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Researches in the Calculus of Variations—Part V., The Discrimination of Maxima and Minima Values of Integrals with Arbitrary Values of the Limiting Variations. By E. P. Culverwell, M.A., F.T.C.D. Received May 8th, 1894. Read May 10th, 1894.

1. Discussions of true maxima and minima of integrals with variable limits, as distinguished from merely stationary solutions, are rare in the standard text-books. Moigno has none; Jellett, Todhunter, and Carll have each obtained different and erroneous results in the one example they all give, that of the maximum solid of revolution for given superficial area (see Jellett, *Cal. of Var.*, pp. 161–165; Todhunter, *History of Cal. of Var.*, p. 408; Carll, *Cal. of Var.*, pp. 122 and 129). The only other problem with variable limits I can find attempted in those text-books is one selected by Mr. Todhunter in his *History*, p. 328, in order to show that the ordinary method is insufficient when the limits themselves enter into the quantity to be integrated. Mr. Carll adopts Mr. Todhunter's view, insisting even more strongly on the inadequacy of the ordinary method. But the ordinary method, though clumsy, is in every case adequate.

The absence of examples is doubtless due to the fact that writers on the calculus of variations have considered the variability of the constants as introducing only a problem of the differential calculus, and have contented themselves by saying that, if the stationary value of the integral be expressed in terms of the arbitrary constants, the rule for ascertaining whether the solution is a maximum or a minimum is well known. But the direct solution springs so naturally from the equations of the calculus of variations that reference to the differential calculus is superfluous.

2. Following the notation on pp. 242–244 of *Proc. of the Lond. Math. Soc.*, Vol. XXIII., I use Δ to denote variations of the *limiting* values of x and y. Then, if U be the integral, we may write the reduced form of the first variation as

$$\delta U = \Big|_{x'}^{x''} \left(u\Delta x + A_0 \Delta y + A_1 \Delta \dot{y} + \&c. + A_{n-1} \Delta y^{(n-1)}\right) + \int_{x'}^{x''} M \delta y \, dx \quad \ldots (1).$$

Now, if we suppose S to be the stationary value of the integral expressed as a function of the limiting values of x, y, $\dot{y}$, ... $y^{(n-1)}$, then the change in the stationary value as we pass from one set of limiting values to another is clearly

$$\Delta S = \frac{dS}{dx'} \Delta x' + \frac{dS}{dx''} \Delta x'' + \frac{dS}{d\dot{y}'} \Delta \dot{y}' + \&c. + \frac{dS}{dy''^{(n-1)}} \Delta y''^{(n-1)} \ldots\ldots (2).$$

Since therefore each stationary value makes $M = 0$, we get from comparing (1) and (2) the following values of the partial differential coefficients of S with regard to limits :—

$$\frac{dS}{dx'} = -u', \quad \frac{dS}{dy'} = -A_0', \quad \frac{dS}{d\dot{y}'} = -A_1', \ \&c.,$$

$$\frac{dS}{dx''} = +u'', \quad \frac{dS}{dy''} = +A_0'', \quad \frac{dS}{d\dot{y}''} = +A_1'', \ \&c.,$$

Hence $\dfrac{d^2S}{dx'^2}$, $\dfrac{d^2S}{dx'dy'}$, &c., are obtained by differentiating these values, and the ordinary method of calculating the stationary values by direct integration, and then finding by differentiation the first and second partial differential coefficients, requires us to take a lot of trouble to obtain what the equations of the calculus give us at once.

3. If the stationary value S is a maximum for fixed limits, and if S itself is a maximum when the limits are variable, then evidently S is a true maximum among all neighbouring values of the integral. But if while S is a maximum among integrals with the *same* limits, it is a minimum among *stationary* integrals with *consecutive* limits, then it is neither a maximum nor a minimum among *all* consecutive integrals.

4. *Example I.*—To find the curve which generates the solid of revolution of maximum volume for given superficial area, one extremity $x'y'$ to lie on the axis of revolution, and the other $x''y''$ to lie on the curve $y = \theta(x)$. (The references to Jellett, Todhunter, and Carll on this problem have already been given.)

Here, taking $y = 0$ as the axis of revolution, we get for U

$$U = \int_{x'}^{x''} (y^2 + ay\sqrt{1+\dot{y}^2})\,dx,$$

a being Euler's multiplier, and from $\delta U = 0$ we get

$$y^2 + \frac{ay}{\sqrt{1+\dot{y}^2}} = C \quad\ldots\ldots\ldots\ldots(3);$$

but, since $y' = 0$, we get $C = 0$, which gives for the solution

$$y^2 + (x-b)^2 = a^2,$$

and the Jacobian condition, as extended to this case (see *Proc. Lond. Math. Soc.*, Vol. XXIII., p. 249) shows that for fixed limits U is a true maximum.

The limiting terms are

$$\Big|_{x'}^{x''} \left\{ (y^2 + ay\sqrt{1+\dot{y}^2})\,\Delta x + \frac{ay\dot{y}}{\sqrt{1+\dot{y}^2}}(\Delta y - \dot{y}\Delta x) \right\} \quad\ldots\ldots\ldots(4);$$

and when we substitute for $\dot{y}$ its value $\sqrt{a^2-y^2}/y$, from (3), the coefficients of $\Delta x'$ and $\Delta x''$ disappear, and, since $\Delta y'$ is always zero by the conditions of the problem, the limiting terms reduce to

$$ay''\dot{y}''/\sqrt{1+\dot{y}''^2}\,\Delta y'';$$

or

$$y''\sqrt{a^2-y''^2}\,\Delta y''.$$

But, if $2\pi k^2$ be the given value of the superficial area, we have

$$a^2 = \frac{k^4}{2k^2-y^2};$$

and therefore, finally,

$$\frac{dS}{dy''} = \frac{y''(k^2-y''^2)}{\sqrt{(2k^2-y''^2)}}.$$

Hence $y'' = 0$ and $y'' = \pm k$ give us stationary solutions. To ascertain if the stationary solutions give maxima values to the integral, take $\frac{d^2S}{dy''^2}$, and substitute these values. Now the sign of

d^2S/dy''^2 is easily seen to be the same as that of

$$\frac{k^2-3y''^2}{\sqrt{(2k^2-y''^2)}},$$

in which the positive sign is to be given to the square root. Hence $y''=0$ gives a minimum value to S, and therefore neither a maximum nor a minimum value to U, but $y''=k$ gives a maximum value to S, and therefore also to U.

So far, we have not considered the form of the limiting curve $y=\theta(x)$. To find for what values of x'' we get maxima values of U we must express S as a function of x''. Of course, we get

$$\frac{dS}{dx''}=\frac{dS}{dy''}\frac{dy''}{dx''}=\theta'(x'')\frac{dS}{dy''}.$$

Hence we get stationary values for the same values of y'' as before, and also when $\theta'(x'')=0$. Again,

$$\frac{d^2S}{dx''^2}=\theta''(x'')\frac{dS}{dy''}+\left[\theta'(x'')\right]^2\frac{d^2S}{dy''^2}.$$

Therefore, if the curve $y=\theta(x)$ has at any point P a *minimum* ordinate *greater* than k, or a *maximum* ordinate less than k, then these ordinates give maxima values to U, for dS/dy'' is negative if $y>k$, and positive if $y<k$.*

Example II.—To find the brachistochrone for a particle descending from a curve $y'=\theta(x')$ to another curve $y''=\varphi(x'')$, the initial velocity being that due to a height h.

Mr. Todhunter gives this as a problem in which the fact that the limit appears in the quantity to be integrated introduces a difficulty not provided for in the ordinary method, and after three pages of work he leaves the second variation in a form which cannot be calculated because it still contains arbitrary variations under the integral

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* Prof. Jellett's result is that, if the ordinate of $y=\theta(x)$ be a minimum, the volume will be a true maximum, while, if the ordinate be a maximum, the volume will only be a maximum compared with others obtained by the revolution of a *circular arc*. He does not give his work, nor does he give the value $y=k$ at all.

Mr. Todhunter, in amending Prof. Jellett's conclusions, seems to have forgotten that, if S be a minimum, U is not also a minimum, a point which Prof. Jellett evidently had in view, and he must also have made some error in his clerical work, since his result is that, if $y=k$ be a maximum or a minimum ordinate, U is also a maximum or a minimum respectively, whereas $y=k$ always makes U a maximum, independently of the curve $y=\theta(x)$.

Mr. Carll, in reproducing Mr. Todhunter's result, says that he has carefully checked the work, but he also has fallen into some clerical error in calculating his d^4v/dx^4 (p. 130), which does not change sign, as he states, but is always negative.

sign. He then abandons the problem as insoluble by known methods (see his *History*, p. 328).

Taking the axis of x vertically downwards, the velocity at any point x is proportional to $\sqrt{h+x-x'}$, which I shall write as $\sqrt{H}$. Hence the problem is to make

$$U = \int_{x'}^{x''} \frac{\sqrt{1+\dot{y}^2}}{\sqrt{H}}\, dx$$

a minimum.

The stationary curve gives

$$\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = \frac{\sqrt{H}}{\sqrt{a}} \quad \text{.............................(4)},$$

where a is a constant of integration; and the limiting conditions give

$$\Delta y' = \theta'(x')\,\Delta x' \quad \text{and} \quad \Delta y'' = \phi'(x'')\,\Delta x''.$$

Hence, eliminating $\dot{y}'$ and $\dot{y}''$ from the limiting terms by means of (4), we easily reduce δU to the form

$$\delta U = \left(-\frac{\theta'(x')}{\sqrt{a}} + \frac{\sqrt{a-H''}}{\sqrt{aH''}}\right)\Delta x' + \left(\frac{\phi'(x'')}{\sqrt{a}} + \frac{\sqrt{a-H''}}{\sqrt{aH''}}\right)\Delta x'' + \int_{x'}^{x''} M\delta y\, dx.$$

$$\text{Hence } \frac{dS}{dx'} = -\left(\frac{\theta'(x')}{\sqrt{a}} - \frac{\sqrt{a-H''}}{\sqrt{aH''}}\right), \quad \frac{dS}{dx''} = \frac{\phi'(x'')}{\sqrt{a}} + \frac{\sqrt{a-H''}}{\sqrt{aH''}} \ldots (5),$$

and it is only necessary to find $\frac{d^2S}{dx'^2}$, $\frac{d^2S}{dx'\,dx''}$, and $\frac{d^2S}{dx''^2}$, and put their values into

$$\frac{d^2S}{dx'^2}\frac{d^2S}{dx''^2} - \left(\frac{d^2S}{dx'\,dx''}\right)^2,$$

in order to complete the solution. Thus the work of the calculus of variations is complete.

But the further differentiation is very complicated, because the constant a is a function of x' and x'' determined by the condition that the solution passes through the points $x'y'$ and $x''y''$. The equation which determines a is

$$\phi(x'') - \theta(x') + \sqrt{aH''-H''^2} - \sqrt{ah-h^2}$$
$$-\tfrac{1}{2}a\left(\text{versin}^{-1}\frac{2H''}{a} - \text{versin}^{-1}\frac{2h}{a}\right) = 0 \ldots (6),$$

and the utmost simplification we can make is to get rid of transcendental functions of a from the second differential coefficients; a itself cannot be eliminated completely. I have calculated them out in this form, but, as the result is still extremely complicated, it is not given here.

The ordinary method appears to promise well. The time along the stationary curve is

$$t = \sqrt{a}\left(\text{versin}^{-1}\frac{2H''}{a} - \text{versin}^{-1}\frac{2h}{a}\right),$$

and the problem is to make this a minimum where a is a function of x' and x'' determined by (6). The problem is more difficult than it looks, however, the work required to find even the first differential coefficients in (5) being very long.

5. The criteria for distinguishing maxima and minima values for fixed limits of x and y when s is the independent variable, and the length of the curve is not given, were not included in the previous paper in Vol. XXIII., and the discussion there promised must now be given.

Let
$$U = \int_Q^P u\,ds\,;$$

then the extended Jacobian criterion only applies when we suppose s, as well as the xy-limits, to be unchanged by the variation given, and therefore even for fixed limits of P and Q it is necessary to investigate the effect of a variation of s. The paper referred to enables us to ascertain whether the stationary curve is a maximum when compared with any other curve of the same length, and therefore, if we express the stationary value, S, of the integral for a curve of given length with the given xy-limits, we have only to ascertain whether that stationary value is a maximum when the length be varied, that is, we have to find the signs of

$$\frac{d^2S}{ds'^2} \text{ and } \left(\frac{d^2S}{ds'\,ds''}\right)^2 - \frac{d^2S}{ds'^2}\,\frac{d^2S}{ds''^2}.$$

We can see, too, that this must include that part of the criterion for fixed xy-limits which relates to the conjugate point (see Vol. XXIII., p. 247). For the value of the integral, taken along a stationary curve from any point to the conjugate point, differs only from its value taken along a consecutive stationary curve by terms of the *third* order. Hence the portion quadratic in Δs must vanish when the integration extends from any point to its conjugate, *i.e.*, the second differential coefficients of S with regard to the s-limits change sign, and, if the maximum property held within those limits, it holds no longer when the integral is extended beyond them.

The following problem illustrates this.

Example III.—To find the minimum surface of revolution round an axis $y = 0$ which passes through two points whose common distance from the axis is y'. Here

$$U \equiv \int_{s'}^{s''} \{ y + \tfrac{1}{2}\lambda (\dot{x}^2 + \dot{y}^2 - 1) \} \, ds,$$

and $\Delta x'$, $\Delta y'$, &c., are zero, since the limits of x and y are fixed,

$$\delta U = \Big|_{s'}^{s''} (y - \lambda) \, \Delta s + \int_{s'}^{s''} \left\{ \left(1 - \frac{d}{ds} (\lambda \dot{y}) \right) \delta y - \frac{d}{ds} (\lambda \dot{x}) \, \delta x \right\} ds.$$

Hence $$1 - \frac{d}{ds} (\lambda \dot{y}) = 0, \quad \frac{d}{ds} (\lambda \dot{x}) = 0,$$

from which we obtain

$$y - \lambda = a, \quad (y - a)^2 = (s + b)^2 + c^2,$$
$$x + d = c \log \{ (s + b) + \sqrt{(s + b)^2 + c^2} \},$$

a, b, c, and d being constants. Taking the x-axis symmetrically, $d = 0$; also $2b = -s'' - s'$, so that we obtain

$$(y' - a)^2 = \tfrac{1}{4} (s'' - s')^2 + c^2 \quad \text{......................(7)},$$
$$s'' - s' = c \, (e^{x'/c} - e^{-x'/c}) \quad \text{......................(8)},$$
$$y' - a = \lambda = \frac{c}{2} (e^{x'/c} + e^{-x'/c}) \quad \text{......................(9)}.$$

From (9) we see that λ is positive throughout, so that, when all limits are fixed, the integral is a maximum. Again, from the expression for δU, we have

$$dS/ds'' = +(y' - \lambda) = a.$$

Hence $$\frac{d^2 S}{ds''^2} = \frac{da}{ds''}, \quad \frac{d^2 S}{ds'^2} = -\frac{da}{ds'}, \quad \frac{d^2 S}{ds' ds''} = \frac{da}{ds'} = -\frac{da}{ds''},$$

and the sign of the first of these coefficients gives us all we require.

Differentiating (7) and (8), we obtain

$$-2 (y' - a) \frac{da}{ds''} = \tfrac{1}{2} (s'' - s') + 2c \frac{dc}{ds''},$$
$$1 = \frac{dc}{ds''} \left\{ e^{x'/c} - e^{-x'/c} - \frac{x'}{c} (e^{x'/c} + e^{-x'/c}) \right\}.$$

Hence, putting $a = 0$, and eliminating y' by (9), we obtain, after one or two reductions,

$$\frac{d^2S}{ds''^2} = \frac{da}{ds''} = -\frac{c}{2}\,\frac{c\,(e^{x'/c}+e^{-x'/c})-x'\,(e^{x'/c}-e^{-x'/c})}{c\,(e^{x'/c}-e^{-x'/c})-x'\,(e^{x'/c}+e^{-x'/c})}.$$

The denominator of this fraction is always negative (this is evident when x' is greater than c, and when x' is less than c it may be seen by expanding). The numerator is the well-known quantity whose sign distinguishes whether the "conjugate point" has been included in the integration, in which case the minimum property no longer holds.

In the following problem all the limits are variable. It is a modification of the well known problem of describing the curve which, with given perimeter, shall contain the greatest area, and is remarkable in that it has an indefinite number of maxima and minima solutions.

To make $U = \int_{s'}^{s''} \left(\frac{s^2}{l} - 2s + \frac{2y-y'-y''}{2}\,\dot{x}\right) ds$ a maximum or a minimum.

Adding $\frac{1}{2}\lambda\,(\dot{x}^2+\dot{y}^2-1)$ to the bracket, we get for δU,

$$\delta U = \Big|_{s'}^{s''}\left(\frac{s^2}{l}-2s-\lambda\right)\Delta s + \Big|_{s'}^{s''}\left(\frac{2y-y'-y''}{2}+\lambda\dot{x}\right)\Delta x + \Big|_{s'}^{s''}\lambda\dot{y}\,\Delta y$$

$$-\frac{x''-x'}{2}\,(\Delta y'+\Delta y'') + \int_{s'}^{s''}\left[\delta x\,\frac{d}{ds}\,(-y-\lambda\dot{x}) + \delta y\,\frac{d}{ds}\,(x-\lambda\dot{y})\right] ds \text{...(10)}.$$

Hence, from equating to zero the quantity under the integral sign, we get

$$y+\lambda\dot{x} = b, \quad x-\lambda\dot{y} = a \quad \text{......................(11)},$$

a and b being constants of integration. Combined with

$$\dot{x}^2+\dot{y}^2 = 1,$$

these give readily enough

$$\lambda = r, \quad x = r\cos\left(\frac{s}{r}+c\right)+a, \quad y = r\sin\left(\frac{s}{r}+c\right)+(b) \quad \text{...(12)},$$

r, a, b, and c being the four constants of integration, which with s' and s'' enable us to satisfy the six conditions at the limits. Using (11) and (12) to simplify the limiting terms in (10), and writing, as before, S for the stationary value of U, we get for the six conditions at the limits the equations

$$\frac{dS}{ds'} = 0, \quad \frac{dS}{ds''} = 0, \quad \&c. = 0,$$

where
$$\frac{dS}{ds'} = -\left(\frac{s'^2}{l} - 2s' - r\right), \quad \frac{dS}{ds''} = \frac{s''^2}{l} - 2s'' - r \quad \text{.........(13)},$$

$$\frac{dS}{dx'} = +\frac{y'+y''}{2}\,b = -\frac{dS}{dx''} \text{.....................(14)},$$

$$\frac{dS}{dy'} = -\lambda y' - \frac{x''}{2} + \frac{x'}{2} = a - \frac{x'+x''}{2} = -\frac{dS}{dy''} \quad \text{.........(15)}.$$

Hence, for the stationary value, we get, taking $s'' > s'$,

$$s'' = l + \sqrt{l^2 + lr}, \quad s' = l - \sqrt{l^2 + lr} \quad \text{...............(16)}.$$

(The fact that s' and s'' are given by the same quadratic equation arises from the constancy of λ, and is, of course, a mere accidental peculiarity in the problem.)

Again, (14) and (15) taken with (12) give us

$$\cos\left(\frac{s''}{r} + c\right) + \cos\left(\frac{s'}{r} + c\right) = 0 \text{..................(17)},$$

$$\sin\left(\frac{s''}{r} + c\right) + \sin\left(\frac{s'}{r} + c\right) = 0 \text{..................(18)}.$$

These equations combined give us

$$\frac{s''-s'}{r} = (2n+1)\,\pi \text{..........................(19)}.$$

From (19) and (16), we get

$$2\sqrt{l^2+lr} = (2n+1)\,r\pi \text{........................(20)},$$

from which we obtain approximately, if we take $n = 0$,

$$r = \cdot 86 \times l \quad \text{or} \quad r = -\cdot 46 \times l \quad \text{..................(21)},$$

and, whatever value we take for n, we always get one positive and one negative root.

When the limits are *fixed*, we always get either a maximum or a minimum value of U, which is evident geometrically, since the problem is then to draw between two points a curve of given length which shall contain with the chord the greatest area, or the least area (*i.e.*, the maximum *negative* area). The solution, as is well known, is a circular arc, and when rds or $r(s''-s')$ is positive it gives the minimum, and when negative the maximum, value. Hence, by (19), when n is a positive integer the solution is a minimum, and when n is a negative integer it is a maximum.

There is, however, nothing in the problem to fix the positions or directions of the axes of x and y coordinates, and it is evidently permissible to fix the origin at the centre of the circle which gives the solution of the problem, and to take y' and y'' as zero, *i.e.*, to choose the y-axis so as to pass through the two extremities of the semicircle. If also we choose the positive direction of the x-axis from x' to x'', we get for the constants a, b, and c the following equations:—

$$a = 0, \quad b = 0 \quad \text{.........................(22)},$$

$$\sin\left(\frac{s'}{r}+c\right) = 0, \quad \sin\left(\frac{s''}{r}+c\right) = 0 \quad \text{..............(23)},$$

$$\cos\left(\frac{s'}{r}+c\right) = -1, \quad \cos\left(\frac{s''}{r}+c\right) = +1 \quad \text{.........(24)};$$

but when r is negative the signs on the right-hand of the equation (24) must be changed if x'' is to be positive, and x' negative. That the four equations (22) and (23) are consistent is evident either by the geometry of the solution or from (17), (18), and (19). These equations, with

$$x''-x' = +2\sqrt{r^2}, \quad y''-y' = 0 \quad \text{..................(26)},$$

enable us greatly to simplify the work of obtaining the second differential coefficients of S. We do not require to determine the constant c.

In testing whether the solution is a maximum or a minimum, we must, of course, use the expressions for $\frac{dS}{ds'}$, &c., in (13), (14), and (15) in their *unreduced* forms, reducing by the values corresponding to the stationary solution only when all the differentiations have been performed. Before differentiating (13), (14), and (15), it is convenient to obtain the differential coefficients of r, a, and b, with respect to the limits.

From (12), we get

$$\sqrt{(x''-x')^2+(y''^2-y')^2} = 2r\sin\left(\frac{s''-s'}{2r}\right) \quad \text{...........(27)};$$

therefore $0 = \frac{dr}{ds'}\left(2\sin\frac{s''-s'}{r} - \frac{s''-s'}{r}\cos\frac{s''-s'}{2r}\right) - \cos\frac{s''-s'}{2r}$;

but, by (19), $\cos\frac{s''-s'}{2r} = 0,$

wherefore, for the stationary value

$$\frac{dr}{ds'} = \frac{dr}{ds''} = 0 \quad\text{..............................(28)},$$

Again, from (27),

$$\frac{x''-x'}{\sqrt{(x''-x')^2+(y''-y')^2}} = -\frac{dr}{dx'}\left(2\sin\frac{s''-s'}{2r} - \frac{s''-s'}{r}\cos\frac{s''-s'}{2r}\right),$$

which becomes for the stationary value, by (19) and (26),

$$\frac{dr}{dx'} = -\tfrac{1}{2} = -\frac{dr}{dx''} \quad\text{........................(28)};$$

again, for the stationary value

$$\frac{dr}{dy'} = -\frac{dr}{dy''} = -\frac{y''-y'}{4r} = 0 \quad\text{..................(29)}.$$

Again, from (12), we get

$$x''+x'-r\left[\cos\left(\frac{s''}{r}+c\right)+\cos\left(\frac{s'}{r}+c\right)\right] = 2a.$$

Hence

$$1-\frac{dr}{dx'}\left[\cos\left(\frac{s''}{r}+c\right)+\cos\left(\frac{s'}{r}+c\right)+\frac{s''}{r}\sin\left(\frac{s''}{r}+c\right)+\frac{s'}{r}\sin\left(\frac{s'}{r}+c\right)\right]$$
$$+r\frac{dc}{dx'}\left[\sin\left(\frac{s''}{r}+c\right)+\sin\left(\frac{s'}{r}+c\right)\right] = 2\frac{da}{dx'},$$

in which the quantities inside the brackets vanish by (23) and (24).

Hence $$\frac{da}{dx'} = \frac{da}{dx''} = \tfrac{1}{2} \quad\text{..............................(30)},$$

and, similarly, we show that

$$\frac{da}{dy'} = \frac{da}{dy''} = 0 \quad\text{..............................(31)}.$$

Again, $y''+y'-2r\left[\sin\left(\frac{s''}{r}+c\right)+\sin\left(\frac{s'}{r}+c\right)\right] = 2b;$

therefore

$$-\frac{dr}{dx'}\left[\sin\left(\frac{s''}{r}+c\right)+\sin\left(\frac{s'}{r}+c\right)-\frac{s''}{r}\cos\left(\frac{s''}{r}+c\right)-\frac{s'}{r}\cos\left(\frac{s'}{r}+c\right)\right]$$
$$-2r\frac{dc}{dx'}\left[\cos\left(\frac{s''}{r}+c\right)+\cos\left(\frac{s'}{r}+c\right)\right]=2\frac{db}{dx},$$

Reducing this by (19), (23), (24), and (28), we get

$$\left.\begin{aligned}\frac{db}{dx'}&=-\frac{s''-s'}{4r}=-\frac{\pi}{4}\\ \frac{db}{dx''}&=\frac{\pi}{4}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(32),$$

and, similarly, we may easily show that

$$\frac{db}{dy'}=\frac{db}{dy''}=0\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(33).$$

It is now easy to find the second differential coefficients. From (13), we get

$$\frac{d^2S}{ds'^2}=-2\frac{S'}{l}+2+\frac{dr}{ds'}=-2\frac{s'}{l}+2=2\frac{\sqrt{l^2+lr}}{l}=\frac{r\pi}{l}=\cdot 86\pi$$
$$\ldots\ldots\ldots\ldots(34),$$

taking the positive root. This is also the value of d^2S/ds''^2, and

$$\frac{d^2S}{ds'ds''}=-\frac{dr}{ds''}=0\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35),$$

Again, from (13) and (28), we get

$$\frac{d^2S}{ds'dx'}=+\frac{dr}{dx'}=-\tfrac{1}{2}=\frac{d^2S}{ds''dx''}\ldots\ldots\ldots\ldots\ldots\ldots(37).$$

Similarly, we get

$$\frac{d^2S}{ds'dx''}=\frac{d^2S}{ds''dx'}=+\tfrac{1}{2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(38),$$

and, from (14) and (32),

$$\frac{d^2S}{dx'^2}=-\frac{d^2S}{dx'dx''}=\frac{d^2S}{dx''^2}=-\frac{db}{dx'}=\frac{\pi}{4}\ldots\ldots\ldots\ldots(39),$$

while it is easy to see that the remaining coefficients, which all have dy' or dy'' in the denominator, vanish.

Hence, from (34) to (39), the part depending on the limits may be written as

$$\left.\begin{aligned} &\cdot 86\pi\Delta s'^2+\Delta s'(\Delta x'-\Delta x'')+\frac{\pi}{8}(\Delta x'-\Delta x'')^2 \\ +&\cdot 86\pi\Delta s''^2+\Delta s''(\Delta x''-\Delta x')+\frac{\pi}{8}(\Delta x''-\Delta x')^2 \end{aligned}\right\} \quad \ldots\ldots\ldots(40),$$

each line of which is evidently positive.

Hence the integral is a true minimum for these values of the limits.

There are, however, other solutions with $n>0$ or $n<0$, and these must be examined. Since we take $s''-s'$ positive, it is evident from (20) that $2n+1$ and r must have the same sign, and hence, if n be taken positively, only the positive root of r can be used. If we work out the condition analogous to (40), we shall find that every positive root for r got from putting $n=1, 2, 3$, &c., in (20) gives a new minimum solution in which the integration extends over an arc 3π, 5π, &c., round the circle, so that the area may be counted many times over in the integral, while, similarly, each negative value of r got from (20) by putting $n=-1$, $n=-2$, &c., gives us a maximum value of the integral. Thus there are to this integral an indefinite number of maxima and minima solutions.

As this result is in direct opposition to the principles laid down in Moigno, Jellett, and other text-books, which state that, if the limits are all variable, it is impossible that there should be a maximum or a minimum value, it is well to point out the error in the arguments used by these authors, and thus remove a *primâ facie* doubt as to the correctness of the above solution. Jellett gives two reasons: first, he counts the disposable constants and finds them insufficient in number, but the insufficiency is due to his leaving out the two disposable limits x' and x'' (or in the above problem s' and s''); secondly, he says that we may see *a priori* that there can be no maximum or minimum because the integral must be susceptible of all ranges of values if everything but its form be arbitrary, an argument which is obviously invalid, as it would equally show that the expression $(y-a)(y-b)(y-c)$ had no maximum or minimum. Moigno's reason is that when all the limits are variable the solution requires $x'=x''$, and the whole integral disappears, the error here being that he has omitted all the other solutions which give x' different from x''; in the present example we have taken a root s' different from the s'' root. But, besides this fatal objection, there are two others: first, that even if we admit his statement, the resulting value zero for the integral would be a true stationary value, and might be a true maximum or minimum; and, second, that there are many integrals in which the initial and final coordinates do not appear symmetrically.

6. When we examine the maxima and minima of double integrals with variable limits, we find an entirely new problem before us, because the ordinary rules of the calculus of variations for ascertaining the maxima and minima values of single integrals are not applicable to the single integrals which we obtain as limiting terms in dealing with double integrals. In fact, none of the rules for the discrimination of maxima and minima values already given, either in the present paper or the preceding one, are applicable where y is a single-valued function of x.

Thursday, June 14th, 1894.

Mr. A. B. KEMPE, F.R.S., President, in the Chair.

There not being the number of members present required by Rule XLIV. to constitute a "Special" Meeting, Mr. Tucker was called upon to communicate abstracts of the following papers which had been received:—

The Solutions of

$$\sinh\left(\lambda\frac{d}{dx}\right)y = f(x),\quad \cosh\left(\lambda\frac{d}{dx}\right)y = f(x),$$

λ a Constant: Mr. F. H. Jackson.

A Theorem in Inequalities: Mr. A. R. Johnson.

Some Properties of Two Circles: Mr. Tucker.

Note on Four Special Circles of Inversion of a System of "Generalized Brocard" Circles of a Plane Triangle: Mr. J. Griffiths.

On the Order of the Eliminant of Two or More Equations: Dr. R. Lachlan.

Impromptu communications were then made by Prof. Greenhill (on a Gyrostatic Top), Dr. Larmor, and Prof. M. J. M. Hill.

There being at this time a quorum, the meeting was made

SPECIAL.

The President read out five resolutions which had been approved by the Council, and invited discussion upon them, and upon the "Memorandum and Articles of Association of the London Mathematical Society," printed copies of which were put into the hands of the members present.

After some discussion, the resolutions, having been slightly amended, were submitted to the meeting and carried unanimously in the following forms:—

1. That the London Mathematical Society be incorporated under § 23 of the Companies Act, 1867, with the Memorandum and Articles of Association submitted by the Council.
2. That before such Incorporation all the liabilities of the present Society be discharged by the Treasurer, and all persons dealing with the Society as creditors be informed that the Society will be incorporated under § 23 of the Companies Act, 1867, and the present constitution be terminated.
3. That the Incorporation be effected by the Council.
4. That immediately after the Incorporation of the London Mathematical Society all the property now held by trustees for the benefit of the Society be transferred to the Society itself, and that the Council take all necessary steps to effect this transfer.
5. That upon such incorporation and transfer being completed, the present constitution of the Society shall be terminated.

The following presents were received:—

"Proceedings of the Royal Society," Vol. LV., No. 333.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XVIII., St. 5; Leipzig, 1894.

"Proceedings of the Edinburgh Mathematical Society," Vol. I., Session 1883.

"Berichte über die Verhandlungen der Koniglich Sächsischen Gesellschaft der Wissenschaften zu Leipzig," 1894, I.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. VIII., No. 2; Manchester, 1893-4.

"Nyt Tidsskrift for Mathematik," A. Femte Aargang, Nos. 2, 3; and B. Femte Aargang, No. 1.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXVII., Livr. 4 and 5, and Tome XXVIII., Livr. 1; Harlem, 1894.

"Jahrbuch über die Fortschritte der Mathematik," Bd. XXIII., Heft 2; Berlin, 1894.

"Bulletin de la Société Mathématique de France," Tome XXII., Nos. 3 and 4; Paris, 1894.

"Bulletin of the New York Mathematical Society," Vol. III., No. 8; May, 1894.

"Rendiconti del Circolo Matematico di Palermo," Tomo VIII., Fasc. 1, 2, and 3; 1894.

"Atti della Reale Accademia dei Lincei—Rendiconti," Serie 5, Vol. III., Fasc. 8, Sem. 1; Roma, 1894.

D'Ocagne, M.—"Sur la Composition des Lois d'Erreurs de Situation d'un Point" (from *Comptes Rendus*).

"Educational Times," June, 1894.

"Indian Engineering," Vol. XV., Nos. 16–20.

"Royal Society Catalogue of Scientific Papers," Gir.-Pet, Vol. X., 4to; London, 1894.

Mannheim, Le Col. A.—"Principes et Développements de Géométrie Cinématique," 4to; Paris, 1894.

"Memorie della Regia Accademia di Scienze, Lettere ed Arti in Modena," Serie II, Vol. IX.; Modena, 1893.

Note on Four Special Circles of Inversion of a System of Generalized Brocard Circles of a Plane Triangle. By John Griffiths, M.A. Received May 26th, 1894. Read June 14th, 1894.

Connected with a system of generalized Brocard circles of a triangle there are four circles—say, J, J_1, J_2, J_3—with respect to each of which the inverse of every circle of the system is a circle of the same system. Or we may briefly say that a system of generalized Brocard circles is self-inverse with regard to four different centres.

In a note recently communicated by me to the Society (see *Proceedings*, Vol. XXV., Nos. 479, 480), it was shown that a triangle ABC has three systems of what may be called generalized Brocard circles, or, shortly, G.B. circles. Every circle in each of the three systems in question possesses properties analogous to the Brocard circle of ABC, and can be constructed by means of a certain number of points dependent on a variable primary point U, taken on one of three given circles connected with the triangle ABC.

If U be a point on the circular arc BUC which touches AC in C,* and the angle UBC be denoted by ω, the equation in isogonal coordinates of the G.B. circle of the first system corresponding to U is

$$GBC\,(x,\, y,\, z,\, \cot\omega) = \lambda x + \mu y + \nu z - \delta = 0,$$

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* See Fig., page 381.