Note on Dirichlet's Formula for the Number of Classes of Binary Quadratic Forms for a Complex Determinant. *By* Professor G. B. Mathews.

Read January 14th, 1892. Received, in amended form, February 15th, 1892.

In the memoir on binary quadratic forms with complex coefficients (*Crelle*, xxiv., p. 291, or *Werke*, i., p. 612), Dirichlet has given a formula for the number of classes for a given determinant D. In the simplest case, when the determinant is not divisible by any square, and

$$D = \alpha + \beta i,$$

with

$$\alpha \equiv 1 \pmod{4}, \quad \beta \equiv 0 \pmod{2},$$

the formula in question reduces to

$$h = \frac{8\,|D|}{\pi \log \sigma} \cdot \Sigma \left(\frac{\lambda\alpha + \nu\beta}{m}\right) \frac{1}{\lambda^2 + \nu^2},$$

where $m = \alpha^2 + \beta^2$, $|D| = +\sqrt{\alpha^2+\beta^2}$, $\left(\frac{\lambda\alpha+\nu\beta}{m}\right)$ is a Legendrian symbol of reciprocity, and λ, ν assume all real integral values, such that

$$\lambda \equiv 1 \pmod{4}, \quad \nu \equiv 0 \pmod{2}.$$

Moreover, σ denotes the norm of $(T + U\sqrt{D})$, where (T, U) is the fundamental solution of

$$T^2 - DU^2 = 1.$$

Dirichlet observes that the value of h may be expressed in finite terms by means of the elliptic functions of modulus $\sqrt{\tfrac{1}{2}}$, and then proceeds to say that although the result may be obtained without any difficulty, the appreciation of its true character depends upon the algebraical theory of the division of the period for these particular transcendents, the discussion of which he reserves for the second part of the memoir. This second part was never published, if indeed it was ever written; and I am not aware that anything has been done towards completing Dirichlet's investigation on the lines that he has indicated. In the following note, the explicit value of h, in terms of elliptic functions, is obtained by elementary methods for the particular case above given; there would be no particular difficulty in treating the other cases in a similar manner.

If r denotes $e^{2\pi i/m}$, then, by a formula due to Gauss (*Summatio quarumdam serierum*, &c.),

$$\left(\frac{\lambda a+\nu\beta}{m}\right)\sqrt{m}=\Sigma\left(\frac{k}{m}\right)r^{(\lambda a+\nu\beta)\kappa},$$

$$[\kappa=1, 2, 3 \ldots (m-1)],$$

with the convention that $\left(\frac{k}{m}\right)$ is to be put equal to zero, when k is not prime to m.

Hence $$h=\frac{8}{\pi\log\sigma}\Sigma\left(\frac{k}{m}\right)\frac{e^{2k\pi i(\lambda a+\nu\beta)/m}}{\lambda^2+\nu^2},$$

and this is easily transformed into

$$h=\frac{16}{\pi\log\sigma}\Sigma\left(\frac{k}{m}\right)\frac{\cos\frac{2\lambda ak\pi}{m}\cos\frac{2\nu\beta k\pi}{m}}{\lambda^2+\nu^2},$$

where λ now assumes all positive odd values, and ν all positive even values; and it is understood that when $\nu=0$, the corresponding term of the series is multiplied by $\frac{1}{2}$ instead of 1.

Now consider the elliptic functions for which

$$K'=K,\quad \kappa=\kappa'=\sqrt{\tfrac{1}{2}},\quad q=e^{-\pi};$$

then (Jacobi, *Werke*, I., p. 155),

$$\log\operatorname{dn}\frac{2Kx}{\pi}=\log\sqrt{\kappa'}+\frac{2\cos 2x}{\sinh\pi}+\frac{2\cos 6x}{3\sinh 3\pi}+\ldots$$

Write successively

$$x=\frac{(a+\beta i)\pi}{m},\qquad x=\frac{(a-\beta i)\pi}{m},$$

and add; thus $\log\left\{\operatorname{dn}\frac{2(a+\beta i)K}{m}\operatorname{dn}\frac{2(a-\beta i)K}{m}\right\}$

$$=\log\kappa'+\frac{4\cosh\frac{2\beta\pi}{m}}{\sinh\pi}\cos\frac{2a\pi}{m}+\frac{4\cosh\frac{6\beta\pi}{m}}{3\sinh 3\pi}\cos\frac{6a\pi}{m}+\ldots$$

Changing β into $\frac{m}{2}-\beta, =\beta'$ say, and making use of the formula

$$\operatorname{dn}(u-iK')=-\frac{i\operatorname{cn}u}{\operatorname{sn}u},$$

we obtain

$$\text{constant}+\log\left\{\frac{\operatorname{cn}\frac{2(a+\beta i)K}{m}\operatorname{cn}\frac{2(a-\beta i)K}{m}}{\operatorname{sn}\frac{2(a+\beta i)K}{m}\operatorname{sn}\frac{2(a-\beta i)K}{m}}\right\}$$

$$=\log\kappa'+\frac{4\cosh\frac{2\beta'\pi}{m}}{\sinh\pi}\cos\frac{2a\pi}{m}+\frac{4\cosh\frac{6\beta'\pi}{m}}{3\sinh 3\pi}\cos\frac{6a\pi}{m}+\dots.$$

Adding this to the previous formula, and writing, for convenience,

$$\psi(u)=\frac{\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u},$$

we have

$$\log\left\{\psi\left(\frac{\overline{a+\beta i}.K}{m}\right)\psi\left(\frac{\overline{a-\beta i}.K}{m}\right)\right\}$$

$$=C+\frac{4\left\{\cosh\frac{2\beta\pi}{m}+\cosh\frac{2\beta'\pi}{m}\right\}}{\sinh\pi}\cos\frac{2a\pi}{m}$$

$$+\frac{4\left\{\cosh\frac{6\beta\pi}{m}+\cosh\frac{6\beta'\pi}{m}\right\}}{3\sinh 3\pi}\cos\frac{6a\pi}{m}+\dots,$$

where C is a constant.

Now (Todhunter's *Int. Calc.*, p. 300, 4th edition),

$$\frac{\cosh a(\pi-x)+\cosh ax}{a\sinh a\pi}=\frac{2}{\pi}\left\{\frac{1}{a^2}+\frac{2\cos 2x}{2^2+a^2}+\frac{2\cos 4x}{4^2+a^2}+\dots\right\};$$

consequently

$$\log\left\{\psi\left(\frac{\overline{a+\beta i}.K}{m}\right)\psi\left(\frac{\overline{a-\beta i}.K}{m}\right)\right\}$$

$$=C+\frac{8}{\pi}\left\{\frac{1}{1^2}+\frac{2\cos\frac{4\beta\pi}{m}}{2^2+1^2}+\frac{2\cos\frac{8\beta\pi}{m}}{4^2+1^2}+\dots\right\}\cos\frac{2a\pi}{m}$$

$$+\frac{8}{\pi}\left\{\frac{1}{3^2}+\frac{2\cos\frac{12\beta\pi}{m}}{2^2+3^2}+\frac{2\cos\frac{24\beta\pi}{m}}{4^2+3^2}+\dots\right\}\cos\frac{6a\pi}{m}$$

$$+\dots\dots.$$

Comparing this with the expression for h, and observing that, for a complete set of residues, $\Sigma\left(\frac{k}{m}\right) = 0$, so that C need not be determined, we find that

$$h \log \sigma = \Sigma\left(\frac{k}{m}\right) \log \left\{ \psi\left(\frac{k(a+\beta i)K}{m}\right) \psi\left(\frac{k(a-\beta i)K}{m}\right) \right\},$$

$$\left[k = 1, 2, 3 \dots (m-1)\right].$$

If a stands for a quadratic residue of m, and b for a non-residue, we may write

$$h \log \sigma = \log \frac{\Pi\psi\left(\frac{a(a+\beta i)K}{m}\right)\psi\left(\frac{a(a-\beta i)K}{m}\right)}{\Pi\psi\left(\frac{b(a+\beta i)K}{m}\right)\psi\left(\frac{b(a-\beta i)K}{m}\right)}.$$

It is interesting to see that we thus obtain, in an elliptic function form, an expression for

$$Nm\left(T_h + U_h\sqrt{D}\right),$$

where (T_h, U_h) is a solution of $T^2 - DU^2 = 1$.

Objections might no doubt be raised to the details of the preceding analysis; in fact, it cannot be said to be rigorous; still I believe the result is correct, and that the process may be justified to some extent by observing that the transcendental series equated to each other present exactly the same kind of discontinuities at the same critical points.

It ought to be said that the series for h in its first form is similar to those considered by Kronecker (Berlin *Sitzungsber.*, July, 1885), so that the result finally obtained ought to agree with his results (*l.c.*, p. 764); but, on account of the difference of notation, and the fact that Dirichlet's λ, ν do not agree with Kronecker's m, n, the identification is very troublesome.

Anyone who is inclined to undertake a numerical verification might consult with advantage Schwering's memoir on the "Multiplication of the Lemniscate Function" (*Crelle*, CVII., p. 196); it should be observed, however, that Schwering's sn u does not correspond to the sn u of the present note, but is Gauss's sinlemn u.

Thursday, February 11th, 1892.

Professor GREENHILL, F.R.S., President, in the Chair.

Messrs. E. T. Dixon and R. Holmes were admitted into the Society. Mr. James William Nicholson, M.A., President and Professor of Mathematics, Louisiana State University, U.S.A., was elected a member.

The following communications were made:—

On the Logical Foundations of Applied Mathematical Science: Mr. Dixon.

Note on the inadmissibility of the usual reasoning by which it appears that the Limiting Value of the Ratio of two Infinite Functions is the same as the ratio of their first derived, with instances in which the result obtained by it is erroneous: Mr. Culverwell.

On Saint-Venant's Torsion of Prisms: Mr. Basset.

The following presents were received:—

"Proceedings of the Mathematical Society of Edinburgh," Vols. II., III., IV., V., VI., VIII., IX.

"Beiblätter zu den Annalen der Physik und Chemie," Band XV., St. 12, 1891 Band XVI., St. 1, 1892.

"Royal Society—Proceedings," Vol. L., No. 304, January, 1892.

"Nyt Tidsskrift for Mathematik," B. Anden Aargang, Nos. 2, 3, 4.

"Leipzig. Gesells. der Wissenschaften — Berichte über die Verhandlungen," Math. Phys. Classe, 1891, III.

"Bulletin des Sciences Mathématiques," 2e Série, Tome XV., December, 1891.

"Leipzig. Gesells. der Wissenschaften — Ueber einen eigenthümlichen Fall Elektrodynamischer Induction," II.

"Educational Times," February, 1892.

"Atti della R. Accad. dei Lincei—Rendiconti," Vol. VII., Fasc. 12, e Indice del Volume, 2° Semestre, 1891; Roma.

"Cambridge Philosophical Society—Proceedings," Vol. VII., Pt. 5.; "Transactions," Vol. XV., Pt. 2.

"Toulouse—Annales de la Faculté des Sciences," Tome V., 1891, Fasc. 3, 4.

"Napoli—Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. V., Fasc. 1-12, 1891.

"Surveyor," Vol. I., Nos. 1, 2, 3, 1892.

"Œuvres Complètes de Christiaan Huygens," Vol. IV.; Le Haye, 1891.

"Royal Society's Catalogue of Scientific Papers," 1874-1883, Vol. IX.

"Mendizábal Tamborrel (J. de), Tables des Logarithmes," Paris, 1891.

Thursday, March 10th, 1892.

Professor GREENHILL, F.R.S., President, in the Chair.

Abinas Chandra Basu, Professor of Mathematics, Agra College, Agra, was elected a member.

The President and Mr. S. Roberts spoke of the services rendered by the late Dr. Hirst to the Society, and upon the loss sustained by the mathematical world in consequence of his decease.

The President announced that the Council had just selected the six following mathematicians for the compliment of foreign membership—viz., Messrs. Poincaré, Hertz, Mittag-Leffler, Schwarz, Beltrami, and Willard Gibbs; and, in accordance with Rule 29, he nominated them for election at this the first subsequent ordinary meeting, with a view to their being ballotted for at the April meeting.

The following communications were made:—

The simplest Equivalent of a given Optical Path, and the observations required to determine it: Dr. J. Larmor.

On cases in which a Hyperelliptic Integral of the First Order can be expressed as the sum of two Elliptic Integrals: Professor W. Burnside.

On the Analytical Theory of the Congruency: Professor Cayley.

Notes on Dualistic Differential Transformations: Mr. E. B. Elliott.

On certain Curves of the Fourth Order and the Porism of the Inscribed and Circumscribed Polygon: Mr. R. A. Roberts.

Professor M. J. M. Hill made a few remarks on Singular Solutions, and the President spoke on the Rectification of the Cartesian Oval.

A Cabinet Likeness of Mr. S. Roberts was presented by that gentleman to the Album.

The following presents were received:—

"Beiblätter zu den Annalen der Physik und Chemie," Band XVI., St. 2; Leipzig, 1892.

"Actuaries—Journal of Institute of," Vol. XXIX., Pt. 6, January, 1892.

"Jahrbuch über die Fortschritte der Mathematik," Jahrgang 1889, Band XXI., Heft 1; Berlin, 1892.

"Nieuw Archief voor Wiskunde," Deel XIX., St. 1–2; Amsterdam, 1892.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. X., No. 4; Coimbra, 1892.

"Bulletin des Sciences Mathématiques," 2nd Series, Tome xvi., January, 1892.

"Sitzungsberichte der K. Preuss. Akademie der Wissenschaften zu Berlin," 41–53, and Jahrgang 1891; Berlin, 1891.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. i., Fasc. 1–2, Sem. 1; Roma, 1892.

"Educational Times," March, 1892.

"Annals of Mathematics," Vol. vi., No. 4; University of Virginia, January, 1892.

"Journal für die reine und angewandte Mathematik," Band cix, H. 2.

"Bollettino delle Pubblicazione Italiano ricevute per diritto di stampa," 1892, No. 148, in duplicato; Firenze.

The Simplest Specification of a given Optical Path, and the Observations required to determine it. By J. Larmor. Read March 10th, 1892.

1. The complete specification of an optical path through a heterogeneous medium like the atmosphere, or through a combination of transparent substances which may form an optical instrument, requires not merely the form of the curve which a narrow beam or filament of light traverses, but also a statement of the character of the modification impressed upon the filament by traversing that path. The mode of division of a beam of light into filaments, or linear elements of waves, in the way thus suggested, corresponds much more closely to the physical reality than the more ordinary analysis which splits these filaments up into the rays of which they are supposed to be constituted; and it is fortunate that, by making use of the dioptrical methods introduced by Sir W. R. Hamilton, and further developed more recently by Maxwell, the consideration of filaments leads to a deeper and more coordinated analysis than that of rays. That this is the case follows from the fact that the effects impressed upon all filaments by traversing a given path may be determined in a simple manner by observation or calculation relating to a few cases, and may be considered as expressed in terms of certain optical properties of the path. In addition to the geometrical gain which results from recognising that the treatment of a group of rays is nearly as simple as that of a single ray, there is the fact that all