with deep sincerity for all his services rendered. His extreme modesty has always made him prefer to figure as if his work for us were light and his mathematical distinction inconsiderable. But we have known far better, in the one matter and the other. Mr. Tucker remains to us, to begin his thirty-third year on the Council and his thirty-second as Secretary. Nearly all the papers which the Society has ever received have caused him correspondence. Sixteen Presidents have, like myself, found their office free from anxiety because of his and his colleague's assiduity. There has been no limit to the burdens he would willingly take upon himself in his absolutely unselfish devotion to the interests of the Society. Such a use of what might have been the leisure of half a life-time has put mathematical science under an obligation for which no gratitude would be excessive. May he long be good enough, and have the health and strength, to add to this load of obligation!

On the Functions Y and Z which satisfy the Identity $4(x^p-1)/(x-1) = Y^2 \pm pZ^2$, *where p is a Prime of the Form* $4k \pm 1$.

By L. J. ROGERS.

Received November 7th, 1898. Communicated November 10th, 1898.

§ 1.

In Prof. Mathews's *Theory of Numbers*, pp. 215–219, a very full account is given of the resolution of $4X, = 4(x^p-1)/(x-1)$ into the form $Y^2-e_1pZ^2$, where Y and Z are integral functions and

$$e_1 = (-1)^{\frac{1}{2}(p-1)},$$

according to his notation on pp. 216 and 217.

On these latter pages a method is given for calculating successively the several coefficients which occur in Y and Z, with a remark that it would be desirable to discover a method of writing down their general values, without having to calculate the preceding coefficients. The object of this present paper is to show how this may be done, and to further point out certain properties of the functions Y and Z, whereby the coefficients may be more easily deduced.

It is well known that

$$\left. \begin{aligned} Y-\sqrt{e_1 p}\, Z &= 2\,(x-r^{g^2})(x-r^{g^4})\ldots \\ Y+\sqrt{e_1 p}\, Z &= 2\,(x-r^{g})\,(x-r^{g^3})\ldots \end{aligned} \right\}, \qquad (1)$$

where r is any complex p^{th} root of unity, and g is a primitive root of p.

Denoting by $\dot{Y}$ and $\dot{Z}$ the functions $\dfrac{dY}{dx}$ and $\dfrac{dZ}{dx}$, we have

$$\frac{\dot{Y}-\sqrt{e_1 p}\,\dot{Z}}{Y-\sqrt{e_1 p}\,Z} = \frac{1}{x-r^{g^2}}+\frac{1}{x-r^{g^4}}+\frac{1}{x-r^{g^6}}+\ldots,$$

$$\frac{\dot{Y}+\sqrt{e_1 p}\,\dot{Z}}{Y+\sqrt{e_1 p}\,Z} = \frac{1}{x-r^{g}}+\frac{1}{x-r^{g^3}}+\frac{1}{x-r^{g^5}}+\ldots,$$

whence, by subtraction,

$$2\sqrt{e_1 p}\,\frac{\dot{Y}Z-Y\dot{Z}}{4X} = \sum_{h=1}^{p-1} \frac{(h/p)}{x-r^h}$$

[where (h/p) is the usual Legendrian symbol]

$$= \frac{1}{x}\Sigma\,(h/p)+\frac{1}{x^2}\Sigma\,(h/p)\,r^h+\frac{1}{x^3}\Sigma\,(h/p)\,r^{2h}+\ldots.$$

Now $\quad \Sigma\,(h/p)\,r^h = \sqrt{e_1 p} \quad$ and $\quad \Sigma\,(h/p)\,r^{mh} = (m/p)\sqrt{e_1 p},$

or, using the notation before referred to,

$$\frac{\dot{Y}Z-Y\dot{Z}}{2X} = \frac{1}{x^2}+\frac{e_2}{x^3}+\frac{e_3}{x^4}+\ldots \text{ to infinity},$$

where $e_m = (m/p)$, when $m>1$ and prime to p, while $e_{mp}=0$. Thus

$$\frac{\dot{Y}Z-Y\dot{Z}}{2X} = \frac{1}{x^p-1}\,(x^{p-2}+e_2 x^{p-3}+\ldots),$$

i.e., $\quad x\,(x-1)(\dot{Y}Z-Y\dot{Z}) = 2\,(x^{p-1}+e_2 x^{p-2}+\ldots e_{p-1}x). \qquad (2)$

It is, however, somewhat more convenient to reverse the polynomials Y and Z so as to consider the successive coefficients of ascending powers of x, and, in the infinite series which follow, to suppose $x<1$. These reversed series will be denoted by y and z, with the supposition that their leading terms are positive. By the known reciprocal properties of Y and Z, we have then

$$y = e_1 Y \quad \text{and} \quad z = Z,$$

while $$x^{p-1}+e_2x^{p-2}+\ldots = e_1(x+e_2x^2+e_3x^3+\ldots e_{p-1}x^{p-1}).$$

Hence (2) becomes

$$x(x-1)(\dot{y}z-y\dot{z}) = 2(x+e_2x^2+e_3x^3+\ldots). \tag{3}$$

Now let

$$\left.\begin{aligned} y &= 2\sqrt{X}\cos\theta \\ \sqrt{-e_1p}\,z &= 2\sqrt{X}\sin\theta \end{aligned}\right\}; \tag{4}$$

so that (3) becomes

$$-4Xx(x-1)\,\dot{\theta} = 2\sqrt{-e_1p}\,(x+e_2x^2+\ldots),$$

$$2\dot{\theta} = \sqrt{-e_1p}\,(1+e_2x+\ldots)/(1-x^p)$$

$$= \sqrt{-e_1p}\,(1+e_2x+e_3x^2+\ldots \text{ to infinity}),$$

and $$2\theta = \sqrt{-e_1p}\,(x+\tfrac{1}{2}e_2x^2+\tfrac{1}{3}e_3x^3+\ldots), \tag{5}$$

no constant being added, since, when $x=0$, $z=0$, $\theta=0$. Substituting in (4), we have

$$y = 2(1-x^p)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}\left\{\begin{aligned} &1+\tfrac{1}{4}e_1p\,(x+\tfrac{1}{2}e_2x^2+\ldots)^2 \\ &+\frac{1}{2^4}\,\frac{1}{24}p^2\,(x+\tfrac{1}{2}e_2x^2+\ldots)^4 \\ &+\ldots \end{aligned}\right\},$$

$$z = (1-x^p)^{\frac{1}{2}}(1-x^{-\frac{1}{2}})\left\{\begin{aligned} &(x+\tfrac{1}{2}e_2x^2+\ldots) \\ &+\tfrac{1}{4}\cdot\tfrac{1}{6}e_1p\,(x+\tfrac{1}{2}e_2x^2+\ldots)^3 \\ &+\frac{1}{2^4}\cdot\frac{1}{5!}p^2\,(x+\tfrac{1}{2}e_2x^2+\ldots)^5 \\ &+\ldots \end{aligned}\right\}.$$

From these equations any coefficient in y or z can be calculated without the knowledge of the preceding coefficients, and the results are the same in form as those obtained in Prof. Mathews's treatise, p. 217, These formulæ are, however, extremely intricate, and, considering that the coefficients reduce in all cases to integers, are remarkably fractional. Their ultimate simple form no doubt depends upon some inherent property relating to the order of the quadratic residues of p, but it is not easy to see how such a reduction in general can be effected.

It may be interesting, then, to point out two other methods for obtaining the required coefficients.

§ 2.

Let $$U \equiv 1+2(x+x^{\alpha}+x^{\beta}+\dots),$$

where 1, α, β are the quadratic residues of p which lie between 0 and p, so that the right-hand side of § 1 (3), is $2(U-X)$.

Now, when $x=r^{g^2}$, $U=\sqrt{e_1 p}$,

but, when $x=r^{g}$, $U=-\sqrt{e_1 p}$.

Again, when $x=r^{g^2}$, $Y=\sqrt{e_1 p}\,Z$,

but, when $x=r^{g}$, $Y=-\sqrt{e_1 p}\,Z$.

Hence $UY-e_1pZ$ and $UZ-Y$ are zero for all values of r, and therefore contain X as a factor.

Moreover, both functions $\equiv 0$ (mod 2), since

$$U\equiv 1 \quad\text{and}\quad Y\equiv e_1pZ\equiv Z.$$

Let us write then
$$\left.\begin{aligned} UY-e_1pZ &= 2MX \\ UZ-\;Y &= 2NX \end{aligned}\right\}; \tag{1}$$

or, what is the same thing,

$$\left.\begin{aligned} 2U &= MY-e_1pNZ \\ 2 &= MZ-NY \end{aligned}\right\}. \tag{2}$$

Similarly, if $V=2X-U$, *i.e.*, if $\frac{1}{2}(V-1)$ be the sum of powers of x whose indices are $<p$, and non-residues of p, then

$$\left.\begin{aligned} &VY+e_1pZ \text{ may be written } 2XM' \\ \text{and}\quad &VZ-Y \qquad\;\;\text{,,}\qquad\quad 2XN' \end{aligned}\right\}, \tag{3}$$

so that
$$M+M'=Y,$$
$$N+N'=Z,$$
$$2V=M'Y-e_1pN'Z.$$

Now, if $e_1=-1$, then $p-1$ is not a residue, so that U is of lower order than X. Consequently M is of lower order than Y, and N of lower order than Z. On the other hand, if $e_1=1$, then $p-1$ is a residue, and V is of lower order than X. In this case, M', N' are respectively of lower order than Y, Z.

In all cases therefore it is possible to find numerically integral polynomials in x, say μ and ν, such that

$$\mu Z-\nu Y=2.$$

Moreover, if Y and Z are known, M and N may be very easily deduced by differentiation, for

$$x(x-1)(\dot{Y}Z-Y\dot{Z}) = 2e_1(U-X), \quad \text{by §1, (2),}$$

while $$(x-1)(Y\dot{Y}-e_1pZ\dot{Z}) = 2p-2xX+2p(x-1)X,$$

by differentiating the identity

$$Y^2-e_1pZ^2 = 4X.$$

Hence $4Xx(x-1)\dot{Y} = 2pY-2xXY+2p(x-1)XY-2UZp+2XZp,$

$$4Xx(x-1)\dot{Z} = 2pZ-2xXZ+2p(x-1)XZ-2e_1Y(U-X).$$

Substituting for U, from (1), we have

$$\begin{aligned} 2e_1M &= p(x-1)Z-xZ+e_1Y-2x(x-1)\frac{dZ}{dx} \\ &= p(x-1)Z+xZ+e_1Y-2x\frac{d}{dx}(x-1)Z, \end{aligned}$$

which is a form more adapted for the numerical calculation of M. Similarly,

$$2pN = p(x-1)Y+xY+pZ-2x\frac{d}{dx}(x-1)Y.$$

When $e_1 = 1$, it is better to use the equivalent equations

$$2e_1M' = -p(x-1)Z-xZ+e_1Y+2x\frac{d}{dx}(x-1)Z,$$

$$2pN' = -p(x-1)Y-xY+pZ+2x\frac{d}{dx}(x-1)Y.$$

§ 3.

We may combine the results of the previous section by writing

$$2W = (U+V)-e_1(U+V),$$

$$2\mu = (M+M')-e_1(M-M'),$$

$$2\nu = (N+N')-e_1(N-N'),$$

so that $W, \mu, \nu = U, M, N$ or V, M', N' according as $e_1 = -1$ or $+1$.

With this notation, § 2, (2) and (4) give us

$$\left.\begin{aligned} 2W &= \mu Y-e_1p\nu Z \\ 2 &= \mu Z-\nu Y \end{aligned}\right\}, \qquad (1)$$

where μ, ν are integral and of lower order than Y, Z respectively.

Now let Z/Y be converted into a chain-fraction of the form

$$\frac{1}{2x+b_1+}\,\frac{n_2}{m_2\phi_2(x)+}\,\frac{n_3}{m_3\phi_3(x)+}\cdots\frac{n_h}{m_h\phi_h(x)},\tag{2}$$

where the m's and n's are positive or negative integers, and the leading coefficients in $\phi_2(x)$, $\phi_3(x)$, &c., are all unity.

If $\frac{P_1}{Q_1}, \frac{P_2}{Q_2}, \ldots, \frac{P_h}{Q_h}$ be the successive convergents, derived without numerical reduction from the equations

$$P_r = m_r\phi_r(x)\,P_{r-1}+n_rP_{r-2},$$

$$Q_r = m_r\phi_r(x)\,Q_{r-1}+n_rQ_{r-2},$$

then, evidently,

$$\left.\begin{aligned}Q_h &= m_2m_3\ldots m_hY\\ P_h &= m_2m_3\ldots m_hZ\end{aligned}\right\}.$$

Since, moreover, the fraction

$$\frac{n_{h-1}}{m_{h-1}\phi_{h-1}(x)+}\,\frac{n_h}{m_h\phi_h(x)} = \frac{n_{h-1}m_h}{m_{h-1}m_h\phi_{h-1}(x)+}\,\frac{n_h}{\phi_h(x)},$$

it is obvious that we can with equal generality suppose that $m_h=1$, so that

while

$$\left.\begin{aligned}Q_h &= m_2m_3\ldots m_{h-1}Y\\ P_h &= m_2m_3\ldots m_{h-1}Z\\ Q_{h-1} &= m_2m_3\ldots m_{h-1}\mu\\ P_{h-1} &= m_2m_3\ldots m_{h-1}\nu\end{aligned}\right\}.\tag{3}$$

Again, by the laws of chain fractions, we have

$$\frac{\mu}{Y}=\frac{Q_{h-1}}{Q_h}=\frac{1}{\phi_h(x)+}\,\frac{n_h}{m_{h-1}\phi_{h-1}(x)+}\cdots\frac{n_2}{2x+b_1},$$

$$\frac{\nu}{Z}=\frac{P_{h-1}}{P_h}=\frac{1}{\phi_h(x)+}\,\frac{n_h}{m_{h-1}\phi_{h-1}(x)+}\cdots\frac{n_3}{m_2\phi_2(x)}.$$

Now consider the fraction

$$F\equiv\frac{1}{\phi_h(x)+}\,\frac{n_h}{m_{h-1}\phi_{h-1}(x)+}\cdots\frac{n_2}{2x+b_1-}\,\frac{e_1p}{2x+b_1+}\,\frac{n_2}{m_2\phi_2(x)+}\cdots\frac{1}{\phi_h(x)},$$

which differs from $\frac{\mu}{Y}$, by writing $2x+b_1-\frac{e_1pZ}{Y}$ instead of $2x+b_1$, in deriving $\frac{\mu}{Y}$ from its two preceding convergents. Thus

$$F = \frac{\mu - \nu \frac{e_1 pZ}{Y}}{Y - Z\frac{e_1 pZ}{Y}} = \frac{\mu Y - e_1 p\nu Z}{Y^2 - e_1 pZ^2} = \frac{W}{2X}.$$

Hence, if $\frac{W}{2X}$ be chain-fractionized, the converging denominators of degree $\frac{1}{2}(p-3)$ and $\frac{1}{2}(p-1)$ will be numerically proportional to Z and Y; while the corresponding numerators are multiples of μ and ν.

As an example we may put

$$p = 23,$$

$$W = U = 2(x^{18}+x^{16}+x^{13}+x^{12}+x^9+x^8+x^6+x^4+x^3+x^2+x)+1,$$

$$\frac{U}{2X} = \frac{1}{x^4+x^3+1-}\,\frac{4}{2x^2+3-}\,\frac{2}{2x-1+}\,\frac{1}{2x+7+}\,\frac{32}{x-5-}\,\frac{1}{2x-1-}$$
$$\frac{8}{2x-1+}\,\frac{23}{2x-1+}\,\dots,$$

whence $\quad \mu = 2x^7-x^6-4x^5-4x^4-5x^3+2x^2+10x-1,$

and $\quad \nu = x^6-x^3-2x^2+1,$

while the values of Y and Z agree with those given in Prof. Mathews's treatise, p. 218.

It is easy to see that, although we have a means of calculating Y and Z which is very simple in theory, yet in practice it involves great labour, and work of such a kind as to give chances of numerical errors.

§ 4.

The most practical way of determining the coefficients of Y and Z is derived from the equation (3) in § 1, by eliminating $\dot{z}$.

We have, namely,

$$e_1 pz\,(x+e_2x^2+\dots)$$
$$= \tfrac{1}{2}e_1 px\,(x-1)(\dot{y}z^2 - yz\dot{z})$$
$$= \tfrac{1}{2}x\,(x-1)\,\dot{y}\,(Y^2-4X) - \tfrac{1}{2}x\,(x-1)\,y\,(y\dot{y}-2\dot{X})$$
$$= 2x\,(1-x^p)\,\dot{y} - x\,(1-x)\,\dot{X}\,y.$$

But $\quad (1-x)\,\dot{X} - X = -px^{p-1},$

so that $2x\,(1-x)(1-x^p)\,\dot{y} - x\,(1-x^p)\,y + px^p(1-x)\,y$

$$= e_1 pz\,(1-x)(x+e_2x^2+e_3x^3+\dots). \qquad (1)$$

Now, as we are only concerned with coefficients in y and z as far as $x^{\frac{1}{2}(p-2+e_1)}$, we may neglect powers higher than this, and replace (1) by

$$2x(1-x)\dot{y}-xy \equiv e_1 pz(1-x)(x+e_2x^2+\dots) \quad (\text{mod } x^p-1).$$

Similarly, $\quad 2x(1-x)\dot{z}-xy \equiv y(1-x)(x+e_2x^2+\dots.$

If, then, $\quad y = 2+x+c_2x^2+c_3x^3+\dots,$

$$z = x+f_2x^2+f_3x^3+\dots,$$

we get

$$x^2(4c_2-3)+x^3(6c_3-5c_2)+x^4(8c_4-7c_3)+\dots$$
$$= e_1 p(x+f_2x^2+f_3x^3+\dots)\{x+(e_2-1)x^2+(e_3-e_2)x^3+\dots\}, \quad (2)$$

and

$$2x+x^2(4f_2-3)+x^3(6f_3-5f_2)+\dots$$
$$= (2+x+c_2x^2+c_3x^3+\dots)\{x+(e_2-1)x^2+(e_3-c_2)x^3+\dots\}. \quad (3)$$

From these two equations we can very easily deduce $c_2, f_2, c_3, f_3, \dots$ successively, the advantages of the method resting on the integral form of all the terms on the right-hand sides, while the necessarily integral forms ultimately obtained for the c's and f's avoid the possibility of numerical errors.

For instance, if $p = 67$, $e_1 = -1$, and the quadratic residues as far as 18 numerically are $1, -2, -3, 4, -5, 6, -7, -8, 9, 10, -11, -12, -13, 14, 15, 16, 17, -18, \dots$, so that

$$x+(e_2-1)x^2+\dots$$
$$= x-2x^2+4x^4-2x^5+2x^6-2x^7+2x^9-2x^{11}+2x^{14}-2x^{18}+\dots,$$

whence, deducing $c_2 = -16$, $f_2 = 0$, $c_3 = 9$, $f_3 = -3$, &c., we get

$$Y = 2x^{33}+x^{32}-16x^{31}+9x^{30}+33x^{29}-44x^{28}-18x^{27}+79x^{26}-39x^{25}-48x^{24}$$
$$+75x^{23}-35x^{22}-14x^{21}+69x^{20}-89x^{19}+10x^{18}+106x^{17}-$$

with the reciprocal terms $-106x^{16}-10x^{15}-\dots$;

$$Z = x^{32}-3x^{30}+3x^{29}+4x^{28}-8x^{27}+x^{26}+9x^{25}-8x^{24}-x^{23}+7x^{22}-8x^{21}$$
$$+5x^{20}+5x^{19}-14x^{18}+8x^{17}+$$

with the reciprocal terms $8x^{16}-14x^{15}+\dots$.

Thursday, December 8th, 1898.

Lt.-Col. CUNNINGHAM, R.E., Vice-President, in the Chair.

Fifteen members present.

The minutes of the last meeting were read and confirmed.

The following gentlemen were elected members :—Robert Judson Aley, A.B., A.M., Ph.D., Professor of Mathematics, Indiana University, Bloomington, Indiana, U.S.A.; Ernest William Barnes, B.A., Fellow of Trinity College, Cambridge, R.M. Academy, Woolwich; John Hilton Grace, B.A., Fellow of St. Peter's College, Cambridge; Frank Morley, Sc.D. Cambridge, Professor of Mathematics in Haverford College, Pennsylvania, U.S.A.; Charles Almeric Rumsey, B.A., formerly Scholar of Trinity College, Cambridge; John Thomas Walley, M.A., Fellow of Jesus College, Cambridge, Assistant Professor of Mathematics, Aberystwyth.

A letter from the Auditor, Mr. Gallop, announcing that he had duly audited the accounts of the Society for the Session 1897–98, was read. On the motion of Prof. Hudson, seconded by Mr. Berry, a vote of thanks to the Auditor was carried unanimously.

Major MacMahon communicated a discovery he had recently made in the Theory of Compound Partitions.

Mr. J. E. Campbell read a paper "On Simultaneous Partial Differential Equations."

Messrs. Hammond and Berry made remarks on the communications.

The following papers were communicated in abstract :—

On Hyperplane Coordinates: Mr. W. H. Young.

Two Problems of Wave Propagation at the Surface of an Elastic Solid, and The Influence of Gravity on Waves in an Elastic Solid, with especial reference to the Earth: Mr. T. J. Bromwich.

On a Theorem in Determinants allied to Laplace's: Prof. W. H. Metzler.

Lt.-Col. Allan Cunningham, R.E. (Mr. Tucker, *pro tem.*, in the Chair), drew attention to the three following exceptionally high numbers :—

$$N_1, N_2 = \left[2^{213}(2^{209}\pm 1)^3 \mp (2^{211}\pm 1)^3\right] = (2^{210}\pm 1)(2^{210}\mp 1)^3,$$

$$N_3 = \left[\{(2^{105}+1)^4 - 2^{103}(3.2^{104}+1)\}^2 + \{(2^{105}+1)^4 - 2^{213}(2^{106}+3)\}^2\right]$$

$$= 2\,(2^{210}+1)^4.$$

The *complete factorization* of the numbers $(2^{210}\pm 1)$ being known (see Lucas's memoir *Sur la Série récurrente de Fermat*, Rome, 1879, pp. 9, 10), the three large numbers (N) are also *completely factorizable into their prime factors.* The two N_1, N_2 are of order 2^{840}, and therefore contain 253 figures; whilst N_3 is of order 2^{841}, and therefore contains 254 figures. The largest number hitherto *completely factorized* into its prime factors (so far as known to the author) is $(2^{210}+1)$, which contains 64 figures.

The following presents were made to the Library :—

"Mathematical Questions with their Solutions from the 'Educational Times,'" Vol. LXIX., 8vo; London, 1898.

Bashforth, F.—"Replica di Krupp alla protesta del Signor Bashforth," translated by F. Bashforth, B.D., 8vo; Cambridge, 1898.

"Proceedings of the Royal Society," Vol. LXIV., No. 404.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. v., No. 2, November, 1898; New York.

"Transactions of the Ottawa Literary and Scientific Society," No. 1, 1897–8.

"Bulletin des Sciences Mathématiques," Tome XXII., November, 1898; Paris.

"Tōkyō Sūgaku-Butsurigaku Kwai Kiji," Maki No. VIII., Dai 3.

"Atti della reale Accademia dei Lincei—Rendiconti," Vol. VII., Fasc. 9, Sem. 2; Roma, 1898.

"Educational Times," December, 1898.

"Indian Engineering," Vol. XXIV., Nos. 17–20; Oct. 22–Nov. 12, 1898.