peut appliquer la transformation indiquée par la première des formules de l'art. 1: on parvient ainsi à

$$\Sigma(y, z')(v, \zeta') = \Sigma yv \Sigma z \zeta - \Sigma zv \Sigma y \zeta.$$

Ce dernier membre pouvant être assimilé à la forme  $(y, z')$ , il

\* Meant for  $\Sigma ad$ .

en résulte que le produit d'un nombre quelconque de fonctions, telles que  $\Sigma(y, z')(v, \zeta')$ , est lui-même de la forme  $(y, z')$ ."

The application here of the identity

$$\Sigma ab' = \Sigma a \Sigma b - \Sigma ab$$

requires a little attention. The result of multiplication and classification of the terms is

$$\Sigma yv.z'\zeta' - \Sigma zv.y'\zeta',$$

or, as it might preferably be written,

$$\Sigma\{\overline{yv} . \overline{z'\zeta'}\} - \{\overline{\Sigma zv} . \overline{y'\zeta'}\};$$

and this we know from the said identity

$$= [\Sigma \overline{yv} . \Sigma \overline{z'\zeta'} - \Sigma(\overline{yv} . \overline{z'\zeta'})] - [\Sigma \overline{zv} . \Sigma \overline{y'\zeta'} - \Sigma(\overline{zv} . \overline{y'\zeta'})],$$

which, because of the equality of  $\Sigma(\overline{yv} . \overline{z'\zeta'})$  and  $\Sigma(\overline{zv} . \overline{y'\zeta'})$ , becomes

$$\Sigma \overline{yv} . \Sigma \overline{z'\zeta'} - \Sigma \overline{zv} . \Sigma \overline{y'\zeta'}.$$

The inherent weak points, however, of the mode of demonstration stand out more clearly when the next case comes to be considered, viz., the case for resultants of the third order. From the three sets of  $n$  letters

$$\begin{array}{ccccccc} x, & x', & x'', & . & . & . & . \\ y, & y', & y'', & . & . & . & . \\ z, & z', & z'', & . & . & . & . \end{array}$$

all possible "résultantes à trois lettres" are formed, and each resultant is multiplied by the corresponding resultant formed from other three sets of  $n$  letters,

$$\begin{array}{ccccccc} \xi, & \xi', & \xi'', & . & . & . & . \\ v, & v', & v'', & . & . & . & . \\ \zeta, & \zeta', & \zeta'', & . & . & . & . \end{array}$$

Each of these  $\frac{1}{6}n(n-1)(n-2)$  products consists of 36 terms, there being thus  $6n(n-1)(n-2)$  terms in all. But these  $6n(n-1)(n-2)$  terms are found to be separable into six groups, viz.

$$+ \Sigma\{x\xi . y'v' . z''\zeta''\} , + \Sigma\{y\xi . z'v' . x''\zeta''\} , \dots$$

so that the result which we are able to register at this point is

$$\begin{aligned} \Sigma(x, y', z'')(\xi, v', \zeta'') = & \Sigma x\xi . y'v' . z''\zeta'' + \Sigma y\xi . z'v' . x''\zeta'' \\ & + \Sigma z\xi . x'v' . y''\zeta'' - \Sigma x\xi . z'v' . y''\zeta'' \\ & - \Sigma y\xi . x'v' . z''\zeta'' - \Sigma z\xi . y'v' . x''\zeta'' . \end{aligned}$$

To the right hand member of this the substitution

$$\Sigma ab'e'' = \Sigma a \Sigma b \Sigma c + 2 \Sigma abc - \Sigma a \Sigma bc - \Sigma b \Sigma ca - \Sigma c \Sigma ab$$

is now applied six times in succession ; that is to say, for

$$\Sigma x\xi . y'v' . z''\zeta''$$

and the five other term-aggregates which follow, we substitute

$$\begin{aligned} & \Sigma x\xi \Sigma yv \Sigma z\zeta + 2 \Sigma (x\xi . yv . z\zeta) \\ & - \Sigma x\xi \Sigma (yv . z\zeta) - \Sigma yv \Sigma (z\zeta . x\xi) - \Sigma z\zeta \Sigma (x\xi . yv) \end{aligned}$$

and five other like expressions. By this means we arrive, "toute réduction faite," at

$$\begin{aligned} \Sigma (x, y', z'') (\xi, v', \zeta'') = & \Sigma x\xi \Sigma yv \Sigma z\zeta + \Sigma y\xi \Sigma zv \Sigma x\zeta + \Sigma z\xi \Sigma xv \Sigma y\zeta \\ & - \Sigma x\xi \Sigma zv \Sigma y\zeta - \Sigma y\xi \Sigma xv \Sigma z\zeta - \Sigma z\xi \Sigma yv \Sigma x\zeta, \end{aligned}$$

which is the result desired.

It is easy to imagine the troubles in store for any one who might have the hardihood to attempt to establish the next case in the same manner.

If Binet's multiplication-theorem be described as expressing a sum of products of resultants as a single resultant, his next theorem may be said to give a sum of products of sums of resultants as a sum of resultants. The paragraph in regard to it is a little too much condensed to be perfectly clear, and must therefore be given verbatim. It is (p. 288)—

"Désignons par  $S(y', z'')$  une somme de résultantes, telle que

$$(y', z'') + (y'', z'') + (y''', z'') + \&c. ;$$

c'est-à-dire,

$$y'z'' - z'y'' + y''z''' - z''y''' + y'''z'''' - z'''y'''' + \&c. ;$$

et continuons d'employer la caractéristique  $\Sigma$  pour les intégrales relatives aux accens supérieurs des lettres. L'expression  $\Sigma[S(y', z'') . S(v, \zeta'')]$  devient par le développement de chacun de ses termes, et en vertu de la première formule de l'art. 1 ou de celle du no. 4,

$$\begin{aligned} & \Sigma yv \Sigma z\zeta - \Sigma zv \Sigma y\zeta + \Sigma y''v \Sigma z''\zeta - \Sigma z''v \Sigma y''\zeta + \&c. \\ & + \Sigma y'''v \Sigma z'''\zeta - \Sigma z'''v \Sigma y'''\zeta + \Sigma y''''v \Sigma z''''\zeta - \Sigma z''''v \Sigma y''''\zeta + \&c. \\ & + \&c. \end{aligned}$$

En indiquant donc par  $S_1$  des intégrales qui supposent, dans chaque terme, les mêmes accens inférieurs aux lettres du même alphabet, ces accens pouvant être ou non les mêmes pour celles des alphabets différens, on pourra écrire la précédente suite, en faisant usage de ce signe, ce qui donne

$$\Sigma[S(y, z')S(v, \zeta')] = S_1[\Sigma yv \Sigma z\zeta - \Sigma zv \Sigma y\zeta].$$

Cette nouvelle quantité est encore de la forme  $S(y', z'')$ , en sorte qu'on peut dire que le produit de fonctions, telles que

$$\Sigma\{S(y, z') S(v, \zeta')\},$$

sera lui-même de la forme  $S(y', z'')$ .

This, if I understand it correctly, may be paraphrased and expanded as follows:—

Take the product of two sums of  $s$  resultants, viz.

$$\begin{aligned} & \{|y_1^1 z_1^2| + |y_2^1 z_2^2| + |y_3^1 z_3^2| + \dots + |y_s^1 z_s^2|\} \\ & \times \{|v_1^1 \zeta_1^2| + |v_2^1 \zeta_2^2| + |v_3^1 \zeta_3^2| + \dots + |v_s^1 \zeta_s^2|\} \end{aligned}$$

$$\text{or} \quad \sum_{s=1}^{s=s} |y_s^1 z_s^2| \cdot \sum_{s=1}^{s=s} |v_s^1 \zeta_s^2|,$$

where, it will be observed, all the resultants in the first factor are obtained from the first resultant  $|y_1^1 z_1^2|$  by merely changing the lower indices into 2, 3, . . . ,  $s$  in succession, and that the second factor is got from the first by writing  $v$  for  $y$  and  $\zeta$  for  $z$ . Then form all the like products whose first factors are

$$|y_1^1 z_1^3|, |y_1^1 z_1^4|, \dots, |y_1^{n-1} z_1^n|;$$

these being along with  $|y_1^1 z_1^2|$  the  $\frac{1}{2}n(n-1)$  resultants derivable from the two sets of  $n$  quantities

$$\begin{aligned} & y_1^1, y_1^2, y_1^3, \dots, y_1^n \\ & z_1^1, z_1^2, z_1^3, \dots, z_1^n. \end{aligned}$$

The sum of these  $\frac{1}{2}n(n-1)$  products may be represented, if we choose, by

$$\sum_{\substack{n=n \\ m=2 \\ m < n}}^{n=n} \left[ \sum_{s=1}^{s=s} |y_s^m z_s^n| \cdot \sum_{s=1}^{s=s} |v_s^m \zeta_s^n| \right].$$

Now if the multiplications be performed, there will be  $s^2$  terms in each product, and the theorem we are concerned with has its origin in the fact that the sum of all the first terms of the products is

expressible as a resultant by applying the multiplication-theorem, likewise the sum of all the second terms, and so on, the result being an aggregate of  $s^2$  resultants. For if we fix upon a particular term of the first product, say the term

$$|y_h^1 z_h^2| \cdot |v_k^1 \zeta_k^2|$$

which arises from the multiplication of the  $h^{\text{th}}$  term of the first factor by the  $k^{\text{th}}$  term of the second factor, then take the corresponding term of the other products, and write down their sum

$$|y_h^1 z_h^2| \cdot |v_k^1 \zeta_k^2| + |y_h^1 z_h^3| \cdot |v_k^1 \zeta_k^3| + \dots + |y_h^{n-1} \zeta_h^n| \cdot |v_k^{n-1} \zeta_k^n|,$$

it is manifest that this sum is by the multiplication-theorem

$$= \begin{vmatrix} y_h^1 v_k^1 + y_h^2 v_k^2 + \dots + y_h^n v_k^n & z_h^1 v_k^1 + z_h^2 v_k^2 + \dots + z_h^n v_k^n \\ y_h^1 \zeta_k^1 + y_h^2 \zeta_k^2 + \dots + y_h^n \zeta_k^n & z_h^1 \zeta_k^1 + z_h^2 \zeta_k^2 + \dots + z_h^n \zeta_k^n \end{vmatrix}.$$

Consequently since  $h$  may be any integer from 1 to  $s$ , and  $k$  likewise any integer from 1 to  $s$ , the theorem arrived at is accurately expressed in modern notation as follows:—

$$\sum_{\substack{n=s \\ m=2 \\ m < n}}^{n=s} \left[ \sum_{s=1}^{s=s} |y_s^m \zeta_s^n| \cdot \sum_{s=1}^{s=s} |v_s^m \zeta_s^n| \right] \\ = \sum_{k=1}^{k=s} \sum_{h=1}^{h=s} \begin{vmatrix} y_h^1 v_k^1 + y_h^2 v_k^2 + \dots + y_h^n v_k^n & z_h^1 v_k^1 + z_h^2 v_k^2 + \dots + z_h^n v_k^n \\ y_h^1 \zeta_k^1 + y_h^2 \zeta_k^2 + \dots + y_h^n \zeta_k^n & z_h^1 \zeta_k^1 + z_h^2 \zeta_k^2 + \dots + z_h^n \zeta_k^n \end{vmatrix},$$

or

$$\sum_{k=1}^{k=s} \sum_{h=1}^{h=s} \begin{vmatrix} y_h^1 & y_h^2 & \dots & y_h^n \\ z_h^1 & z_h^2 & \dots & z_h^n \end{vmatrix} \cdot \begin{vmatrix} v_k^1 & v_k^2 & \dots & v_k^n \\ \zeta_k^1 & \zeta_k^2 & \dots & \zeta_k^n \end{vmatrix}.$$

It is easily seen to be true of resultants of any order, as Binet himself points out. (xxx.)

When  $s$  is put equal to 1, it degenerates into the multiplication-theorem.

The theorem which follows upon this, but which is quite unconnected with it, may be at once stated in modern notation. It is—

If  $\Sigma |x_1 y_2 z_3|$  denote the sum of the resultants obtainable from the three sets of  $n$  quantities

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & \dots & x_n \\ y_1 & y_2 & y_3 & \dots & y_n \\ z_1 & z_2 & z_3 & \dots & z_n, \end{array}$$



and  $\Sigma|x_1y_2|$  denote the like sum obtainable from the first two sets, then

$$\Sigma|x_1y_2z_3| = \Sigma x. \Sigma|y_1z_2| + \Sigma y. \Sigma|z_1x_2| + \Sigma z. \Sigma|x_1y_2| \quad (\text{xxxI.})$$

This is arrived at by writing out the terms of  $\Sigma|y_1z_2|$ , of  $\Sigma|z_1x_2|$ , and of  $\Sigma|x_1y_2|$  in parallel columns, thus

$$\begin{array}{ccc} |y_1 z_2| & |z_1 x_2| & |x_1 y_2| \\ |y_1 z_3| & |z_1 x_3| & |x_1 y_3| \\ \vdots & \vdots & \vdots \\ |y_{n-1} z_n| & |z_{n-1} x_n| & |x_{n-1} y_n| \end{array};$$

then deriving  $n$  results from the members of the first row by multiplying by  $x_1, y_1, z_1$  respectively and adding, multiplying by  $x_2, y_2, z_2$ , and adding, and so on; then treating the second and remaining rows in the same way; and then finally adding all the  $n \cdot \frac{1}{2}n(n-1)$  results together. Each of these results is a vanishing or non-vanishing resultant of the 3<sup>rd</sup> order, and it will be found that each non-vanishing resultant occurs twice with the sign + and once with the sign -.

This process is readily seen to be simply the same as performing the multiplications indicated in the right-hand member of (xxxI.), i.e.,

$$\begin{aligned} & (x_1 + x_2 + \dots + x_n) (|y_1z_2| + |y_1z_3| + \dots + |y_{n-1}z_n|) \\ & + (y_1 + y_2 + \dots + y_n) (|z_1x_2| + |z_1x_3| + \dots + |z_{n-1}x_n|) \\ & + (z_1 + z_2 + \dots + z_n) (|x_1y_2| + |x_1y_3| + \dots + |x_{n-1}y_n|), \end{aligned}$$

summing every three corresponding terms in the products, and writing the sum as a vanishing or non-vanishing resultant. There would be  $n \cdot \frac{1}{2}n(n-1)$  resultants in all; but as each suffix occurs  $n-1$  times in the second factors and once in the first factors, there must be in each product  $n-1$  terms having the said suffix occurring twice: consequently there must be  $n-1$  resultants vanishing on account of this recurrence, and therefore altogether  $n(n-1)$  vanishing resultants. Of the non-vanishing resultants,—in number equal to  $n \cdot \frac{1}{2}n(n-1) - n(n-1)$ , or  $\frac{1}{2}n(n-1)(n-2)$ ,—each one of the form

$$|x_h y_k z_l| \quad \text{where } h < k < l$$

must be accompanied by two others,

$$|x_k y_h z_l| \text{ and } |x_l y_h z_k|,$$

and the sum of these is

$$|x_k y_h z_l| - |x_h y_k z_l| + |x_h y_l z_k|,$$

i.e.,

$$|x_k y_h z_l|.$$

The final result is thus the sum of the resultants of the form

$$|x_k y_h z_l| \text{ where } h < k < l, \text{ and } l = 3, 4, \dots, n,$$

the number of them, as we may see from two different standpoints, being

$$\frac{1}{6}n(n-1)(n-2).$$

Returning to the series of identities,

$$\begin{aligned} x_3|y_1z_2| + y_3|z_1x_2| + z_3|x_1y_2| &= |x_1y_2z_3|, \\ x_4|y_1z_2| + y_4|z_1x_2| + z_4|x_1y_2| &= |x_1y_2z_4|, \\ &\&c. \qquad \qquad \&c. \end{aligned}$$

which by addition give the result

$$\Sigma x \Sigma |y_1z_2| + \Sigma y \Sigma |z_1x_2| + \Sigma z \Sigma |x_1y_2| = \Sigma |x_1y_2z_3|,$$

Binet next raises both sides of all of them to the second power, and obtains

$$\left. \begin{aligned} 3\Sigma |x_1y_2z_3|^2 &= \Sigma x^2 \Sigma |y_1z_2|^2 + \Sigma y^2 \Sigma |z_1x_2|^2 + \Sigma z^2 \Sigma |x_1y_2|^2 \\ &\quad + 2\Sigma yz \Sigma (|z_1x_2| \cdot |x_1y_2|) + 2\Sigma zx \Sigma (|x_1y_2| \cdot |y_1z_2|) \\ &\quad + 2\Sigma xy \Sigma (|y_1z_2| \cdot |z_1x_2|). \end{aligned} \right\} \text{(XXXII.)}$$

Substituting for  $\Sigma |y_1z_2|^2$ ,  $\Sigma |z_1x_2|^2$ ,  $\dots$ , their equivalents as given by the multiplication-theorem, he then deduces

$$\left. \begin{aligned} \Sigma |x_1y_2z_3|^2 &= \Sigma x^2 \Sigma y^2 \Sigma z^2 + 2\Sigma yz \Sigma x \Sigma xy - \Sigma x^2 (\Sigma z^x)^2 \\ &\quad - \Sigma y^2 (\Sigma x^y)^2 - \Sigma z^2 (\Sigma xy)^2, \end{aligned} \right\}$$

not failing to note that this is not a fresh result, but merely a case of the multiplication-theorem in which the factors are equal.

By putting the right-hand member here into the form

$$\begin{aligned} &\Sigma y^2 \{ \Sigma z^2 \Sigma x^2 - (\Sigma yz)^2 \} + \Sigma z^2 \{ \Sigma x^2 \Sigma y^2 - (\Sigma xy)^2 \} \\ &\quad - \Sigma x^2 \{ \Sigma y^2 \Sigma z^2 - (\Sigma yz)^2 \} + 2\Sigma yz \{ \Sigma x \Sigma xy - \Sigma yz \Sigma x^2 \}, \end{aligned}$$

there is next arrived at the first identity of the set

$$\begin{aligned} & \Sigma |x_1 y_2 z_3|^2 \\ = & \Sigma y^2 \Sigma |z_1 x_2|^2 + \Sigma z^2 \Sigma |x_1 y_2|^2 - \Sigma x^2 \Sigma |y_1 z_2|^2 + 2 \Sigma yz \Sigma |z_1 x_2| |x_1 y_2|, \\ = & \Sigma z^2 \Sigma |x_1 y_2|^2 + \Sigma x^2 \Sigma |y_1 z_2|^2 - \Sigma y^2 \Sigma |z_1 x_2|^2 + 2 \Sigma zx \Sigma |x_1 y_2| |y_1 z_2|, \\ = & \Sigma x^2 \Sigma |y_1 z_2|^2 + \Sigma y^2 \Sigma |z_1 x_2|^2 - \Sigma z^2 \Sigma |x_1 y_2|^2 + 2 \Sigma xy \Sigma |y_1 z_2| |z_1 x_2|, \end{aligned} \quad \left. \vphantom{\begin{aligned} & \Sigma |x_1 y_2 z_3|^2 \\ = & \Sigma y^2 \Sigma |z_1 x_2|^2 + \Sigma z^2 \Sigma |x_1 y_2|^2 - \Sigma x^2 \Sigma |y_1 z_2|^2 + 2 \Sigma yz \Sigma |z_1 x_2| |x_1 y_2|, \\ = & \Sigma z^2 \Sigma |x_1 y_2|^2 + \Sigma x^2 \Sigma |y_1 z_2|^2 - \Sigma y^2 \Sigma |z_1 x_2|^2 + 2 \Sigma zx \Sigma |x_1 y_2| |y_1 z_2|, \\ = & \Sigma x^2 \Sigma |y_1 z_2|^2 + \Sigma y^2 \Sigma |z_1 x_2|^2 - \Sigma z^2 \Sigma |x_1 y_2|^2 + 2 \Sigma xy \Sigma |y_1 z_2| |z_1 x_2|, \end{aligned}} \right\} \text{(XXXIII.)}$$

and immediately from these the set

$$\begin{aligned} \Sigma |x_1 y_2 z_3|^2 = & \Sigma x^2 \Sigma |y_1 z_2|^2 + \Sigma zx \Sigma |x_1 y_2| \cdot |y_1 z_2| + \Sigma xy \Sigma |y_1 z_2| \cdot |z_1 x_2|, \\ = & \Sigma y^2 \Sigma |z_1 x_2|^2 + \Sigma xy \Sigma |y_1 z_2| \cdot |z_1 x_2| + \Sigma yz \Sigma |z_1 x_2| \cdot |x_1 y_2|, \\ = & \Sigma z^2 \Sigma |x_1 y_2|^2 + \Sigma yz \Sigma |z_1 x_2| \cdot |x_1 y_2| + \Sigma zx \Sigma |x_1 y_2| \cdot |y_1 z_2|. \end{aligned} \quad \left. \vphantom{\begin{aligned} \Sigma |x_1 y_2 z_3|^2 = & \Sigma x^2 \Sigma |y_1 z_2|^2 + \Sigma zx \Sigma |x_1 y_2| \cdot |y_1 z_2| + \Sigma xy \Sigma |y_1 z_2| \cdot |z_1 x_2|, \\ = & \Sigma y^2 \Sigma |z_1 x_2|^2 + \Sigma xy \Sigma |y_1 z_2| \cdot |z_1 x_2| + \Sigma yz \Sigma |z_1 x_2| \cdot |x_1 y_2|, \\ = & \Sigma z^2 \Sigma |x_1 y_2|^2 + \Sigma yz \Sigma |z_1 x_2| \cdot |x_1 y_2| + \Sigma zx \Sigma |x_1 y_2| \cdot |y_1 z_2|. \end{aligned}} \right\} \text{(XXXIV.)}$$

We may note in passing that either of these sets leads at once to the initial theorem

$$\begin{aligned} 3 \Sigma |x_1 y_2 z_3|^2 = & \Sigma x^2 \Sigma |y_1 z_2|^2 + \Sigma y^2 \Sigma |z_1 x_2|^2 + \Sigma z^2 \Sigma |x_1 y_2|^2 \\ & + 2 \Sigma yz \Sigma |z_1 x_2| \cdot |x_1 y_2| + 2 \Sigma zx \Sigma |x_1 y_2| \cdot |y_1 z_2| \\ & + 2 \Sigma xy \Sigma |y_1 z_2| \cdot |z_1 x_2|, \end{aligned}$$

and that with the multiplication-theorem already established this reverse order would be the more natural.

The next step taken is the formation of resultants of the 2<sup>nd</sup> order from elements which are themselves resultants of the 2<sup>nd</sup> order; viz., just as from the three rows of  $n$  quantities

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & \dots & x_n \\ y_1 & y_2 & y_3 & \dots & y_n \\ z_1 & z_2 & z_3 & \dots & z_n \end{array}$$

there were formed the three other rows of  $\frac{1}{2}n(n-1)$  quantities

$$\begin{aligned} & |y_1 z_2|, |y_1 z_3|, \dots, |y_1 z_n|, |y_2 z_3|, \dots, |y_{n-1} z_n|, \\ & |z_1 x_2|, |z_1 x_3|, \dots, |z_1 x_n|, |z_2 x_3|, \dots, |z_{n-1} x_n|, \\ & |x_1 y_2|, |x_1 y_3|, \dots, |x_1 y_n|, |x_2 y_3|, \dots, |x_{n-1} y_n|, \end{aligned}$$

so from the latter three other rows of quantities

$$\begin{aligned} & \left[ \begin{array}{cc} |z_1 x_2| & |z_1 x_3| \\ |x_1 y_2| & |x_1 y_3| \end{array} \right], \dots, \left[ \begin{array}{cc} |z_{n-2} x_n| & |z_{n-1} x_n| \\ |x_{n-2} y_n| & |x_{n-1} y_n| \end{array} \right], \\ & \left[ \begin{array}{cc} |x_1 y_2| & |x_1 y_3| \\ |y_1 z_2| & |y_1 z_3| \end{array} \right], \dots, \left[ \begin{array}{cc} |x_{n-2} y_n| & |x_{n-1} y_n| \\ |y_{n-2} z_n| & |y_{n-1} z_n| \end{array} \right], \\ & \left[ \begin{array}{cc} |y_1 z_2| & |y_1 z_3| \\ |z_1 x_2| & |z_1 x_3| \end{array} \right], \dots, \left[ \begin{array}{cc} |y_{n-2} z_n| & |y_{n-1} z_n| \\ |z_{n-2} x_n| & |z_{n-1} x_n| \end{array} \right], \end{aligned}$$

are formed, the number in each new row being clearly

$$\frac{1}{2}\{\frac{1}{2}n(n-1)\}\{\frac{1}{2}n(n-1)-1\}$$

i.e.,  $\frac{1}{8}n(n-1)(n-2)(n+1).$

The new quantities are, of course, not written by Binet in the form

$$\begin{vmatrix} | & & | & & | & \\ | & & | & & | & \\ | & & | & & | & \end{vmatrix},$$

but the fact that they are resultants of the 2<sup>nd</sup> order is carefully noted. Each of them is shown to be transformable, by a theorem which may be viewed as an extension of a result given by Lagrange, so as to have two of the elements resultants of the 3<sup>rd</sup> order, and the others resultants of the 1<sup>st</sup> order. This is done by taking, for example, the identities

$$\begin{aligned} x_h|y_i z_j| + y_h|z_i x_j| + z_h|x_i y_j| &= |x_h y_i z_j|, \\ x_k|y_i z_j| + y_k|z_i x_j| + z_k|x_i y_j| &= |x_k y_i z_j|, \end{aligned}$$

multiplying both sides of the first by  $x_k$ , and both sides of the second by  $x_h$ , subtracting, and writing the result in the form

$$\begin{aligned} |x_k y_h| |z_i x_j| + |x_k z_h| |x_i y_j| &= x_k |x_h y_i z_j| - x_h |x_k y_i z_j|, \\ &= \begin{vmatrix} x_k & x_h \\ |x_k y_i z_j| & |x_h y_i z_j| \end{vmatrix}, \end{aligned}$$

where of course it has to be noted that in many cases one of the resultants of the 3<sup>rd</sup> order will vanish. The quantities, therefore, to be dealt with, are

$$\begin{aligned} x_1|x_1 y_2 z_3|, \dots, x_k|x_k y_i z_j| - x_h|x_k y_i z_j|, \dots, x_n|x_{n-2} y_{n-1} z_n|; \\ y_1|x_1 y_2 z_3|, \dots, y_k|y_h z_i x_j| - y_h|y_k z_i x_j|, \dots, y_n|x_{n-2} y_{n-1} z_n|; \\ z_1|x_1 y_2 z_3|, \dots, z_k|z_h x_i y_j| - z_h|z_k x_i y_j|, \dots, z_n|x_{n-2} y_{n-1} z_n|. \end{aligned}$$

By raising each of the elements of the first row to the second power, taking the sum and simplifying, we could, we are told, show that the result would be

$$\sum x_1^2 \sum |x_1 y_2 z_3|^2.$$

Very prudently, however, another process is chosen. It is recalled that the quantities in the third triad of rows are related to those in the second as those in the second are related to those in the first, and that consequently the required sum of squares of resultants is, by the multiplication-theorem itself, expressible as a resultant, viz.,

$$\Sigma \left| |z_1 x_2|, |x_1 y_3| \right|^2 = \Sigma |z_1 x_2|^2 \cdot \Sigma |x_1 y_3|^2 - (\Sigma |z_1 x_2| |x_1 y_3|)^2,$$

where the elements of the resultant on the right are sums of products of quantities in the second triad of rows. Then the same theorem is used to make a further step backwards, viz., to express each of these three sums of products of resultants as a resultant whose elements are sums of products of the quantities in the first triad of rows, the effect of the substitution being

$$\Sigma \left| |z_1 x_2|, |x_1 y_3| \right|^2 = \{ \Sigma z_1^2 \Sigma x_1^2 - (\Sigma z_1 x_1)^2 \} \{ \Sigma x_1^2 \Sigma y_1^2 - (\Sigma x_1 y_1)^2 \} \\ - \{ \Sigma z_1 x_1 \Sigma x_1 y_1 - \Sigma y_1 z_1 \Sigma x_1^2 \}^2.$$

Simple multiplication transforms this into

$$\Sigma x_1^2 \left\{ \Sigma x_1^2 \Sigma y_1^2 \Sigma z_1^2 - \Sigma y_1^2 (\Sigma z_1 x_1)^2 - \Sigma z_1^2 (\Sigma x_1 y_1)^2 \right\} \\ + 2 \Sigma y_1 z_1 \Sigma z_1 x_1 \Sigma x_1 y_1 - \Sigma x_1^2 (\Sigma y_1 z_1)^2 \},$$

which, by still another use of the multiplication-theorem, we know is equal to

$$\Sigma x_1^2 \Sigma |x_1 y_2 z_3|^2.$$

The set of six results of which this is one, is

$$\left. \begin{aligned} \Sigma X_1^2 &= \Sigma x_1^2 \Sigma |x_1 y_2 z_3|^2, \\ \Sigma Y_1^2 &= \Sigma y_1^2 \Sigma |x_1 y_2 z_3|^2, \\ \Sigma Z_1^2 &= \Sigma z_1^2 \Sigma |x_1 y_2 z_3|^2, \\ \Sigma Y_1 Z_1 &= \Sigma y_1 z_1 \Sigma |x_1 y_2 z_3|^2, \\ \Sigma Z_1 X_1 &= \Sigma z_1 x_1 \Sigma |x_1 y_2 z_3|^2, \\ \Sigma X_1 Y_1 &= \Sigma x_1 y_1 \Sigma |x_1 y_2 z_3|^2, \end{aligned} \right\} \quad (\text{xxxv.})$$

if, for shortness, we denote the quantities of the third triad of rows by

$$\begin{aligned} X_1, \quad X_2, \quad \dots \dots \dots \\ Y_1, \quad Y_2, \quad \dots \dots \dots \\ Z_1, \quad Z_2, \quad \dots \dots \dots \end{aligned}$$

Following these, and deduced by means of them, is an equally noteworthy theorem regarding the sums of squares of all the resultants of the third order, which can be formed from the quantities of the second triad of rows. Denoting these quantities temporarily by

$$\begin{aligned} \xi_1, \quad \xi_2, \quad \dots \dots \dots \\ \eta_1, \quad \eta_2, \quad \dots \dots \dots \\ \zeta_1, \quad \zeta_2, \quad \dots \dots \dots \end{aligned}$$

we know (xxxii.) that

$$\begin{aligned} 3\Sigma|\xi\eta_2\zeta_3|^2 &= \Sigma X_1^2 \Sigma \xi_1^2 + \Sigma Y_1^2 \Sigma \eta_1^2 + \Sigma Z_1^2 \Sigma \zeta_1^2 \\ &\quad + 2\Sigma Y_1 Z_1 \cdot \Sigma \eta_1 \zeta_1 + 2\Sigma Z_1 X_1 \cdot \Sigma \zeta_1 \xi_1 \\ &\quad + 2\Sigma X_1 Y_1 \cdot \Sigma \xi_1 \eta_1 ; \end{aligned}$$

whence, by using the set of six results just obtained, we have

$$\begin{aligned} &3\Sigma|\xi_1\eta_2\zeta_3|^2 \\ &= \Sigma|x_1y_2z_3|^2 \left\{ \begin{aligned} &\Sigma \xi_1^2 \Sigma x_1^2 + \Sigma \eta_1^2 \Sigma y_1^2 + \Sigma \zeta_1^2 \Sigma z_1^2 \\ &+ 2\Sigma \eta_1 \xi_1 \cdot \Sigma y_1 z_1 + 2\Sigma \zeta_1 \xi_1 \cdot \Sigma z_1 x_1 + 2\Sigma \xi_1 \eta_1 \cdot \Sigma x_1 y_1 \end{aligned} \right\} \end{aligned}$$

and therefore, again by (xxxii.)

$$\Sigma|\xi_1\eta_2\zeta_3|^2 = \{\Sigma|x_1y_2z_3|^2\}^2. \quad (\text{xxxvi.})$$

It is finally pointed out that from the third triad of rows there might, in like manner, be formed a fourth triad, and analogous identities obtained; also that, instead of starting with three rows, we might start with *four*,

$$\begin{array}{ccccccc} t_1, & t_2, & t_3, & \dots, & t_n \\ x_1, & x_2, & x_3, & \dots, & x_n \\ y_1, & y_2, & y_3, & \dots, & y_n \\ z_1, & z_2, & z_3, & \dots, & z_n, \end{array}$$

form from them other four

$$\begin{array}{l} |x_1y_2z_3|, \dots\dots\dots \\ |y_1z_2t_3|, \dots\dots\dots \\ |z_1t_2x_3|, \dots\dots\dots \\ |t_1x_2y_3|, \dots\dots\dots, \end{array}$$

thence in the same way a third four, and in connection therewith establish the identity

$$\Sigma t_1 \Sigma |x_1y_2z_3| - \Sigma x_1 \Sigma |y_1z_2t_3| + \Sigma y_1 \Sigma |z_1t_2x_3| - \Sigma z_1 \Sigma |t_1x_2y_3| = 0 \quad (\text{xxxvi. 2})$$

and other analogues.

(xxxvii. 2 + xxxv. 2.)

The rest of the memoir, 52 pages, consists of geometrical applications of the series of theorems thus obtained.

## CAUCHY (1812).

[Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions

opérées entre les variables qu'elles renferment. *Journ. de l'Ec. Polyt.*, x. Cah. 17, pp. 29–112.]

This masterly memoir of 84 pages was read to the Institute on the same day (30th November) as Binet's memoir, of which we have just given an account. The coincidence of date has to be carefully noted, because the memoirs have in part a common ground, and because there is a presumption that the authors, knowing this beforehand, had, in a friendly way, arranged for simultaneous publicity. Binet's words on the matter are—

“Ayant en dernière occasion de parler à M. Cauchy, ingénieur des ponts et chaussées, du théorème générale que j'ai énoncé ci-dessus, il me dit être parvenu, dans des recherches analogues à celles de M. Gauss, à des théorèmes d'analyse qui devaient avoir rapport aux miens. Je m'en suis assuré, en jetant les yeux sur ces formules : mais j'ignore si elles ont la même généralité que les miennes : nous y sommes arrivés, je crois, par des voies très-différentes.”

And Cauchy's corroboration is (p. 111)—

“J'avais rencontré l'été dernier, à Cherbourg, où j'étais fixé par les travaux de mon état, ce théorème et quelques autres du même genre, en cherchant à généraliser les formules de M. Gauss. M. Binet, dont je me félicite d'être l'ami, avait été conduit aux mêmes résultats par des recherches différentes. De retour à Paris, j'étais occupé de poursuivre mon travail, lorsque j'allai le voir. Il me montra son théorème qui était semblable au mien. Seulement il désignait sous le nom de *résultante* ce que j'avais appelé *déterminant*.”

Cauchy prefaces his memoir by another, entitled

*Sur le nombre des valeurs qu'une fonction peut acquérir lorsqu'on y permute de toutes les manières possibles les quantités qu'elle renferme.*

This latter must to a certain extent be taken into account, because it serves to show the point of view which he considered most natural for examining the subject, and also the exact position held by the functions now called determinants, when functions in

general come to be classified according to the number of values they are able to assume in certain circumstances.

At the outset of it the writings of Lagrange, Vandermonde, and Ruffini are referred to; the fact is recalled that the maximum number of values which a function can acquire by interchanges among its  $n$  variables is  $1.2.3 \dots n$ ; also that when the maximum is not obtained, the actual number must be a factor of the maximum; and then proof is given of the very notable theorem that *the number of values cannot be less than the greatest prime contained in  $n$  without being equal to 2*. It is pointed out likewise that functions capable of having only two values are known from Vandermonde to be constructible for any number of variables. For example, the number of variables being three,  $a_1, a_2, a_3$ , all that is needed is to form their difference-product

$$(a_3 - a_2)(a_3 - a_1)(a_2 - a_1)$$

or

$$a_3^2 a_2 + a_2^2 a_1 + a_1^2 a_3 - (a_3^2 a_1 + a_2^2 a_3 + a_1^2 a_2),$$

when it is found that either of the parts

$$a_3^2 a_2 + a_2^2 a_1 + a_1^2 a_3,$$

or

$$a_3^2 a_1 + a_2^2 a_3 + a_1^2 a_2,$$

is an instance of a function capable of only two values by permutation of the variables; the result indeed of any permutation being merely that the one function passes into the other. Further, the whole expression

$$a_3^2 a_2 + a_2^2 a_1 + a_1^2 a_3 - (a_3^2 a_1 + a_2^2 a_3 + a_1^2 a_2)$$

is another example, the difference between the two values which it can assume being however a difference of sign merely. As a reference to the title of the memoir of November 1812 will show, it is functions of this latter class which Cauchy there considers.

At the commencement he contrasts them with functions which suffer no change whatever by permutation of variables, that is to say, *symmetric* functions: and, noting the fact, afterwards ascertained, that the new functions consist of terms alternately + and -, and that were it not for this alternation of sign they would be symmetric functions, he decides to extend the term "symmetric" to them, and having done so, seeks to distinguish them from ordi-



nary symmetric functions by calling them “fonctions symétriques alternées,” and calling the other “fonctions symétriques permanentes.” Cauchy’s view of determinants may therefore now be described by saying that he considered them as a *special class of alternating symmetric functions*.

To include them, however, either the adoption of a convention is necessary, or an extension of the definition must be made. For example,  $a_1 b_2 - a_2 b_1$  is not an alternating function, unless the elements be so related that the interchange of  $a_1$  and  $a_2$  necessitates the interchange of  $b_1$  and  $b_2$  at the same time; or unless the definition be so worded that interchange shall refer to *suffixes*, not to letters. Cauchy selects the former course, his words being (p. 30)

“ . . . . concevons les diverses suites de quantités

$$\begin{array}{ccccccc} a_1, & a_2, & . & . & . & . & a_n \\ b_1, & b_2, & . & . & . & . & b_n \\ c_1, & c_2, & . & . & . & . & c_n \\ . & . & . & . & . & . & . \end{array}$$

tellement liées entre elles, que la transposition de deux indices pris dans l’une des suites, nécessite la même transposition dans toutes les autres ; alors, les quantités

$$b_1, c_1, . . . , b_2, c_2, . . . , b_3, c_3, . . .$$

pourront être considérées comme des fonctions semblables de

$$a_1, a_2, a_3, . . . ;$$

et par suite, les fonctions de

$$a_1, b_1, c_1, . . . , a_2, b_2, c_2, . . . , a_n, b_n, c_n, . . .$$

qui ne changeront pas de valeur, mais tout au plus de signe, en vertu de transpositions opérées entre les indices 1, 2, 3, . . .  $n$ , devront être rangées parmi les fonctions symétriques de  $a_1, a_2, . . . , a_n$  ou, ce que revient au même, des indices 1, 2, 3, . . . ,  $n$ . Ainsi

$$\begin{aligned} & a_1^2 + a_2^2 + 4a_1 a_2, \\ & a_1 b_1 + a_2 b_2 + a_3 b_3 + 2c_1 c_2 c_3, \\ & a_1 b_2 + a_2 b_3 + a_3 b_1 + a_2 b_1 + a_3 b_2 + a_1 b_3, \\ & \cos (a_1 - a_2) \cos (a_1 - a_3) \cos (a_2 - a_3), \end{aligned}$$

seront des fonctions symétriques permanentes, la première du second ordre et les autres du troisième ; et au contraire,

$$a_1b_2 + a_2b_3 + a_3b_1 - a_2b_1 - a_1b_3 - a_3b_2, \\ \sin(a_1 - a_2) \sin(a_1 - a_3) \sin(a_2 - a_3)$$

seront des fonctions symétriques alternées du troisième ordre."

The question of nomenclature being settled there next arises the question of notation. This also is decided on the ground of the resemblance of the functions to symmetric functions. It being known that any symmetric function is representable by a typical term preceded by a symbol indicating permutation of the variables, *e.g.*

$$S(a_1b_2) \text{ or } S^2(a_1b_2) \text{ standing for } a_1b_2 + a_2b_1$$

$$\text{and } S^3(a_1b_2) \text{ standing for } a_1b_2 + a_2b_3 + a_3b_1 + a_2b_1 + a_3b_2 + a_1b_3;$$

also, that any non-symmetric function may be taken as the typical term of a symmetric function, the question arises whether the like may not be true of alternating functions. A lengthy examination of the latter point leads to the conclusion that any non-symmetric function *K cannot* be the originating or typical term of an alternating function unless it satisfies a certain condition, viz., that it be such that any value of it obtained by an even number of transpositions of indices will be different from any other value obtained by an odd number of transpositions. Should, however, this condition be satisfied, and  $K_\alpha, K_\beta, K_\gamma, \dots$  be all the values of the former kind, and  $K_\lambda, K_\mu, K_\nu, \dots$  all the values of the latter kind, then

$$(K_\alpha + K_\beta + K_\gamma + \dots) - (K_\lambda + K_\mu + K_\nu + \dots)$$

is an alternating function and is appropriately representable by

$$S(\pm K)$$

if the indices appearing in *K* alone are to be permuted, and by

$$S^n(\pm K)$$

if the indices to be permuted be 1, 2, 3, . . . , *n*. For example, taking the typical term  $a_1b_2$  we have

$$S(\pm a_1b_2) = a_1b_2 - a_2b_1,$$

$$\text{and } S^3(\pm a_1b_2) = a_1b_2 + a_2b_3 + a_3b_1 - a_2b_1 - a_3b_2 - a_1b_3, \\ = S^3(\mp a_2b_1) = S^3(\mp a_1b_3) = \dots$$

$S_4(\pm a_1 b_2)$  is an impossibility, as when there are four indices  $a_1 b_2$  does not satisfy the condition required of a typical term; indeed, Cauchy notes that the number of indices in any term must either be the total number or 1 less.

The number of permutations being even, it is clear that *the number of + terms  $K_\alpha, K_\beta, \dots$  is the same as the number of negative terms  $K_\lambda, K_\mu$ ,* (x. 2)  
a generalisation of a remark of Vandermonde's.

Further, since  $K_\alpha, K_\beta, \dots$  are all the terms that arise from an even number of transpositions, and  $K_\lambda, K_\mu, \dots$  all those that arise from an odd number of transpositions, it is plain that any single transposition performed upon each of the terms of the function

$$(K_\alpha + K_\beta + K_\gamma + \dots) - (K_\lambda + K_\mu + K_\nu + \dots)$$

must change it into

$$(K_\lambda + K_\mu + K_\nu + \dots) - (K_\alpha + K_\beta + K_\gamma + \dots)$$

—this is, in fact, the proof that it is an alternating function—consequently each of the parts

$$K_\alpha + K_\beta + K_\gamma + \dots,$$

$$K_\lambda + K_\mu + K_\nu + \dots,$$

belongs to the class of functions which have only two different values.

Also it is evident that *if throughout the function any particular index be changed into another and no further alteration made, the resulting expression must be equal to zero,* (xii. 5)

a theorem regarding alternating functions which is the generalisation of a theorem of Vandermonde's.

We have lastly to note, that the criterion which determines whether a particular  $K$  belongs to the class  $K_\alpha, K_\beta, \dots$  or to the class  $K_\lambda, K_\mu, \dots$  is incidentally shown to be reducible to a more practical form. For example, if the term be  $K_\theta$ , and it be derivable from  $K$ , say, by the change of the suffixes 1, 2, 3, 4, 5, 6, 7 into 3, 2, 6, 5, 4, 1, 7, that is to say, in Cauchy's language by means of the substitution

$$\begin{pmatrix} 1, 2, 3, 4, 5, 6, 7 \\ 3, 2, 6, 5, 4, 1, 7 \end{pmatrix},$$

we transform this substitution into a "product" of "circular" substitutions, viz., into

$$\begin{pmatrix} 1, & 3, & 6 \\ 3, & 6, & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

and subtracting the number of "factors," 4, from the total number of suffixes 7, make the sign + or - according as this difference is even or odd.

Here the subject of general alternating functions may be left for the present. What remains of the first part of the memoir, refers to special cases, which naturally fall to be considered in another chapter of our history. At the close of the part, Cauchy says (p. 51)—

"Je vais maintenant examiner particulièrement une certaine espèce de fonctions symétriques alternées qui s'offrent d'elles-mêmes dans un grand nombre de recherches analytiques. C'est au moyen de ces fonctions qu'on exprime les valeurs générales des inconnues que renferment plusieurs équations du premier degré. Elles se représentent toutes les fois qu'on a des équations à former, ainsi que dans la théorie générale de l'élimination."

The writings of Laplace, Vandermonde, Bézout, and Gauss are referred to, and from the latter the name "déterminant" is adopted.

The second part bears the title—

*Des fonctions symétriques alternées désignées sous le nom de déterminans.*

and opens with the following explanatory definition (p. 51)—

"Soient  $a_1, a_2, \dots, a_n$  plusieurs quantités différentes en nombre égal à  $n$ . On a fait voir ci-dessus qu'en multipliant le produit de ces quantités, ou

$$a_1 a_2 a_3 \dots a_n$$

par le produit de leurs différences respectives, ou par

$$(a_2 - a_1)(a_3 - a_1) \dots (a_n - a_1)(a_3 - a_2) \dots (a_n - a_2) \dots (a_n - a_{n-1})$$

on obtenait pour résultat la fonction symétrique alternée

$$S(\pm a_1 a_2 a_3^3 \dots a_n^n),$$

qui par conséquent se trouve toujours égale au produit

$$a_1 a_2 a_3 \dots a_n \\ \times (a_2 - a_1)(a_3 - a_1) \dots (a_n - a_1)(a_3 - a_2) \dots (a_n - a_2) \dots (a_n - a_{n-1}).$$

Supposons maintenant que l'on développe ce dernier produit, et que dans chaque terme du développement on remplace l'exposant de chaque lettre par un second indice égale à l'exposant dont il s'agit, en écrivant par exemple  $a_{r,s}$  au lieu de  $a_r^s$ , et  $a_{s,r}$  au lieu de  $a_r^r$ , on obtiendra pour résultat une nouvelle fonction symétrique alternée, qui, au lieu d'être représentée par

$$S(\pm a_1^1 a_2^2 a_3^3 \dots a_n^n)$$

sera représentée par

$$S(\pm a_{1,1} a_{2,2} a_{3,3} \dots a_{n,n}),$$

le signe S étant relatif aux premiers indices de chaque lettre. Telle est la forme la plus générale des fonctions que je désignerai dans la suite sous le nom de *déterminans*. Si l'on suppose successivement \*

$$n=1, n=2, \&c. \dots$$

on trouvera

$$\begin{aligned} S(\pm a_{1,1} a_{2,2}) &= a_{1,1} a_{2,2} - a_{2,1} a_{1,2}, \\ S(\pm a_{1,1} a_{2,2} a_{3,3}) &= a_{1,1} a_{2,2} a_{3,3} + a_{2,1} a_{3,2} a_{1,3} + a_{3,1} a_{1,2} a_{2,3} \\ &\quad - a_{1,1} a_{3,2} a_{2,3} - a_{3,1} a_{2,2} a_{1,3} - a_{2,1} a_{1,2} a_{3,3}. \\ &\&c. \dots \end{aligned}$$

pour les déterminans du second, du troisième ordre, &c. . . . ”

In regard to this it is important to notice that there are really two definitions given us. The latter, viz., that involved in the symbolism of alternating functions,

$$S(\pm a_{1,1} a_{2,2} a_{3,3} \dots a_{n,n})$$

contains nothing more than Leibnitz's rule of formation and an improved rule of signs. The former is new and may be paraphrased as follows:—

*If the multiplications indicated in the expression*

$$a_1 a_2 a_3 \dots a_n \\ \times (a_2 - a_1)(a_3 - a_1) \dots (a_n - a_1)(a_3 - a_2) \dots (a_n - a_2) \dots (a_n - a_{n-1})$$

\*  $n=2, n=3, \&c.$  is meant.

be performed, and in the result every index of a power be changed into a second suffix, e.g.,  $a_r^s$  into  $a_{r,s}$ , the expression so obtained is called a determinant (VIII. 2), and is denoted by

$$S(\pm a_{1,1}a_{2,2}a_{3,3} \dots a_{n,n}) \quad (\text{VII. 5}).$$

In this definition the rule of signs and the rule of term-formation are inseparable—a peculiarity already observed in the case of Bézout's rule of 1764.

After the definitions various technical terms are introduced. The  $n^2$  different quantities involved in

$$S(\pm a_{1,1}a_{2,2}a_{3,3} \dots a_{n,n})$$

are arranged thus

$$\left\{ \begin{array}{l} a_{1,1}, \quad a_{1,2}, \quad a_{1,3}, \quad \dots \quad a_{1,n} \\ a_{2,1}, \quad a_{2,2}, \quad a_{2,3}, \quad \dots \quad a_{2,n} \\ a_{3,1}, \quad a_{3,2}, \quad a_{3,3}, \quad \dots \quad a_{3,n} \\ \&c. \quad \dots \quad \dots \\ a_{n,1}, \quad a_{n,2}, \quad a_{n,3}, \quad \dots \quad a_{n,n} \end{array} \right.$$

“sur un nombre égal à  $n$  de lignes horizontales et sur autant de colonnes verticales,” and as thus arranged are said to form a *symmetric system* of order  $n$ . The individual quantities  $a_{1,1}$ , &c. are called the *terms* of the system, and the letter  $a$  when free of suffixes the *characteristic*. The “terms” in a horizontal line are said to form a *suite horizontale*, in a vertical column a *suite verticale*. *Conjugate terms* are defined as those whose suffixes (“indices”) differ in order, e.g.,  $a_{2,3}$  and  $a_{3,2}$ ; and terms which are self-conjugate, e.g.,  $a_{1,1}$ ,  $a_{2,2}$ , . . . are called *principal terms*. The determinant is said to *belong* to the system, or to be the determinant of the system. The parts of the expanded determinant which are connected by the signs + and – are called *symmetric products*, and the product

$$a_{1,1}a_{2,2}a_{3,3} \dots a_{n,n}$$

of the principal “terms” is called the *principal product*. The “principal product,” however, is also called the *terme indicatif* of the determinant, and thus an awkward double use of the word “terme” is brought into prominence. The system

$$\left\{ \begin{array}{l} a_{1.1} \quad a_{2.1} \quad a_{3.1} \quad . \quad . \quad . \quad a_{n.1} \\ a_{1.2} \quad a_{2.2} \quad a_{3.2} \quad . \quad . \quad . \quad a_{n.2} \\ a_{1.3} \quad a_{2.3} \quad a_{3.3} \quad . \quad . \quad . \quad a_{n.3} \\ \&c. \quad . \quad . \quad . \quad . \\ a_{1.n} \quad a_{2.n} \quad a_{3.n} \quad . \quad . \quad . \quad a_{n.n} \end{array} \right.$$

derived from the previous system by interchanging the suffixes of each "terme" is said to be *conjugate* to the previous system. A symbol for each of these systems is got by taking the last "terme" of its first "suite horizontale," and enclosing the "terme" in brackets: in this way we are enabled to say that  $(a_{1.n})$  and  $(a_{n.1})$  are *conjugate systems*.

In the course of these explanations a modification of the rule of term-formation is incidentally noted, the form taken being specially applicable when the quantities of the system have been disposed in a square. Cauchy's wording of this now familiar rule is (p. 55)—

..... "pour former chacun des termes dont il s'agit, il suffira de multiplier entre elles  $n$  quantités différentes prises respectivement dans les différentes colonnes verticales du système, et situées en même temps dans les diverses lignes horizontales de ce système."

Here we may note in passing that the disposal of the "termes" in a square might have enabled Cauchy to point out (which he did not do) the difference between Gauss' use of the word "determinant" and his own, by saying that the "determinant of a form" had its conjugate "termes" equal.

The rule of signs applicable to alternating functions in general is modified for the special case of determinants, and takes the following form (p. 56):—

"Étant donné un produit symétrique quelconque, pour obtenir le signe dont il est affecté dans le déterminant

$$S(\pm a_{1.1} a_{2.2} a_{3.3} \dots a_{n.n})$$

il suffira d'appliquer la règle qui sert à déterminer le signe d'un terme pris à volonté dans une fonction symétrique alternée. Soit

$$a_{\alpha.1} a_{\beta.2} \dots a_{\zeta.n}$$

le produit symétrique dont il s'agit, et désignons par  $g$  le

nombre des substitutions circulaires équivalentes à la substitution

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \alpha & \beta & \gamma & \dots & \zeta \end{pmatrix}.$$

Ce produit devra être affecté du signe +, si  $n-g$  est un nombre pair, et du signe - dans le cas contraire." (III. 18).

Thus if the sign of the term

$$a_{6\cdot1} a_{8\cdot2} a_{3\cdot3} a_{1\cdot4} a_{9\cdot5} a_{2\cdot6} a_{5\cdot7} a_{4\cdot8} a_{7\cdot9}$$

in the determinant

$$S(\pm a_{1\cdot1} a_{2\cdot2} a_{3\cdot3} \dots a_{9\cdot9}),$$

be wanted, we write the series of first suffixes 6, 8, . . . under the corresponding suffixes of the "principal product," that is to say, under the series 1, 2, 3 . . . 9, obtaining the interchange

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 8 & 3 & 1 & 9 & 2 & 5 & 4 & 7 \end{pmatrix};$$

this we separate into circular interchanges, finding them three in number, viz.,

$$\begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 & 7 & 9 \\ 9 & 5 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 4 & 6 & 8 \\ 6 & 8 & 1 & 2 & 4 \end{pmatrix};$$

and the determinant being of the 9<sup>th</sup> order, we thence conclude that the desired sign is  $(-)^{9-3}$ , i.e., +. In connection with this subject a modification of Cramer's rule is given, no reference being made to "dérangements" at all. Put into the fewest possible words it is—*The sign of the term  $a_{\alpha\cdot1} a_{\beta\cdot2} \dots a_{\zeta\cdot n}$  is the same as the sign of the difference-product of the first suffixes, that is, the sign of*

$$(\beta - \alpha) (\gamma - \alpha) \dots (\zeta - \alpha) (\gamma - \beta) \dots \quad (\text{III. 19}).$$

For example, the sign of

$$a_{6\cdot1} a_{8\cdot2} a_{3\cdot3} a_{1\cdot4} a_{9\cdot5} a_{2\cdot6} a_{5\cdot7} a_{4\cdot8} a_{7\cdot9},$$

above sought, is the sign of the difference-product of

$$6, 8, 3, 1, 9, 2, 5, 4, 7$$

i.e., the sign of

$$\begin{aligned} & (7-4) (7-5) (7-2) (7-9) (7-1) (7-3) (7-8) (7-6) \\ & \times (4-5) (4-2) \dots \dots \dots (4-6) \\ & \times (5-2) \dots \dots \dots (5-6) \\ & \dots \dots \dots \times (8-6) \end{aligned}$$



The object which Cauchy had in view in stating the rule in this unnecessarily complex form was doubtless to show its essential identity with the rule implied in his new definition. He says (p. 58)—

“On démontre facilement cette règle par ce qui précède, attendu qu’une transposition opérée entre deux indices change toujours, comme on l’a fait voir, le signe du produit

$$(a_{\beta} - a_{\alpha}) (a_{\gamma} - a_{\alpha}) \dots (a_{\zeta} - a_{\alpha}) (a_{\gamma} - a_{\beta}) \dots ,$$

et par conséquent celui du produit

$$(\beta - \alpha) (\gamma - \alpha) \dots (\zeta - \alpha) (\gamma - \beta) \dots ”$$

The way having thus been prepared, the propositions of determinants are entered on. Those known to his predecessors we may dispose of rapidly, giving little, if anything, more than the enunciation of them, in order that the new garb in which they appear may be seen.

... “le déterminant du système  $(a_{n,1})$  est égal à celui du système  $(a_{1,n})$  . . . . En conséquence, dans l’expression

$$S(\pm a_{1,1} a_{2,2} \dots a_{n,n})$$

on peut supposer indifféremment, ou que le signe  $S$  se rapporte aux premiers indices, ou qu’il se rapporte aux seconds : (ix. 2).

Si l’on échange entre elles deux suites horizontales ou deux suites verticales du système  $(a_{1,n})$  de manière à faire passer dans une des suites tous les termes de l’autre et réciproquement on obtiendra un nouveau système symétrique, dont le déterminant sera évidemment égal mais de signe contraire à celui du système  $(a_{1,n})$ . Si l’on répète la même opération plusieurs fois de suite, on obtiendra divers systèmes symétriques dont les déterminans seront égaux entre eux, mais alternativement positifs et négatifs. On peut faire la même remarque à l’égard du système  $(a_{n,1})$  (xl. 3).

... si l’on développe la fonction symétrique alternée

$$S[\pm a_{n,n} S(\pm a_{1,1} a_{2,2} \dots a_{n-1,n-1})]$$

tous les termes du développement seront des produits symétriques de l’ordre  $n$ , qui auront l’unité pour coefficient. Ces

termes seront donc respectivement égaux à ceux qu'on obtient en développant le déterminant

$$D_n = S(\pm a_{1,1} a_{2,2} \dots a_{n,n});$$

et comme le produit principal  $a_{1,1} a_{2,2} \dots a_{n,n}$  est positif de part et d'autre, on aura nécessairement

$$\begin{aligned} D_n &= S[\pm a_{n,n} S(\pm a_{1,1} a_{2,2} \dots a_{n-1,n-1})] & (\text{VI. } 3) \\ &= a_{n,n} b_{n,n} + a_{n-1,n} b_{n-1,n} + \dots + a_{1,n} b_{1,n}. \end{aligned}$$

En général, si l'on désigne par  $\mu$  l'un des indices 1, 2, 3, ...,  $n$  on trouvera de la même manière

$$D_n = S[\pm a_{\mu,\mu} S(\pm a_{1,1} a_{2,2} \dots a_{\mu-1,\mu-1} a_{\mu+1,\mu+1} \dots a_{n,n})] \quad (\text{VI. } 4).$$

. . . . . Cette dernière équation

$$0 = a_{1,\nu} b_{1,\mu} + a_{2,\nu} b_{2,\mu} + \dots + a_{n,\nu} b_{n,\mu} \quad (\text{XII. } 6)$$

sera satisfaite toutes les fois que  $\nu$  et  $\mu$  seront deux nombres différents l'un de l'autre.

. . . . on aura donc aussi

$$D_n = a_{\mu,1} b_{\mu,1} + a_{\mu,2} b_{\mu,2} + \dots + a_{\mu,n} b_{\mu,n} \quad (\text{VI. } 4)$$

$$0 = a_{\nu,1} b_{\mu,1} + a_{\nu,2} b_{\mu,2} + \dots + a_{\nu,n} b_{\mu,n} \quad (\text{XII. } 6)$$

les indices  $\mu$  et  $\nu$  étant censés inégaux."

The expressions here denoted by  $b_{1,1}, b_{1,2}, \dots$  are spoken of as *adjugate* ("adjointes") to  $a_{1,1}, a_{1,2}, \dots$ ; and the system

$$\left\{ \begin{array}{l} b_{1,1} \quad b_{1,2} \quad \dots \quad b_{1,n} \\ b_{2,1} \quad b_{2,2} \quad \dots \quad b_{2,n} \\ \&c. \quad \dots \quad \dots \\ b_{n,1} \quad b_{n,2} \quad \dots \quad b_{n,n} \end{array} \right.$$

as *adjugate* to the system  $(a_{1,n})$ . Similarly the system  $(b_{n,1})$  is said to be *adjugate* to the system  $(a_{n,1})$ ; and, on the other hand, it is said to be *adjugate and conjugate* to the system  $(a_{1,n})$ .

Up to this point no new property has been brought forward. The following paragraph (p. 68), however, opens new ground, the formula given in it being of some considerable importance in the after development of the theory.

"Si dans le système de quantités  $(a_{1,n})$  on supprime la dernière

suite horizontale et la dernière suite verticale, on aura le système suivant,

$$\begin{cases} a_{1'1}, & a_{2'1} \dots \dots a_{1'n-1}, \\ a_{2'1}, & a_{2'2} \dots \dots a_{2'n-1}, \\ \&c. \dots \dots \\ a_{n-1'1}, & a_{n-1'2} \dots \dots a_{n-1'n-1}, \end{cases}$$

que je désignerai à l'ordinaire par  $(a_{1'n-1})$ .

“Soit maintenant  $(e_{1'n-1})$  le système adjoint au précédent. Si dans l'équation (13) on change  $b$  en  $e$  et  $n$  en  $n-1$ , ou aura en général

$$D_{n-1} = b_{n'n} = a_{\mu'1}e_{\mu'1} + a_{\mu'2}e_{\mu'2} + \dots + a_{\mu'n-1}e_{\mu'n-1}.$$

Pour déduire de cette dernière équation la valeur de  $b_{\mu'n}$ , il suffira en vertu des règles établies, de changer  $a_{\mu'\nu}$  en  $a_{n'\nu}$  dans l'expression précédente de  $b_{n'n}$ , et de changer en outre le signe du second membre : on aura donc généralement

$$b_{\mu'n} = -(a_{n'1}e_{\mu'1} + a_{n'2}e_{\mu'2} + \dots + a_{n'n-1}e_{\mu'n-1}).$$

Si dans cette équation on donne successivement à  $\mu$  toutes les valeurs entières depuis 1 jusqu'à  $n-1$ , et que l'on substitue les valeurs qui en résulteront pour  $b_{1'n}$ ,  $b_{2'n}$ , ...,  $b_{n-1'n}$  dans l'équation

$$D_n = a_{1'n}b_{1'n} + a_{2'n}b_{2'n} + \dots + a_{n'n}b_{n'n},$$

on obtiendra la formule suivante,

$$D_n = a_{n'n}b_{n'n} - \begin{cases} a_{1'n}a_{n'1}e_{1'1} & + a_{2'n}a_{n'2}e_{2'2} + \dots + a_{n-1'n}a_{n'n-1}e_{n-1'n-1} \\ + a_{1'n}(a_{n'2}e_{1'2} & + a_{n'3}e_{1'3} + \dots + a_{n'n-1}e_{1'n-1}) \\ + a_{2'n}(a_{n'1}e_{2'1} & + a_{n'3}e_{2'3} + \dots + a_{n'n-1}e_{2'n-1}) \\ + \&c. \dots \dots \dots \\ + a_{n-1'n}(a_{n'1}e_{n-1'1} & + a_{n'2}e_{n-1'2} + \dots + a_{n'n-2}e_{n-1'n-2}). \end{cases}$$

Cette équation peut être mise sous la forme

$$D_n = a_{n'n}D_{n'1}^* - S^{n-1}S^{n-1}(a_{\nu'n}a_{n'\mu}e_{\nu'\mu}), \quad (\text{XXXVII.})$$

les deux signes  $S$  étant relatifs le premier à l'indice  $\mu$  et le second à l'indice  $\nu$ .”

This is the well-known formula nowadays described as giving

\* Misprint in original, for  $D_{n-1}$ .

the development of a determinant according to binary products of a row and column. The special row here used is the  $n^{\text{th}}$  and the special column the  $n^{\text{th}}$  likewise.

The four pages regarding the application of determinants to the solution of a set of simultaneous equations may be passed over with the remark that they give evidence of the importance attached by Cauchy to his new definition of determinants, the solution in the case of the example

$$\left. \begin{aligned} a_1x_1 + b_1x_2 &= m_1 \\ a_2x_1 + b_2x_2 &= m_2 \end{aligned} \right\}$$

being first put in the form

$$x = \frac{mb(b-a)}{ab(b-a)}, \quad y = \frac{am(m-a)}{ab(b-a)};$$

and similarly in the case of the example

$$a_r x_1 + b_r x_2 + c_r x_3 = m_r \quad (r = 1, 2, 3).$$

The determinant solution of a set of simultaneous equations is put to good use by Cauchy to obtain new properties of the functions. Taking the set of equations

$$(20) \left\{ \begin{aligned} a_{1.1}x_1 + a_{1.2}x_2 + \dots + a_{1.n}x_n &= m_1 \\ a_{2.1}x_1 + a_{2.2}x_2 + \dots + a_{2.n}x_n &= m_2 \\ \&c. \dots \dots \dots \\ a_{n.1}x_1 + a_{n.2}x_2 + \dots + a_{n.n}x_n &= m_n \end{aligned} \right.$$

and solving for  $x_1, x_2, \dots$  he obtains of course the set

$$\left. \begin{aligned} m_1 b_{1.1} + m_2 b_{2.1} + \dots + m_n b_{n.1} &= D_n x_1, \\ m_1 b_{1.2} + m_2 b_{2.2} + \dots + m_n b_{n.2} &= D_n x_2, \\ \&c. \dots \dots \dots \\ m_1 b_{1.n} + m_2 b_{2.n} + \dots + m_n b_{n.n} &= D_n x_n, \end{aligned} \right\}$$

where  $b_{1.1}, b_{2.1}, \dots$  have the signification above indicated, and  $D_n$  stands for  $S(\pm a_{1.1}a_{2.2} \dots a_{n.n})$ . This second set may be treated in the same way as the first set, the quantities  $m_1, m_2, \dots, m_n$  being viewed as the unknowns. To express the result the system of quantities adjugate to  $(b_{1.n})$  is denoted by  $(c_{1.n})$ , and the determinant of the system  $(b_{1.n})$  is denoted by  $B_n$ , the new set thus being

$$(27) \quad \begin{cases} c_{1,1}D_n x_1 + c_{1,2}D_n x_2 + \dots + c_{1,n}D_n x_n = B_n m_1, \\ c_{2,1}D_n x_1 + c_{2,2}D_n x_2 + \dots + c_{2,n}D_n x_n = B_n m_2, \\ \&c. \dots \dots \dots \\ c_{n,1}D_n x_1 + c_{n,2}D_n x_2 + \dots + c_{n,n}D_n x_n = B_n m_n, \end{cases}$$

Cauchy then proceeds (p. 77)—

“Les équations (27) peuvent encore être mises sous la forme suivante,

$$\begin{cases} c_{1,1}\frac{D_n x_1}{B_n} + c_{1,2}\frac{D_n x_2}{B_n} + \dots + c_{1,n}\frac{D_n x_n}{B_n} = m_1, \\ c_{2,1}\frac{D_n x_1}{B_n} + c_{2,2}\frac{D_n x_2}{B_n} + \dots + c_{2,n}\frac{D_n x_n}{B_n} = m_2, \\ \&c. \dots \dots \dots \\ c_{n,1}\frac{D_n x_1}{B_n} + c_{n,2}\frac{D_n x_2}{B_n} + \dots + c_{n,n}\frac{D_n x_n}{B_n} = m_n; \end{cases}$$

et comme celles-ci doivent avoir lieu en même temps que les équations (20), sans que l'on suppose d'ailleurs entre les termes de la suite  $x_1, x_2, \dots, x_n$  et ceux du système  $(a_{1,n})$  aucune relation particulière, il faudra nécessairement que l'on ait quels que soient  $\mu$  et  $\nu$ ,

$$c_{\mu,\nu}\frac{D_n}{B_n} = a_{\mu,\nu},$$

ou

$$c_{\mu,\nu} = \frac{B_n}{D_n} a_{\mu,\nu}. \quad (\text{XXXVIII.})$$

Cette équation établit un rapport constant entre les termes du système  $(a_{1,n})$  et les termes du système adjoint du second ordre  $(c_{1,n})$ .”

More definitely, and in more modern nomenclature, the theorem is

*The ratio of any element of a determinant to the corresponding element of the second adjugate determinant is equal to the ratio of the determinant itself to its first adjugate.* (XXXVIII.)

Attention is next directed to the group of equations—

$$\left. \begin{array}{llll}
 a_{1,1}a_{1,1} + a_{1,2}a_{1,2} + \dots + a_{1,n}a_{1,n} = m_{1,1} & a_{2,1}a_{1,1} + a_{2,2}a_{1,2} + \dots + a_{2,n}a_{1,n} = m_{1,2} & \dots & a_{n,1}a_{1,1} + a_{n,2}a_{1,2} + \dots + a_{n,n}a_{1,n} = m_{1,n} \\
 a_{1,1}a_{2,1} + a_{1,2}a_{2,2} + \dots + a_{1,n}a_{2,n} = m_{2,1} & a_{2,1}a_{2,1} + a_{2,2}a_{2,2} + \dots + a_{2,n}a_{2,n} = m_{2,2} & \dots & a_{n,1}a_{2,1} + a_{n,2}a_{2,2} + \dots + a_{n,n}a_{2,n} = m_{2,n} \\
 \dots & \dots & \dots & \dots \\
 a_{1,1}a_{n,1} + a_{1,2}a_{n,2} + \dots + a_{1,n}a_{n,n} = m_{n,1} & a_{2,1}a_{n,1} + a_{2,2}a_{n,2} + \dots + a_{2,n}a_{n,n} = m_{n,2} & \dots & a_{n,1}a_{n,1} + a_{n,2}a_{n,2} + \dots + a_{n,n}a_{n,n} = m_{n,n}
 \end{array} \right\}$$

Here there are three symmetric systems of quantities

$$(a_{1..n}), (a_{1..n}), (m_{1..n}),$$

the first appearing in every column of equations, the second in every row, and the third only once. The determinants of these systems are denoted by

$$D_n, \delta_n, M_n,$$

respectively : that is to say

$$\begin{aligned} D_n &= S(\pm a_{1..1} a_{2..2} \dots a_{n..n}) \\ \delta_n &= S(\pm a_{1..1} a_{2..2} \dots a_{n..n}) \\ M_n &= S(\pm m_{1..1} m_{2..2} \dots m_{n..n}). \end{aligned}$$

If now in

$$S(\pm a_{1..1} a_{2..2} \dots a_{n..n})$$

there be substituted for  $m_{1..1}, m_{1..2}, \dots$  their values as given by the group of equations, there will be obtained a function of all the  $\alpha$ 's and  $\alpha$ 's, which must be an alternating function with respect to the first indices of the  $\alpha$ 's and also with respect to the first indices of the  $\alpha$ 's. Further, since each of the  $m$ 's is of the first degree in the  $\alpha$ 's and of the first degree also in the  $\alpha$ 's, each term of the development of  $S(\pm m_{1..1} m_{2..2} \dots m_{n..n})$  must evidently be of the form

$$\pm a_{1..\mu} a_{2..\nu} \dots a_{n..\pi} a_{1..\mu} a_{2..\nu} \dots a_{n..\pi}.$$

But the development by reason of its double alternating character cannot contain such a term without containing all the terms of the product

$$\pm S(\pm a_{1..\mu} a_{2..\nu} \dots a_{n..\pi}) S(\pm a_{1..\mu} a_{2..\nu} \dots a_{n..\pi}).$$

Consequently it must equal one or more products of this kind. But again the indices  $\mu, \nu, \dots, \pi$  are either all different or not. If they be different, we have

$$S(\pm a_{1..\mu} a_{2..\nu} \dots a_{n..\pi}) = \pm S(\pm a_{1..1} a_{2..2} \dots a_{n..n}) = \pm \delta_n;$$

and if any two of them be equal

$$S(\pm a_{1..\mu} a_{2..\nu} \dots a_{n..\pi}) = 0.$$

The like is true in regard to  $S(\pm a_{1..\mu} a_{2..\nu} \dots a_{n..\pi})$ . This enables us to conclude that the development of  $M_n$  is equal to one or more products of the form

$$\pm D_n \delta_n:$$

in other words, that

$$M_n = c D_n \delta_n,$$

where  $c$  is a constant. But if we take the very special case where

$$a_{\mu,\mu} = 1, \quad a_{\mu,\mu} = 1, \quad a_{\mu,\nu} = 0, \quad a_{\nu,\mu} = 0,$$

and where consequently

$$m_{\mu,\mu} = 1, \quad m_{\mu,\nu} = 0,$$

we see that

$$M_n = 1, \quad D_n = 1, \quad \delta_n = 1,$$

and that therefore

$$c = 1.$$

Hence the final result is

$$M_n = D_n \delta_n. \quad (\text{xvii. 5}).$$

This, the now well-known multiplication-theorem of determinants, Cauchy puts in words as follows (p. 82):—

*Lorsqu'un système de quantités est déterminé symétriquement au moyen de deux autres systèmes, le déterminant du système résultant est toujours égal au produit des déterminans des deux systèmes composans.* (xvii. 5).

It is quite clear, from what has been said above, that it was discovered independently, and about the same time, by Binet and Cauchy, and ought to bear the names of both. Binet has the further merit of having reached a theorem of which Cauchy's is a special case, and then made an additional generalisation in a different direction; and Cauchy has the advantage over Binet of having produced, along with his special case, a satisfactory proof of it.

From the theorem Cauchy goes on to deduce several results equally important. Substituting for the system  $(a_{1..n})$  the system  $(b_{1..n})$  adjugate to  $(a_{1..n})$  so that

$$\delta_n = S(\pm b_{1.1} b_{2.2} \dots b_{n.n}) = B_n,$$

we know that then

$$m_{\mu,\mu} = D_n \text{ and } m_{\mu,\nu} = 0;$$

that consequently  $M_n$  consists of but a single term, viz.

$$m_{1.1} m_{2.2} \dots m_{n.n}, \text{ i.e. } D_n^n:$$

and that therefore by the theorem

$$D_n^n = B_n D_n,$$

whence

$$B_n = D_n^{n-1}. \quad (\text{xxi. 2}).$$



This result, afterwards so well known, Cauchy translates into words as follows (p. 82):—

*. . . le déterminant du système  $(b_{1,n})$  adjoint au système  $(a_{1,n})$  est égal à la  $(n-1)^{\text{me}}$  puissance du déterminant de ce dernier système. (xxi. 2).*

Again, by returning to the identity,

$$c_{\mu,\nu} = \frac{B_n}{D_n} a_{\mu,\nu}$$

and substituting the value of  $B_n$  just obtained, there is deduced the result

$$c_{\mu,\nu} = D_n^{n-2} a_{\mu,\nu}; \quad (\text{xxxix.})$$

or, in words,

*. . . étant donné un terme quelconque  $a_{\mu,\nu}$  du système  $(a_{1,n})$ , pour obtenir le terme correspondant du système adjoint du second ordre  $(c_{1,n})$  il suffira de multiplier le terme donné par la  $(n-2)^{\text{me}}$  puissance du déterminant du premier système. (xxxix.)*

A considerable amount of space (pp. 82–92) is devoted to the consideration of the adjugate systems of

$$(a_{1,n}), (a_{1,n}), (m_{1,n}),$$

and the adjugates of these adjugates; but nothing new is elicited. The section closes with the manifest identity

$$\begin{aligned} & (a_{1,1} + a_{2,1} + \dots + a_{n,1}) (a_{1,1} + a_{2,1} + \dots + a_{n,1}) \\ & + (a_{1,2} + a_{2,2} + \dots + a_{n,2}) (a_{1,2} + a_{2,2} + \dots + a_{n,2}) \\ & + \&c. \dots \dots \dots \\ & + (a_{1,n} + a_{2,n} + \dots + a_{n,n}) (a_{1,n} + a_{2,n} + \dots + a_{n,n}) \\ = & m_{1,1} + m_{2,1} + \dots + m_{n,1} \\ & + m_{1,2} + m_{2,2} + \dots + m_{n,2} \\ & + \dots \dots \dots \\ & + m_{1,n} + m_{2,n} + \dots + m_{n,n}, \end{aligned}$$

which, using later technical terms, we may express as follows:—

*If there be two determinants, and the sum of the elements of one first column be multiplied by the sum of the elements of the other first column, the sum of the elements of one second column by the sum of the elements of the other second column, and so on, then the sum of these products is equal to the sum of the elements of the product of the two determinants. (xl.)*

The third section breaks entirely fresh ground, its heading being

*Des Systèmes de Quantités dérivées et de  
leurs Déterminans.*

Of the integers 1, 2, 3, . . . ,  $n$  all the possible sets of  $p$  integers are supposed to be taken, and arranged in order on the principle that any one has precedence of any other if the product of the members of the former be less than the product of the members of the latter. The number  $n(n-1) \dots (n-p+1) / 1.2.3 \dots p$  of the said sets being denoted by  $P$ , the  $P^{\text{th}}$  and last set would thus be

$$n-p+1, n-p+2, \dots, n-1, n.$$

Now, any two of the sets being fixed upon, say the  $\mu^{\text{th}}$  and  $\nu^{\text{th}}$ , the system of quantities  $(a_{1,n})$  is returned to, and from it are deleted (1) all the "termes" whose first index is not found in the  $\mu^{\text{th}}$  set, and (2) all the "termes" whose second index is not found in the  $\nu^{\text{th}}$  set. What is left after this action is clearly "un système de quantités symétriques de l'ordre  $p$ ," the determinant of which may be denoted by  $a_{\mu,\nu}^{(p)}$ . For example, if  $\mu=\nu=1$ , all the  $a$ 's would be deleted whose first or second index was not included in the set 1, 2, 3, . . . ,  $p$ , and there would be left the system

$$\left\{ \begin{array}{cccc} a_{1,1} & a_{1,2} & \dots & a_{1,p} \\ a_{2,1} & a_{2,2} & \dots & a_{2,p} \\ \&c. & \dots & \dots & \\ a_{p,1} & a_{p,2} & \dots & a_{p,p} \end{array} \right.$$

of which the determinant would be denoted by

$$a_{1,1}^{(p)}.$$

As any one of the  $P$  sets could be taken along with any other, preparatory to forming such a determinant, there would necessarily be in all  $P \times P$  possible determinants. Arranged in a square as follows:—

$$\left\{ \begin{array}{cccc} a_{1,1}^{(p)} & a_{1,2}^{(p)} & \dots & a_{1,P}^{(p)} \\ a_{2,1}^{(p)} & a_{2,2}^{(p)} & \dots & a_{2,P}^{(p)} \\ \&c. & \dots & \dots & \\ a_{P,1}^{(p)} & a_{P,2}^{(p)} & \dots & a_{P,P}^{(p)} \end{array} \right.$$

they manifestly form "un système symétrique de l'ordre P," the determinant of which, in strict accordance with previous convention, is denoted by

$$(a_{1,P}^p).$$

Cauchy then proceeds (p. 96)—

Si l'on donne successivement à  $p$  toutes les valeurs

$$1, 2, 3, \dots, n-3, n-2, n-1$$

P prendra les valeurs suivantes,

$$n, \frac{n(n-1)}{1 \cdot 2}, \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}, \dots, \frac{n(n-1)}{1 \cdot 2}, n,$$

et l'on obtiendra par suite un nombre égal à  $n-1$  de systèmes symétriques différens les uns des autres, dont le premier sera le système donné  $(a_{1,n})$ . Ces différens systèmes seront désignés respectivement par

$$(a_{1,n}), \left[ a_1 \cdot \frac{n(n-1)}{1 \cdot 2} \right], \left[ a_1 \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \right], \dots, \left[ a_1 \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \right], \left[ a_1 \cdot \frac{n(n-1)}{1 \cdot 2} \right], (a_{1,n}^{(n-1)});$$

je les appellerai *systèmes dérivés* de  $(a_{1,n})$ . Parmi ces systèmes, ceux qui correspondent à des valeurs de  $p$  dont la somme est égale à  $n$  sont toujours de même ordre; je les appellerai *systèmes dérivés complémentaires*. Ainsi en général

$$(a_{1,P}^{(p)}) \text{ et } (a_{1,P}^{(n-p)})$$

sont deux systèmes dérivés complémentaires l'un de l'autre, dont l'ordre est égal à

$$P = \frac{n(n-1) \cdot \dots \cdot (n-p+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot p}.$$