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XVI. *On the Problem in Nautical Astronomy for finding the Latitude by Means of two Observations of the Sun's Altitude and the Time elapsed between them.* By JAMES IVORY, *A.M. F.R.S.*

THE method generally practised in the British Navy for solving this problem was invented by Douwes, an examiner of sea officers and pilots at Amsterdam, who proposed it to the English Admiralty in 1740. It is however no more than a very limited solution; since it can only be applied with the desired success to correct the latitude by account when one of the observations is very near the meridian, or when the middle time is very little different from half the interval between the two observations. In all other cases one application of the rules will hardly lead to a result sufficiently near the truth; and a series of approximations obtained by repeated operations generally converges so slowly that the method is of little practical utility unless it be assisted by some other artifice.

Dr. Brinkley of Dublin, so long ago as 1791, gave a method of correcting the result found by one operation of Douwes's rules, which, unless in particular circumstances, is abundantly exact for nautical purposes. The same astronomer has since reconsidered the subject; and, in the Nautical Almanac 1822, has reduced his method to easy formulæ comprehending every case that can occur in practice. If this improved method be liable to objection, it is on account of the length and embarrassment of the calculation.

Delambre in his Astronomy (vol. iii. chap. 26) has examined the different methods that have been proposed for solving this problem with his usual industry and accuracy. After a careful examination of the different processes with respect to the length of the calculation and the exactness of the result, he inclines to reject all the indirect methods, and to give the preference to the direct and rigorous solution obtained by the rules of spherical trigonometry. Two elaborate articles in the *Conn. des Tems*, 1817 and 1822, written by the same astronomer, are intended to add strength to his opinion. In these he particularly examines the effect produced by supposing, as is usually done, that the sun's declination is equal to the mean quantity between the two observations and suffers no variation in the elapsed time; and he shows that the error arising from this source may be equal to the sun's change of declination. The error may no doubt be obviated by allowing for the variation of this element of the calculation *; but a new rule is required for this purpose, which

* Delambre's Astronomy, vol. iii. chap. 26. § 110.

which adds to the length and perplexity of the operations, before too complicated. In the Quarterly Journal of Science, No. 22, Dr. Brinkley has computed an example making allowance for the change of declination; and if a fair comparison be instituted between his calculation and that by the direct method, as applied to the same example in No. 21 of the same Journal, it appears hardly possible to avoid giving the preference in every respect to the latter.

I have now to propose a direct solution of the problem, simpler and easier in the calculation than that recommended by Delambre, and which, I conceive, will be found more convenient in practice than the indirect process commonly used. In explaining this solution I shall first assume that the sun's declination is the same at both the observations and equal to the mean quantity between the two times; and I shall afterwards point out an easy way of correcting the error which this assumption introduces in the result.

The principles of the method are contained in this preliminary proposition.

Lemma. (See figure 3, Plate II.) Let the base AB of a spherical triangle AZB, be bisected in O, and through O draw a great circle perpendicular to AB; then having let fall upon this circle the perpendicular ZD from the vertex of the triangle, we shall have these two formulæ, viz.

$$\text{Sin } ZD = \frac{\cos ZB - \cos ZA}{2 \sin AO},$$

$$\text{Cos } DO = \frac{\text{Cos } ZB + \cos ZA}{2 \cos AO \cos ZD}.$$

Conceive a great circle to pass through the points Z and O: then $\cos ZOB = -\cos ZOA = \sin ZOD$. Now, from the two spherical triangles ZOB and ZOA, we get,

$$\text{Cos } ZB = \cos BO \cos ZO + \sin BO \sin ZO \sin ZOD,$$
$$\text{Cos } ZA = \cos AO \cos ZO - \sin AO \sin ZO \sin ZOD;$$

wherefore, by subtracting,

$$\text{Cos } ZB - \cos ZA = 2 \sin AO \sin ZO \sin ZOD;$$

but, in the right-angled triangle ZOD, $\sin ZD = \sin ZO \sin ZOD$; consequently,

$$\text{Cos } ZB - \cos ZA = 2 \sin AO \sin ZD,$$

from which the first of the two formulæ is derived.

Again, by adding the same two equations, we obtain,

$$\text{Cos } ZB + \cos ZA = 2 \cos AO \cos ZO;$$

but, in the triangle ZOD, $\cos ZO = \cos ZD \times \cos OD$; wherefore,

$$\text{Cos } ZB + \cos ZA = 2 \cos AO \cos ZD \cos DO,$$

from which the second formula is deduced. Now

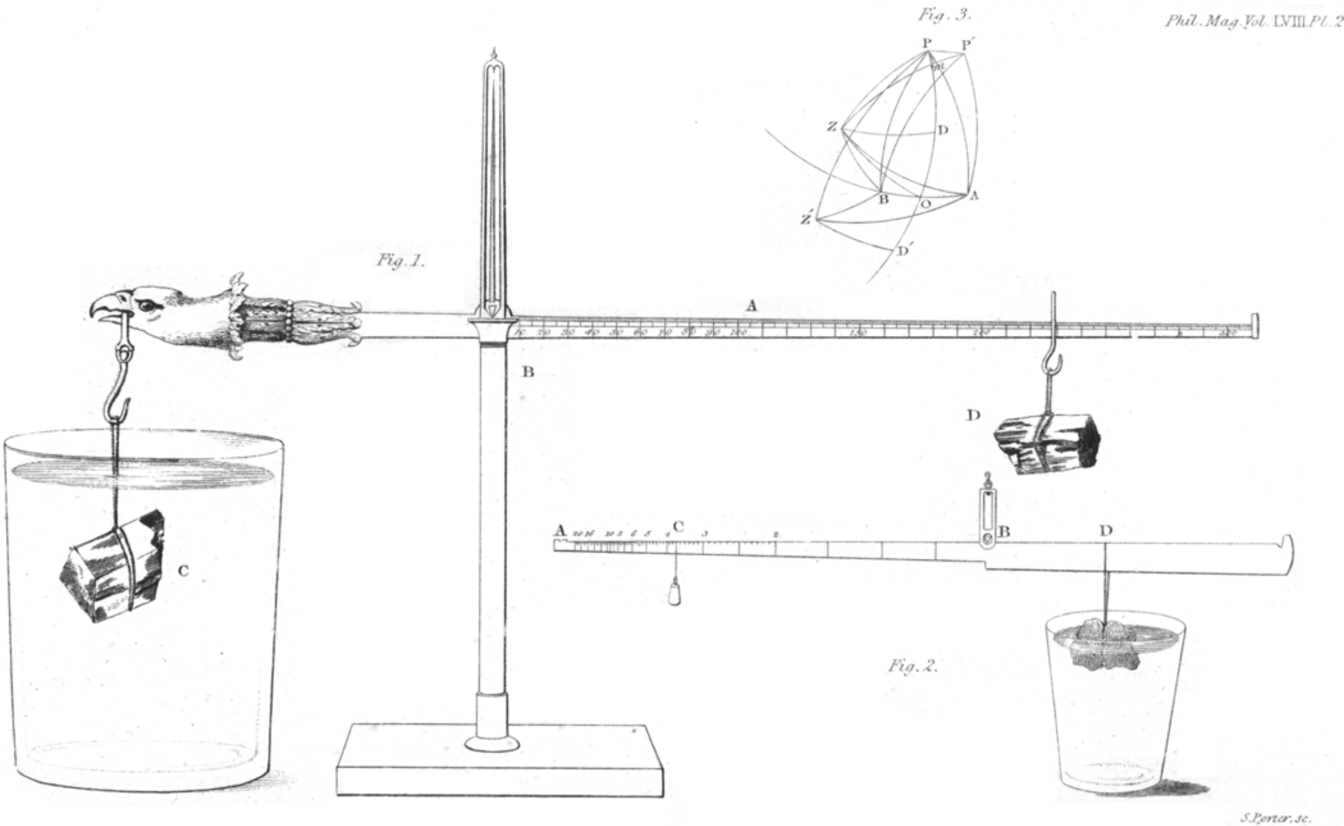
Fig. 3.
P
P′
m
Z
D
B
O
A
Z′
D′
Fig. 1.
A
B
C
D
Fig. 2.
A
C
B
D
S. Porter, sc.

Now let P be the elevated pole; PA and PB the horary circles passing through the sun's centre at the two observations; then, if each of the arcs PA and PB be equal to the polar distance of the sun, or to the complement of his declination at the middle of the elapsed time, A and B will represent the two apparent places of the sun. Conceive two small circles described upon the surface of the sphere about the poles A and B, and at distances from them respectively equal to the complements of the observed altitudes; these circles will intersect in two points Z and Z′ situated at equal distances on opposite sides of the great circle passing through A and B, and, generally speaking, either of the points Z or Z′ may represent the place of observation. The problem therefore admits of two solutions, the latitude sought being the complement of either of the polar distances ZP or Z′P.

Draw the great circle PDOD′ to bisect the vertical angle of the isosceles triangle APB; which circle will therefore intersect the base AB at right angles and will bisect it. Draw the arcs ZD and Z′D perpendicular to the circle PO; these arcs will be equal, because Z and Z′ are similarly situated with regard to both the circles AB and PO. Then the angle APB is known, for it is equal to the elapsed time converted into degrees at the rate of 15° to 1^{h}; wherefore in the right-angled triangle APO, the hypothenuse AP and the angle APO, equal to half APB, being known, the sides AO and OP may be found by the rules of spherical trigonometry. In the triangle AZB, the two sides ZB and ZA, being the complements of the observed altitudes, are known; and as the arc AO, half AB, has been found, we may compute the arcs ZD and DO by the premised lemma. Now PD is the difference, and PD′ the sum, of the arcs PO and OD; and hence the two sides about the right angle are known in each of the triangles ZPD, Z′PD′: wherefore we may find the polar distances ZP and Z′P, and likewise the angles ZPO and Z′PO, which are the horary angles at the middle time, and the problem will be completely solved.

Although the problem is ambiguous in theory, yet, in most cases, it becomes determinate in practice. In the first place there is no ambiguity when the arc Z′P is equal to, or greater than, 90°: for the distance ZP between the place of observation and the elevated pole is always less than a quadrant. In order to find a criterion for determining this point without actually computing both latitudes, it is to be observed that the angle contained between the circles ZO and OP is always less than a right angle; and, because in right-angled spherical triangles the sides are of the same affection with the angles opposite to them, it follows that the arc ZD will be less than 90°. Wherefore, Z′D being less than 90°, the polar distance Z′P will be greater than, or equal to, a quadrant,

 according

according as the known arc PD′, or PO + OD, is greater than, or equal to, a quadrant; in all which cases there is only one solution, by means of the triangle ZPD having the side PD equal to the difference of PO and OD. But when the arc PD′, or DO + OP, is less than 90°, the same pole will be elevated above the horizons of both the zeniths, and recourse must be had to other considerations to distinguish the true solution from the false one. Now, in this case, the zenith Z′ will be always between the great circle AOB and the equator, having a latitude less than the complement of the arc PO: wherefore, if it be known that the latitude of the place of observation is greater than the complement of PO, the ambiguity will be removed, and the true solution will be obtained by means of the triangle ZPD as in the former cases. On the other hand, if the latitude of the ship be less than the complement of PO, both latitudes must be computed; if they be on different sides of the complement of PO, the case will be determined; but if they be both less than that arc, the solution will remain ambiguous unless the latitude by account be known so nearly as to enable the calculator to make a choice. That both the latitudes may be less than the complement of PO, which is the greatest distance between the great circle passing through A and B and the equator, will be obvious if it be considered that the two zeniths may be as near the great circle AB as we please, and may even coincide in one point in its circumference. This ambiguous case can happen but rarely; and when it does occur, the problem will have no pretension to much precision; because the difference between the arcs ZB and ZA, will be so nearly equal to the arc AB, that very small errors in the observed altitudes will occasion a great variation in the position of the points Z and Z′. By means of these observations the ambiguity of the solution is mostly, but not entirely, taken away.

I shall now reduce the foregoing solution into algebraic formulæ of calculation, which will be shorter than giving a rule in words at length. Let h and h' denote the two altitudes, the letter without the accent standing for the greater; D the sun's declination at the mean time between the two observations; and t the angle found by converting half the elapsed time into degrees at the rate of 15° to 1^h: these are the data of the problem. Put also b for half the base, and p for the perpendicular, of the isosceles triangle APB; y for the arc ZD; x for the arc DO; and further, for the sake of abridging, let

$$A = \frac{\sin h + \sin h'}{2},$$

$$B = \frac{\sin h - \sin h'}{2}.$$

Then,

Then, if λ be the latitude, and S the horary angle of the middle time, which are the things sought, we shall obtain the following formulæ, by means of the premised lemma and the rules for solving right-angled spherical triangles.

1. $\text{Sin } b = \cos D \sin t$,
2. $\text{Cos } p = \frac{\sin D}{\cos b}$,
3. $\text{Sin } y = \frac{B}{\sin b}$,
4. $\text{Cos } x = \frac{A}{\cos y \cos b}$,
5. $\text{Sin } \lambda = \cos y \cos (p \mp x)$;
6. $\text{Sin } S = \frac{\sin y}{\cos \lambda}$.

For the sake of illustration, I shall now subjoin some examples; and I have purposely taken them from Dr. Brinkley's Addition to the Nautical Almanac 1822, in order that the two modes of calculation may be more easily compared.

Example I.

Alt. 21° 26′ A.M. } interval, 3^h { ⊙'s decl. 1° N.
Alt. 60° 56′ A.M. } $t = 22° 30′$

Sin h	= 87406	(1)
Sin h'	= 36542	(2)
2A,	123948	
2B,	50864	
A,	61974	
B,	25432	

Cos D,	9·99993	(3)
Sin t,	9·58284	(4)
Sin b,	9·58277	(5)
	$b = 22° 29'·8$	

Cos b,	9·96563	(6)
A . C	10·03437	
Sin D,	8·24186	(7)
Cos p,	8·27623	(8)
	$p = 88° 55'$	

A.C Sin b,	10·41723	
Log B,	9·40538	(9)
Sin y,	9·82261	(10)
	$y = 41° 39'·4$	

Cos b,	9·96563	
Cos y,	9·87340	(11)
	9·83903	
A.C.	10·16097	
Log A,	9·79221	(12)
Cos x;	9·95318	(13)

Cos y,	9·87340	
Cos $(p-x)$,	9·66021	(14)
Sin λ,	9·53361	(15)

$x = 26° 7'·8$
$p-x = 62\ 47\ ·2$

$\lambda = 19° 58'·7$, latitude.

Sin

Sin y, 9·82261
Sec^t λ, 10·02696 (16)
Sin S 9·84957 (17)

S = 45° 0′·7
t = 22 30

S + t, 67° 30·7 } Horary angles at the two
S − t, 22 30·7 } observations.

Here there is no ambiguity, since $p + x$ is greater than 90°.

The exact latitude is 19° 58′ 45″, although the example may have been originally framed by taking it equal to 20°. This is Dr. Brinkley's 2d example (p. 10), who brings out 19° 59′ by one operation of Douwes's rules and the correction by his own method. By the process here followed, to find the latitude requires taking out fifteen numbers from the Tables. Now one operation of Douwes's rules requires taking out twelve numbers, and the correction must double this labour: perhaps it does more, if we consider the length of the calculation, and the embarrassment of having to use different formulæ. Delambre's method requires nineteen different logarithms, besides employing additions and subtractions of the arcs not wanted here.

Example II.

Alt. 76° 6′ A.M. } interval 6h 20′ { ☉'s decl. 20° N.
Alt. 8° 3′ P.M. } t = 47° 30′ { Lat. by account 9° N.

Sin h = 97072
Sin h' = 14004
2 A, 111076
2 B, 83068
A, 55538
B, 41534

Cos D, 9·97299
Sin t, 9·86763
Sin b, 9·84062
b = 43° 51′·2

Cos b, 9·85800
A.C. 10·14200
Sin D, 9·53405
Cos p, 9·67605
p = 61° 41′·2

A.C. sin b, 10·15938
Log B, 9·61840
Sin y, 9·77778
y = 36° 50′

Cos y, 9·90330
Cos b, 9·85800
9·76130
A.C. 10·23870
Log A, 9·74459
Cos x, 9·98329
x = 15° 47′·5

Cos y, 9·90330
Cos $p+x$, 9·33608
Sin λ, 9·23938
λ = 9° 59′·5

$p+x$ = 77 28′·7

Sin

Sin y,	9·77778
Sec^t. λ,	10·00664
Sin S,	9·78442

S = 37° 30′
t, 47 30

S + t, 85°
t − S, 10° } Horary angles.

This is Dr. Brinkley's 1st example, p. 9. The exact latitude is 10°, and he brings out 10° 1′ by the same process as in the last example. This instance admits of two solutions, the arc $p + x$ being less than 90°: but the one near the equator is taken, because the latitude by account is set down 9°. The ambiguity will be removed if the other latitude be computed by the formula, $\cos\lambda = \cos y \cos(p - x)$; it comes out 33° 51′.

Example III.

Alt. 70° 1′, 35 21 } interval 2^h 20′, t = 17° 30′ { ⊙'s decl. 5° 30′.

Sin h	= 93979
Sin h'	= 57857
2 A,	151836
2 B,	36122
A,	75918
B,	18061

Cos D,	9·99800	Cos b =	9·97962
Sin t,	9·47814	A.C.	10·02038
Sin b,	9·47614	Sin D,	8·98157
	b = 17° 25′	Cos p,	9·00195
			p = 84° 14′·1
A.C. sin b,	10·52386,	Cos b,	9·97962
Log B,	9·25674	Cos y,	9·90170
Sin y,	9·78060		9·88132
	y = 37° 6′·8	A.C.	10·11868
		Log A,	9·88034
Cos y,	9·90170	Cos x,	9·99902
Cos $(p-x)$,	9·22279		x = 3° 51′
Sin λ,	9·12449	$p-x$, =	80 23·1
	λ = 7° 39′·2		

Sin y,	9·78060
Sec^t λ,	10·00388
Sin S,	9·78448

S = 37° 30′
t, 17 30

S + t, 55°
S − t, 20 } Horary angles.

This

This is Dr. Brinkley's third example (pp. 11 and 12). It is an unfavourable instance for his rules, requiring several computations and corrections to arrive at a right result. It admits of two solutions but without ambiguity, if the latitude by account be sufficient to ascertain that the true latitude is less than $5^\circ 46'$, the complement of p. The other latitude is $1^\circ 31'\cdot 6$.

These examples will be sufficient for showing the method of calculation. I proceed now to consider the correction required for the sun's change of declination in the interval between the observations. The true place of the pole will now be at P′, without the great circle DO which bisects the arc AB, because the polar distances P′A and P′B are unequal. Draw P′P perpendicular to that great circle, and complete the isosceles triangle APB. The arcs AP and PB make equal angles with the circle P′P; and hence in the small change of place from P to P′, one of the two arcs AP′ and BP′ will increase just as much as the other decreases; and each of the arcs AP and PB will be equal to half the sum of the polar distances P′A and P′B. We shall therefore obtain the arc ZP by the method already explained; and, having drawn Pm perpendicular to ZP′, the correction we are seeking is $mP' = PP' \times \sin P'Pm = PP' \times \sin ZPO = PP' \times \sin S$. Also, by the lemma,

$$PP' = \frac{\cos P'A - \cos P'B}{2 \sin AO}.$$

Now, d being the declination at B the greater altitude, and D the mean declination as before, we have

$$P'A = PA - (D - d),$$
$$P'B = PA + (D - d);$$

And hence, $PP' = (D - d) \times \frac{\sin PA}{\sin AO} = \frac{D - d}{\sin t}$.

Wherefore,

$$Pm = (D - d) \times \frac{\sin S}{\sin t}.$$

The corrected latitude will therefore be

$$\lambda - (D - d) \times \frac{\sin S}{\sin t}:$$

or, independently of S; because $\sin S = \frac{\sin y}{\cos \lambda}$;

$$\lambda - (D - d) \times \frac{\sin y}{\sin t \cos \lambda}.$$

Again the arcs P′A and P′B may be considered as making equal angles with PP′: consequently the horary circle at the middle time, which bisects the angle AP′B, will be perpendicular to PP′. Hence the true horary angle of the middle time is equal to the complement of ZP′P. But from the triangle ZPP′, we get Sin

$$\text{Sin } ZP'P = \sin ZPP' \times \frac{\sin ZP}{\sin ZP'} = \cos S \times \frac{\cos \lambda}{\cos(\lambda - \delta\lambda)},$$

putting $\delta\lambda$ for mP', the variation of λ: wherefore, if $S + \delta S$ be the true horary angle of the middle time, we shall get

$$\text{Cos}(S + \delta S) = \cos S \times \frac{\cos \lambda}{\cos(\lambda - \delta\lambda)} = \frac{\cos S}{1 + \delta\lambda \tan \lambda}:$$

and hence

$$\delta S = \delta\lambda \times \frac{\cos S \tan \lambda}{\sin S} = (D - d) \times \frac{\cos S \tan \lambda}{\sin t}.$$

The corrected horary angle is therefore

$$S + (D - d) \times \frac{\cos S \tan \lambda}{\sin t}.$$

By means of these easy formulæ the change of the sun's declination may be allowed for, when this is thought necessary, without hurting the uniformity of the general calculation.

As an example, I shall take the instance in the Quarterly Journal, No. 22, p. 372.

Example IV.

1st Alt. 42° 14′·1 } interval 3^h { ☉'s declination 8° 15′
2d Alt. 16 5′·8 } $t = 22°\ 30'$ { change in 3^h, +3

$d = 8°\ 15'$
$D = 8°\ 16'\cdot5$

Sin h = 67217
Sin h' = 27726

2 A, 94943
2 B, 39491
A, 47471·5
B, 19745·5

Cos D,	9·99545	Cos b,	9·96639
Sin t,	9·58284	A.C.	10·03361
Sin b,	9·57829	Sin D,	9·15813
	$b = 22°\ 15'\cdot2$	Cos. p,	9·19174
A.C. sin b,	10·42171		$p = 81°\ 3'\cdot2$
Log. B,	9·29547	Cos y,	9·93112
Sin y,	9·71718	Cos b,	9·96639
	$y = 31°\ 25'\cdot6$		9·89751
		A.C.	10·10249
Cos. y,	9·93112	Log A,	9·67643
Cos $p-x$,	9·94594	Cos x,	9·77892
Sin λ,	9·87706		$x = 53°\ 3'\cdot2$
	$\lambda = 48°\ 53'\cdot4$		$p - x = 28\ \ 0$
Sin y,	9·71718		
Sec^t λ,	10·18210		
Sin S,	9·89928		
	S = 52° 28′		

In calculating the corrections of λ and S three places of the logarithms are sufficient.

$$D - d = 1'{\cdot}5$$

Log. 1'·5,	0·176	Log. 1'·5	0·176
Sin S,	9·899	Cos S,	9·785
A.C. sin t,	10·417	Tan λ,	10·059
Log 3·1,	0·492	A.C. sin t,	10·417
		Log 2·7	0·437

λ = 48° 53'·3	S = 52° 28'
− 3'·1	+ 2'·7
48 50'·2	52 30'·7
true latitude.	true hor. angle of M. T.

The method that has been explained requires only the easy lemma for computing the arcs ZD and DO, and the rules for solving right-angled spherical triangles; and it is an advantage that every step is the calculation of some part of the figure, by which circumstance the memory is assisted. The process here followed is also preferable to the other methods in leading to the determination of the problem, or in pointing out which of the two possible solutions is the true one, when this can be done. In the extensive Nautical Tables published by the late Mr. Mendoza, there is one for assisting the direct solution of this problem. It contains the base, and likewise the angle at the base, of the isosceles triangle A P B formed by the two circles of declination. A similar table that should contain the perpendicular P O of the same triangle, and likewise half the base A O, or rather the sine and co-sine of A O, would render the preceding method by far the shortest of any hitherto proposed. But the use of such tables is not free from objection, and ought not to be adopted unless a great advantage is gained.

August 6, 1821. J. Ivory.

XVII. *On the aëriform Compounds of Charcoal and Hydrogen; with an Account of some additional Experiments on the Gases from Oil and from Coal.* By Wm. Henry, *M.D. F.R.S.*

[From the Transactions of the Royal Society.]

The experiments on the aëriform compounds of charcoal and hydrogen, described in the following pages, are supplementary to a Memoir on the same class of bodies, which the Royal Society did me the honour to insert in their Transactions for 1808*, as well as to other papers on the same subject, which have been published in Mr. Nicholson's Journal, and in the Memoirs of the Manchester Society. Of these essays, I beg leave to offer a very

* See Phil. Mag. vol. xxxii. p. 277.

brief