On the Instability of Jets. By LORD RAYLEIGH, F.R.S.

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Many, it may even be said, most of the still unexplained phenomena of Acoustics are connected with the instability of jets of fluid. For this instability there are two causes; the first is operative in the case of jets of heavy liquids, *e.g.*, water, projected into air (whose relative density is negligible), and has been investigated by Plateau in his admirable researches on the figures of a liquid mass, withdrawn from the action of gravity. It consists in the operation of the capillary force, whose effect is to render the infinite cylinder an unstable form of equilibrium, and to favour its disintegration into detached masses whose aggregate surface is less than that of the cylinder. The other cause of instability, which is operative even when the jet and its environment are of the same material, is of a more dynamical character.

With respect to instability due to capillary force, the principal problem is the determination, as far as possible, of the mode of disintegration of an infinite cylinder, and in particular of the number of masses into which a given length of cylinder may be expected to distribute itself. It must, however, be observed that this problem is not so definite as Plateau seems to think it; the mode of falling away from unstable equilibrium necessarily depends upon the peculiarities of the small displacements to which a system is subjected, and without which the position of equilibrium, however unstable, could not be departed from. Nevertheless, in practice, the latitude is not very great, because some kinds of disturbance produce their effect much more rapidly than others. In fact, if the various disturbances be represented initially by $a_1, a_2, a_3, \ldots$, and after a time t by $a_1e^{q_1t}$, $a_2e^{q_2t}$, $a_3e^{q_3t}, \ldots$, the (positive) quantities q_1, q_2, q_3, &c., being in descending order of magnitude, it is easy to see that, when $a_1, a_2, \ldots$ are small enough, the first kind necessarily acquires the preponderance. For example, at time t the ratio of the second kind to the first is $\frac{a_2}{a_1}e^{-(q_1-q_2)t}$, which, independently of the value of $a_2 : a_1$, can be made as small as we please by taking t great enough. But, in order to allow the application of the analytical expressions for so extended a time, it is generally necessary to suppose the whole amount of disturbance to be originally extremely small.†

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* Additions made since the reading of the paper are enclosed in square brackets. *April*, 1879.

† Some of the theorems given in the Proceedings for June 1873 (Theory of Sound, §§ 88, 89), for the periods of vibrations about a configuration of stable equilibrium, are applicable, *mutatis mutandis*, to the times of falling away from unstable equilibrium when various types of displacement are considered. For example, the application of a constraint could never diminish the shortest time previously possible.

Let us, then, taking the axis of z along the axis of the cylinder, suppose that at time t the surface of the cylinder is of the form

$$r = a + \alpha \cos \kappa z \qquad (1);$$

where α is a small quantity variable with the time, and $\kappa = 2\pi\lambda^{-1}$, λ being the *wave-length* of the original disturbance. The information that we require will be readily obtained by Lagrange's method, when we have calculated expressions for the potential and kinetic energies of the motion represented by (1).

The potential energy due to the capillary forces is a question merely of the surface of the liquid. If we denote the surface corresponding (on the average) to the unit length along the axis by σ, we readily find

$$\sigma = 2\pi a + \tfrac{1}{2}\pi a \kappa^2 \alpha^2 \qquad (2).$$

In this, however, we have to substitute for a (which is not strictly constant) its value obtained from the condition that S, the volume enclosed per unit of length, is given. We have

$$S = \pi a^2 + \tfrac{1}{2}\pi \alpha^2 \qquad (3),$$

whence

$$a = \sqrt{\left(\frac{S}{\pi}\right)} \cdot \left(1 - \tfrac{1}{4}\frac{\pi\alpha^2}{S}\right) \qquad (4).$$

Using this in (2), we get with sufficient approximation

$$\sigma = 2\sqrt{(\pi S)} + \frac{\pi\alpha^2}{2a}(\kappa^2 a^2 - 1) \qquad (5);$$

or, if σ_0 be the value of σ for the undisturbed condition,

$$\sigma - \sigma_0 = \frac{\pi\alpha^2}{2a}(\kappa^2 a^2 - 1) \qquad (6).$$

From this we infer that, if $\kappa a > 1$, the surface is greater after displacement than before; so that, if $\lambda < 2\pi a$, the displacement is of such a character that with respect to it the system is stable. We are here concerned only with values of κa less than unity. If T_1 denote the cohesive tension, the potential energy V reckoned per unit of length from the position of equilibrium is

$$V = -T_1 \frac{\pi\alpha^2}{2a}(1 - \kappa^2 a^2) \qquad (7).$$

We have now to calculate the kinetic energy of motion. It is easy to prove that the velocity potential is of the form

$$\phi = A J_0(i\kappa r) \cos \kappa z \qquad (8),$$

J_0 being the symbol of Bessel's functions of zero order, so that

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots \qquad (9).$$

The coefficient A is to be determined from the consideration that the outwards normal velocity at the surface of the cylinder is equal to $\dot{a}\cos\kappa z$. Hence

$$i\kappa A\, J_0'(i\kappa a) = \dot{a} \quad\text{.........................(10)}.$$

Denoting the density by ρ, we have for the kinetic energy the expression

$$T = \tfrac{1}{2}\rho \int 2\pi a \,.\, \phi \,\frac{d\phi}{dr_{(=a)}}\, dz\,;$$

or, if we reckon it in the same way as V per unit of length,

$$T = \tfrac{1}{2}\rho\,\pi\, a^2\, \frac{J_0(i\kappa a)\,\dot{a}^2}{i\kappa a\, J_0'(i\kappa a)} \quad\text{.......................(11)}.$$

Thus, by Lagrange's method, if $a \propto e^{qt}$,

$$q^2 = \frac{T_1}{\rho a^3}\,\frac{(1-\kappa^2 a^2)\,.\,i\kappa a\,.\,J_0'(i\kappa a)}{J_0(i\kappa a)} \quad\text{...................(12)},$$

which determines the law of falling away from equilibrium for a disturbance of wave-length λ. The solutions for the various values of λ and the corresponding energies are independent of one another; and thus, by Fourier's theorem, it is possible to express the condition of the system at time t, after the communication of any infinitely small disturbances symmetrical about the axis. But what we are most concerned with at present is the value of q^2 as a function of κa, and especially the determination of that value of κa for which q^2 is a maximum. That such a maximum must exist is evident *a priori*. Writing x for κa, we have to examine the values of

$$\frac{(1-x^2)\,.\,ix\,.\,J_0'(ix)}{J_0(ix)} \quad\text{..........................(13)}.$$

Expanding in powers of x, we may write, for (13),

$$\tfrac{1}{2}x^2(1-x^2)\left\{1-\frac{x^2}{2^3}+\frac{x^4}{2^4\,.\,3}-\frac{11x^6}{2^{10}\,.\,3}+\frac{19x^8}{2^{11}\,.\,3\,.\,5}+\dots\right\} \quad\text{.....(14)}.$$

or

$$\tfrac{1}{2}\left\{x^2-\tfrac{9}{8}x^4+\frac{7}{2^4\,.\,3}x^6-\frac{25}{2^{10}}x^8+\frac{91}{2^{11}\,.\,3\,.\,5}x^{10}+\dots\right\} \quad\text{......(15)}.$$

Hence, to find the maximum, we obtain by differentiation

$$1-\tfrac{9}{4}x^2+\frac{7}{2^4}x^4-\frac{100}{2^{10}}x^6+\frac{91}{2^{11}\,.\,3}x^8+\dots = 0 \quad\text{...........(16)}.$$

If the last two terms be neglected, the quadratic gives $x^2 = \cdot 4914$. If this value be substituted in the small terms, the equation becomes

$$\cdot 98928 - \tfrac{9}{4}x^2+\tfrac{7}{16}x^4 = 0,$$

whence

$$x^2 = \cdot 4858 \quad\text{................................(17)}.$$

The corresponding value of λ is given by

$$\lambda = 4{\cdot}508 \times 2a \quad\text{.........................}(18),$$

which gives accordingly the ratio of wave-length to diameter for the kind of disturbance which leads most rapidly to the disintegration of the cylindrical mass. The corresponding number obtained by Plateau from some experiments by Savart is 4·38, but this estimate involves a knowledge of the coefficient of contraction of a jet escaping through a small hole in a thin plate, and is probably liable to a greater error than its deviation from 4·51.

The following table exhibits the relationship between x^2 or $\kappa^2 a^2$ and the square root of expression (13) to which q is proportional:—

x^2		x^2	
·05	·1536	·4	·3382
·1	·2108	·5	·3432
·2	·2794	·6	·3344
·3	·3182	·8	·2701
		·9	·2015

In the cases just considered, the cause of instability is statical, and the phenomena are independent of the general translatory motion of the jet; but the other kind of instability has its origin in this very translatory motion. In his work on the discontinuous movements of fluids, Helmholtz* remarks upon the instability of surfaces separating portions of fluid which move discontinuously, and Sir W. Thomson,† in treating of the influence of wind on waves in water, supposed frictionless, has shewn under what conditions a level surface of water is rendered unstable. In the following investigations the method of Thomson's paper is applied to determine the law of falling away from unstable equilibrium in some of the simpler cases of a plane or cylindrical surface of separation.

Let us suppose that the equilibrium position of the surface of separation is represented by $z = 0$, and that on the positive and negative sides of it the velocities of the fluid are parallel to the axis of x, and of magnitudes V and V' respectively. In the absence of friction, the motion consequent upon any deformation of the surface of separation is determinate, in virtue of a well known hydrodynamical law. By Fourier's theorem, any displacement in two dimensions can be resolved into component displacements of the undulatory type, and the effect of any two undulatory displacements may be considered separately. We might, therefore, take as the initial equation of the

* Phil. Mag., Vol. xxxvi. 1868. † Phil. Mag., Nov. 1871.

surface of separation $h = H \cos \kappa x$, in which h denotes the elevation at any point, λ the wave-length of the disturbance, and $\kappa = 2\pi\lambda^{-1}$. But, as in almost all such cases, it is more convenient to use complex expressions, from which the imaginary parts are finally rejected. We will therefore assume

$$h = H e^{int} e^{i\kappa x} \quad \text{.........................(19)};$$

and the principal question which we have to consider is the dependence of n upon κ or λ.

For the velocity potential of the fluid on the positive side, we may take

$$\phi = A e^{int} e^{i\kappa x} e^{-\kappa z} + Vx \quad \text{.....................(20)},$$

in which A is to be determined by equating the value of the normal velocity at the surface of separation with that obtained from (21). Thus (the positive direction of z being downwards)

$$-\frac{d\phi}{dz}_{(z=0)} = \kappa A e^{int} e^{i\kappa x} = \frac{dh}{dt} + V\frac{dh}{dx} = (in + i\kappa V) H e^{int} e^{i\kappa x},$$

whence $$A = i\kappa^{-1}(n+\kappa V) H \quad \text{.....................(22)};$$

so that $$\phi = i\kappa^{-1}(n+\kappa V) H e^{int} e^{i\kappa x} e^{-\kappa z} + Vx \quad \text{..............(23)}.$$

Similarly, for the fluid on the negative side,

$$\phi' = -i\kappa^{-1}(n+\kappa V') H e^{int} e^{i\kappa x} e^{\kappa z} + V'x \quad \text{..............(24)}.$$

We have now to satisfy the condition of the equality of pressures. If σ denote the density, the hydrodynamical equation of pressure for the first fluid is

$$p = C - \sigma\frac{d\phi}{dt} - \tfrac{1}{2}\sigma U^2 \quad \text{.....................(25)};$$

and approximately, when $z = 0$,

$$\tfrac{1}{2}U^2 = \tfrac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \tfrac{1}{2}\left(\frac{d\phi}{dy}\right)^2 = \tfrac{1}{2}V^2 - V(n+\kappa V) H e^{int} e^{i\kappa x} \quad \text{......(26)}.$$

In like manner, $$p' = C' - \sigma'\frac{d\phi'}{dt} - \tfrac{1}{2}\sigma' U'^2 \quad \text{................(27)},$$

where $$\tfrac{1}{2}U'^2 = \tfrac{1}{2}V'^2 + V'(n+\kappa V') H e^{int} e^{i\kappa x} \quad \text{................(28)}.$$

Hence $$\sigma(n+\kappa V)^2 + \sigma'(n+\kappa V')^2 = 0 \quad \text{................(29)},$$

which is the equation by which n and κ are connected.

The simplest case of (29) occurs when $\sigma' = \sigma$, and $V' = -V$, so that the equilibrium motions of the portions of fluid are equal and opposite.

We have then $$n^2 + \kappa^2 V^2 = 0 \quad \text{..........................(30)};$$

and for the elevations $$h = H e^{\mp\kappa Vt} \begin{matrix}\cos \kappa x \\ \sin \kappa x\end{matrix} \quad \text{...................(31)}.$$

If initially $\frac{dh}{dt} = 0$, we get

$$h = H \cosh \kappa V t \begin{matrix} \cos \kappa x \\ \sin \kappa x \end{matrix} \qquad \text{.....................(32)} ;$$

indicating that the waves on the surface of separation are *stationary*, and increase in amplitude with the time according to the law of the hyperbolic cosine. By (31), Fourier's theorem allows us to express the consequences of arbitrary initial values of h and $\frac{dh}{dt}$, within the limits of time imposed by our methods of approximation.

Next, let us suppose that $\sigma' = \sigma$, $V' = 0$. We get from (29)

$$n = \tfrac{1}{2}\kappa V(-1 \pm i) \qquad \text{..........................(33)},$$

whence $$h = He^{\mp \frac{1}{2}\kappa V t} e^{i(-\frac{1}{2}\kappa V t + \kappa x)},$$

of which the real part is

$$h = He^{\mp \kappa V t} \cos \kappa (\tfrac{1}{2} V t - x) \qquad \text{..................... (34)}.$$

In (34) an arbitrary constant may of course bo added to x. It appears that the waves travel in the same direction as the stream, and with *one-half* its velocity. [In the case of the positive exponent, the rapidity with which the amplitude increases is very great. Since $\kappa = 2\pi\lambda^{-1}$, the amplitude is multiplied by e^{π}, or about 23, in the time occupied by the tream in passing over a distance λ. If $\lambda = V\tau$, $e^{\frac{1}{2}\kappa V t} = e^{\pi \frac{t}{\tau}}$, independent of V.]

As a generalised form of (34), we may take

$$h = A \cosh (\tfrac{1}{2}\kappa V t) \cos \kappa (\tfrac{1}{2} V t - x) + B \sinh (\tfrac{1}{2}\kappa V t) \sin \kappa (\tfrac{1}{2} V t - x) \text{...(35)},$$

which gives, when $t = 0$, $h = A \cos \kappa x$.

If $\frac{dh}{dt} = 0$ initially, $B = A$, by which the solution corresponding to a surface of separation initially displaced without velocity is determined.

If initially $h = 0$, and $\frac{dh}{dt}$ is finite, we have, as the appropriate form,

$$h = B \sinh (\tfrac{1}{2}\kappa V t) \cos \kappa (\tfrac{1}{2} V t - x) \qquad \text{.................(36)}.$$

Again, suppose that $\sigma' = \sigma$, $V' = V$. In this case the roots of (29) are equal, but the general solution may be obtained by the usual method. From (29) we have

$$n = \tfrac{1}{2}\kappa [\pm i (V' - V) - (V' + V)] \qquad \text{.................(39)};$$

or, if we put $V' = V(1 + \alpha)$,

$$n = \tfrac{1}{2}\kappa V [\pm i\alpha - (2 + \alpha)] \qquad \text{......................(40)}.$$

The corresponding solution for h is

$$h = e^{i\kappa x} e^{\frac{1}{2}i(2+\alpha)\kappa Vt} \left[A e^{\frac{1}{2}i\kappa Vt.\alpha} + B e^{-\frac{1}{2}i\kappa Vt.\alpha}\right] \quad\text{.............. (41),}$$

where A and B are arbitrary constants.

Passing now to the limit when $\alpha = 0$, and taking new arbitrary constants, we get

$$h = e^{i\kappa x} e^{-i\kappa Vt} \left[A + Bt\right] \quad\text{.......................(42);}$$

or, in real quantities,

$$h = {\cos \atop \sin} \kappa (Vt - x) \; [A + Bt] \quad\text{....................(42).}$$

If initially $h = \cos \kappa x$, $\dfrac{dh}{dt} = 0$,

$$h = \cos \kappa (Vt - x) + \kappa Vt \sin \kappa (Vt - x) \quad\text{...............(43).}$$

The peculiarity of this case is that previous to the displacement there is no real surface of separation at all. Its bearing upon the flapping of sails and flags will be evident.

The proportionality to $V^{-1}\lambda$ of the time of falling away from equilibrium follows from the principle of dynamical similarity, as there is no linear element but λ.

When $V' = V$, the solution is the same, whether $\sigma' = \sigma$ or not. For example, (31), (32), (33) are applicable when $\sigma' = 0$.

In general, the solution of (29) is

$$\frac{n}{\kappa} = -\frac{\sigma V + \sigma' V' \pm i\sqrt{(\sigma\sigma')}\,.\,(V - V')}{\sigma + \sigma'} \quad\text{...............(44).}$$

If $\sigma V + \sigma' V' = 0$, n is a pure imaginary, and the waves are stationary.

We will now suppose that the two portions of fluid are limited by rigid walls whose equations are respectively $z = l$, $z = -l'$. Then, corresponding to $h = He^{int}e^{i\kappa x}$, we get for the velocity potentials, in place of (23), (24),

$$\phi = \quad i\kappa^{-1}(n + \kappa V)\, H \frac{\cosh \kappa (z - l)}{\sinh \kappa l} e^{int} e^{i\kappa x} + Vx \quad\text{.........(45),}$$

$$\phi' = -i\kappa^{-1}(n + \kappa V')\, H \frac{\cosh \kappa (z + l')}{\sinh \kappa l} e^{int} e^{i\kappa x} + V'x \quad\text{.........(46),}$$

and, in place of (29),

$$\sigma (n + \kappa V)^2 \coth \kappa l + \sigma' (n + \kappa V')^2 \coth \kappa l' = 0 \quad\text{......... (47).}$$

If $l' = l$, the result is the same as if l and l' were both infinite.

If l' be infinite, $\coth \kappa l' = 1$; (47) may then be applied to a jet of width $2l$, symmetrical and symmetrically displaced with respect to the line $z = l$, and moving with velocity V in an infinite mass whose velocity is V'. If $V' = 0$, $\sigma' = \sigma$, so that the jet is of the same density as its stationary environment, (47) becomes

$$(n + \kappa V)^2 \coth \kappa l + n^2 = 0 \quad\text{.................... (48),}$$

of which the solution is

$$n = \kappa V \frac{-1 \pm i\sqrt{(\tanh \kappa l)}}{1+\tanh \kappa l} \quad \text{.....................} \quad (49),$$

a generalisation of (33).

Thus $$h = He^{\pm\mu\kappa Vt} \cos \kappa \left[\frac{Vt}{1+\tanh \kappa l} - x\right] \quad \text{..............} \quad (50),$$

where $$\mu = \frac{\sqrt{(\tanh \kappa l)}}{1+\tanh \kappa l} \quad \text{..........................} \quad (51).$$

When κl is very small, we may take in place of (50)

$$h = He^{\pm\sqrt{(\kappa l)}\kappa Vt} \cos \kappa \, (Vt - x) \quad \text{.....................} \quad (52).$$

We see, from (52), that when l is small the time of falling away from equilibrium is increased.

If the condition to be satisfied at $z = l$ be $\phi = 0$, in place of $\frac{d\phi}{dz} = 0$, the value of ϕ is

$$\phi = -i\kappa^{-1}(n+\kappa V) H \frac{\sinh \kappa (z-l)}{\cosh \kappa l} e^{int} e^{i\kappa x} + Vx \quad \text{......} \quad (53);$$

so that, if, as before, $\frac{d\phi'}{dz} = 0$ when $z = -l'$,

$$\sigma (n+\kappa V)^2 \tanh \kappa l + \sigma' (n+\kappa V')^2 \coth \kappa l' = 0 \quad \text{........} \quad (54).$$

If $l' = \infty$, $\sigma' = \sigma$, $V' = 0$,

$$(n+\kappa V)^2 \tanh \kappa l + n^2 = 0 \quad \text{......................} \quad (55).$$

This is applicable to a jet of width $2l$, moving in still fluid with velocity V, and displaced in such a manner that the sinuosities of its two surfaces are parallel.

When κl is small, we have, approximately,

$$h = He^{\pm\sqrt{(\kappa l)}\cdot\kappa Vt} \cos \kappa \, (\kappa l \,.\, Vt - x) \quad \text{.................} \quad (56).$$

By a combination of the solutions represented by (52), (56), we may determine the consequences of any displacements (in two dimensions) of the two surfaces of a thin jet moving with velocity V in still fluid of its own density.

[These solutions may be extended to cases where the surface of separation is not plane, provided that the velocities of the fluids be constant (V, V') along it. Let us suppose that ϕ, ψ are the velocity potential and stream functions for the steady motion of the first fluid, and that the surface of separation corresponds to $\psi = \psi_0$. At $\psi = \psi_1$ let there be a rigid barrier, which of course has no influence upon the steady motion. Then, if the elevation at any point s, measured along the surface of separation, be given by

$$h = He^{i\kappa s} e^{int},$$

the velocity-potential of the disturbed motion is

$$\phi+\delta\phi=\phi-i\kappa^{-1}H(n+\kappa V)\,\frac{\cosh\kappa V^{-1}(\psi-\psi_1)}{\sinh\kappa V^{-1}\psi_1}\,e^{int}e^{i\kappa V^{-1}\phi}.$$

If l be the width of a uniform stream of velocity V, whose whole amount is equal to that of the stream between $\psi=\psi_0$ and $\psi=\psi_1$, and if dashed letters denote the corresponding quantities for the second fluid, we get finally for the equation in n

$$\sigma\coth\kappa l\,(n+\kappa V)^2+\sigma'\coth\kappa l'\,(n+\kappa V')^2=0,$$

which is the same form as (47).]

We will now pass to the consideration of cylindrical surfaces of separation, limiting ourselves for simplicity to the case of disturbances symmetrical about the axis (x). If h denote the increment of distance of any point on the surface from the axis, we may take, as before,

$$h=He^{int}e^{i\kappa x}\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(57),$$

and the corresponding value of the velocity potential for the fluid inside the cylinder is

$$\phi=AJ_0(i\kappa r)\,e^{int}e^{i\kappa x}+Vx\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(58),$$

in which A is to be determined by the condition relating to the normal velocity at the surface ($r=a$). Thus

$$\phi=\kappa^{-1}(n+\kappa V)\,H\,\frac{J_0(i\kappa r)}{J_0'(i\kappa a)}\,e^{int}e^{i\kappa x}+Vx\;\ldots\ldots\ldots\;(59).$$

For the motion of the fluid outside the cylinder, we have, in the first place, the general form

$$\begin{aligned}\phi'=\;&C(-r)^{-\frac{1}{2}}e^{\kappa r}\left\{1-\frac{1}{1.(-8\kappa r)}+\frac{1^2.3^2}{1.2.(-8\kappa r)^2}-\ldots\right\}\\&+D(-r)^{-\frac{1}{2}}e^{-\kappa r}\left\{1+\frac{1}{1.(-8\kappa r)}+\frac{1^2.3^2}{1.2.(-8\kappa r)^2}+\ldots\right\},\end{aligned}$$

in which, however, by the condition at infinity, we are to put $C=0$. Writing for brevity

$$(-r)^{-\frac{1}{2}}e^{-\kappa r}\left\{1+\frac{1}{1.(-8\kappa r)}+\frac{1^2.3^2}{1.2.(-8\kappa r)^2}+\ldots\right\}=\chi(r)\ldots(60),$$

we have accordingly

$$\phi'=B\,\chi(r)\,e^{int}e^{i\kappa x}+V'x\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(61);$$

or, on determining the value of B,

$$\phi'=i(n+\kappa V')\,H\,\frac{\chi(r)}{\chi'(a)}\,e^{int}e^{i\kappa x}+V'x\ldots\ldots\ldots\ldots\;(62).$$

In the same manner as for plane surfaces, the condition of equality of pressures now gives

$$\sigma i\kappa^{-1}(n+\kappa V)^2\,\frac{J_0(i\kappa a)}{J_0'(i\kappa a)}+\sigma'(n+\kappa V')^2\,\frac{\chi(a)}{\chi'(a)}=0\;\ldots\ldots\;(63),$$

as the quadratic by which n is determined.