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THE CHANGES IN ABERRATIONS WHEN THE OBJECT AND STOP ARE MOVED

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ABSTRACT. If the aberrations of any centered optical system are known both for an object which intersects all rays transmitted by the system and also for the centre of the effective stop, the position in the image space of the emergent portion of a given incident ray is known, and the aberrations in the image of any other object for any stop position can be expressed in terms of those for the first object. The investigation aims to express the relations in the second case in terms of those present in the first when the objects are planes normal to the axis of symmetry whatever the order of the aberration may be.

The investigation is based upon a convenient potential function, and the Eikonal, with the principal foci as reference points, is chosen as the standard function. First a distinction is drawn between aberrational and non-aberrational terms, and the discussion leads to the general form taken by the sine-condition in the outer parts of the field of view; it also becomes evident that there is only one stop aberration of each order to be considered, and that this is of spherical aberration form. The transformation is achieved by a change of variables, and to secure simple forms the aberrations of any order are arranged in series, each series forming a separate group as regards transformations. The rays in a plane containing the axis belong entirely to one series, which is called the zero series; in this series the coefficients in general must have finite values depending on the position of the object and of the stop if the aberrations are to be removed. In all the other series the coefficients should be zero for freedom from the aberrations, the conditions thus being independent of the positions of the stop and object. As the number of the series increases the expressions for the calculation of the coefficients become more simple. These facts are illustrated by the conditions for first aberrations, all of which are members of series 0 except the Petzval sum which belongs to series 1.

The formulae obtained enable the aberrations to be expressed in terms of the coefficients of the standard Eikonal or *vice versa*, or alternatively the aberrations for any object and stop positions to be given in terms of those for standard positions. For use in the latter case certain advantages are secured by choosing for the stop the position where the magnification is $+1$ and for the object the surface for which the magnification is -1 .

As a very simple application the principles underlying the use of the formulae to obtain with the minimum of labour the effects of small variations in the constructional data of the system are illustrated.

As the paths of rays are fully known when their intersections with two planes are given, it is possible with any optical system to express the aberrations present for any assigned positions of the image and of the stop in terms of those present when these occupy any other positions. The equations representing these transformations have a number of important applications. To mention only one, aberrations for

any positions may be evaluated in terms of those of any other positions which are regarded as standards, the latter usually being chosen so as to give symmetrical forms or alternatively so that a number of the aberrations will necessarily be zero or at least very small, thus reducing the number of terms which require independent evaluation to a minimum. When the selected positions for standard purposes correspond to the magnifications $+1$ and -1 the constants may be used, with at most a change of sign in certain of them, for the system in its direct and in its reversed form. This reference system is found to be much the most convenient for thin and approximately thin lenses, and assumes particular importance in the theory of close lenses, from which almost all real systems are constructed.

Although the transformations to be investigated can be derived without difficulty by direct geometrical reasoning, there are many advantages in basing the enquiry on the properties of one of the potential functions which have been introduced into the theory of optical instruments. One of the chief advantages of this course is that any assumptions which violate the laws of light propagation are automatically avoided, and at the same time all aberrations are displayed in the lowest terms to which they can be reduced.

As the Eikonal has been extensively employed by the author as the foundation of a symbolic system for aberrations which offers the fullest symmetry, this function will be employed in the present paper. The form of the relations found is independent of the choice of function, though the precise interpretation of the aberrations, when finite, varies slightly with the different functions available.

Let then \mathcal{E} denote the length of the optical path along a particular ray between the feet of the perpendiculars to the incident and emergent portions of the ray from the principal foci for the object and image spaces respectively. The direction cosines of the incident ray will be represented by L, M, N and those of the emergent ray by L', M', N' , the axis of x in either case being coincident with the axis of symmetry. It is convenient to assume that the refractive index of both external media is unity. This involves no loss of generality for the form in which these refractive indices enter into the expression for \mathcal{E} is known and they may if desired be inserted into the final result. Alternatively it is possible to define the aberration coefficients in a form into which these indices do not enter. The Eikonal \mathcal{E} may be regarded as a function of the four direction cosines M, N, M', N' , and the intersections of the ray with the two paraxial focal planes are then given by the equations

$$\left. \begin{aligned} y &= \frac{\partial \mathcal{E}}{\partial M}, & z &= \frac{\partial \mathcal{E}}{\partial N} \\ y' &= -\frac{\partial \mathcal{E}}{\partial M'}, & z' &= -\frac{\partial \mathcal{E}}{\partial N'} \end{aligned} \right\} \dots\dots(1).$$

If the points of intersection with planes normal to the axis distant x and x' from the principal foci are desired, both x and x' being positive when following the direction in which light passes through the system from the object space to the image space, it is only necessary to replace \mathcal{E} in (1) by \mathcal{E}' where

$$\mathcal{E}' = \mathcal{E} - xL + x'L' \dots\dots(2),$$

L and L' being of course regarded as functions of M and N and of M' and N' respectively. In particular if the intersections of the ray with the planes through the conjugate points for magnification G are desired

$$\mathcal{E}_{\kappa} = \mathcal{E}\kappa - \frac{L}{G} - L'G \quad \dots\dots(3),$$

where κ is the power of the system for paraxial rays. If the instrument is a telescope it will be noted that no investigation can be based upon the direction cosines as they fail to identify a ray uniquely, so that no additional limitation is involved in assuming that κ is finite.

The terms representing aberrations in the image.

\mathcal{E}_G will consist of two parts, one of which expresses the variation in the length of the path between the perpendiculars from the axial points of the conjugate planes due to the obliquity of the ray, while the other indicates the presence of path variations which are due to aberrations in the image. It is thus necessary to consider which terms belong to the one part and which to the other in order to have a clear definition of aberrations.

From symmetry \mathcal{E} is a function of (1), (2), and (3) where

$$(1) = M^2 + N^2, \quad (2) = MM' + NN', \quad (3) = M'^2 + N'^2 \quad \dots\dots(4),$$

and as $L = \sqrt{1 - (1)}$, $L' = \sqrt{1 - (3)}$, \mathcal{E}_{κ} is a function of the same variables. Introduce new variables I, II, III defined by

$$\left. \begin{aligned} \text{I } (S - G)^2 &= (1) - 2G(2) + G^2(3) \\ \text{II } (S - G)^2 &= (1) - (S + G)(2) + SG(3) \\ \text{III } (S - G)^2 &= (1) - 2S(2) + S^2(3) \end{aligned} \right\} \quad \dots\dots(5),$$

so that

$$\left. \begin{aligned} (1) &= S^2\text{I} - 2SG\text{II} + G^2\text{III} \\ (2) &= S\text{I} - (S + G)\text{II} + G\text{III} \\ (3) &= \text{I} - 2\text{II} + \text{III} \end{aligned} \right\} \quad \dots\dots(6),$$

S being a number different from G .

The intersections of the ray with the conjugate planes are then given by

$$\left. \begin{aligned} y(S - G)^2 &= (M - GM') (2\mathcal{E}_I + \mathcal{E}_{II}) + (M - SM') (\mathcal{E}_{II} + 2\mathcal{E}_{III}) \\ y'(S - G)^2 &= (M - GM') (2G\mathcal{E}_I + S\mathcal{E}_{II}) + (M - SM') (G\mathcal{E}_{II} + 2S\mathcal{E}_{III}) \end{aligned} \right\} G(7),$$

where differentiation of \mathcal{E} with regard to I, II, III is denoted by the addition of the appropriate suffix, and the G preceding the number of the equations means that the suffix G is to be understood to apply to all the \mathcal{E} 's in the equations. Two similar equations give z and z' if N and N' are substituted for M and M' . From these equations

$$\left. \begin{aligned} (y' - Gy)(S - G) &= (M - GM') \mathcal{E}_{II} + 2(M - SM') \mathcal{E}_{III} \\ \text{and similarly} \quad (z' - Gz)(S - G) &= (N - GN') \mathcal{E}_{II} + 2(N - SN') \mathcal{E}_{III} \end{aligned} \right\} \quad \dots\dots G(8).$$

When aberrations are absent $y' - Gy = z' - Gz = 0$ for all rays, and as the relation

$$\frac{M - GM'}{N - GN'} = \frac{M - SM'}{N - SN'}$$

is only true for rays which are contained in a plane through the axis, it follows that in the absence of aberrations

$$\mathcal{E}_{II} = \mathcal{E}_{III} = 0 \quad \dots G(9),$$

or since the three variables I, II, and III are quite independent, \mathcal{E}_G must be a function of I only. In other words, any term which involves II or III corresponds to an aberration, and when \mathcal{E}_G is expressed as an infinite series the number of independent aberrations in the general case is the number of terms in which II or III are present. The coefficients of these terms, if desired with numerical coefficients also, will measure the magnitudes of the aberrations and may for convenience be referred to as aberrations themselves.

The extended sine law.

It may be noted in passing that since in the absence of aberrations \mathcal{E}_G must be a function of I only, all the rays which start from a given point of the object plane must be refracted so that $M - GM'$ and $N - GN'$ have constant values expressible in terms of the coordinates y and z of this point, as is clearly seen from (7) on putting $\mathcal{E}_{II} = \mathcal{E}_{III} = 0$. This is the form the sine condition takes for the outer parts of the field of view, and is a particular case of a very general law, which the author has called the cosine law, from which the behaviour of rays in the neighbourhood of any given ray may be determined whatever the system may be and whether or not aberrations are present.

Aberrations of the stop.

When there are no aberrations in the image \mathcal{E}_G is an arbitrary function of I. If this function is expressed in a series of ascending powers of I the first power depends only on the focal length of the system, and the other terms may be given such values that any ray passing through the axial point of the object plane corresponding to the magnification S is refracted through the axial point of the corresponding image plane. Departures from these values will evidently correspond to the presence of spherical aberration of various orders in the image of this plane. The real plane corresponding to the magnification S is assumed to be the plane of the aperture stop of the system, and it is necessary to add this one aberration to those previously enumerated to obtain a complete system. It will be noted that the presence of aberrations in the outer parts of the image of the stop is inevitable, as at once follows from the obvious result that if two pairs of planes have a one to one correspondence an infinite number of pairs of planes can be found satisfying the same relation.

To find the form which \mathcal{E}_1 must have for there to be no aberration at the centre

of the stop, the planes corresponding to the magnification S should be taken for reference. Thus

$$\begin{aligned}\mathcal{E}_s \kappa &= \mathcal{E} \kappa - \frac{L}{S} - L'S \\ &= \mathcal{E}_G \kappa + (S - G) \left(\frac{L}{SG} - L' \right)\end{aligned}$$

and the coordinates (η, ζ) , (η', ζ') of intersection of the ray with these planes, when \mathcal{E}_G is a function of I only, satisfy

$$\left. \begin{aligned}\eta (S - G)^2 &= 2 (M - GM') \mathcal{E}_1 - \frac{(S - G)^3}{SG\kappa} \cdot \frac{M}{L} \\ \eta' (S - G)^2 &= 2G (M - GM') \mathcal{E}_1 - \frac{(S - G)^3}{\kappa} \cdot \frac{M'}{L'}\end{aligned} \right\} \dots\dots G(10),$$

with similar equations for ζ and ζ' . If $\eta = 0$ and $\eta' = 0$ are to occur simultaneously the rays concerned must satisfy

$$\frac{M}{L} = \frac{2SG(M - GM')}{(S - G)^3} \mathcal{E}_1 \kappa = S \frac{M'}{L'} \dots\dots G(11),$$

the equality of the first and third quantities expressing the tangent law. Eliminating M and M' from these equations gives

$$(SL - GL') 2G \mathcal{E}_1 \kappa = (S - G)^3$$

and since \mathcal{E}_1 is a function of I only it is only necessary to find the form $SL - GL'$ takes when it is a function of I alone.

If $S^2 = 1$ the solution takes particularly simple forms for evidently

$$M - SM' = N - SN' = L - L' = 0,$$

and

$$\mathcal{E}_1 = \frac{(S - G)^2}{2G\kappa} (1 - I)^{-\frac{1}{2}},$$

or

$$\mathcal{E}_1 = - \frac{(S - G)^2}{G\kappa} (1 - I)^{\frac{1}{2}}.$$

When $S^2 \neq 1$ the equation for \mathcal{E}_1 is a biquadratic which may be solved by successive approximation. The solution in series form may be written

$$\begin{aligned}\frac{G\kappa}{(S - G)^2} \mathcal{E}_G &= \frac{1}{2}I + \frac{1}{8}e_1 I^2 - \frac{1}{16}(e_2 - 2e_1^2) I^3 + \frac{1}{128}(5e_3 - 24e_2 e_1 + 24e_1^3) I^4 \\ &\quad - \frac{1}{256}(7e_4 - 40e_3 e_1 - 18e_2^2 + 132e_2 e_1^2 - 88e_1^4) I^5 \\ &\quad + \frac{1}{1024}(21e_5 - 140e_4 e_1 - 120e_3 e_2 + 520e_3 e_1^2 \\ &\quad + 468e_2^2 e_1 - 1456e_2 e_1^3 + 728e_1^5) I^6 - \dots \dots\dots (12),\end{aligned}$$

where $e_n = \frac{S^{2n+1} - G}{S - G}$. No constant term need be inserted as its value is quite immaterial. This is the value \mathcal{E}_G must have when the system is to be regarded as fully corrected, and if this expression is subtracted in the general case from \mathcal{E}_G , all the terms which remain express aberrations. In the transformation formulae

which are to be found the inclusion of such terms as those given in (12) would be inconvenient, and consequently the formulae relate directly to the terms in the expansion of \mathcal{E}_G without the subtraction of this part. It has then simply to be remembered that the coefficient of the term in I only of any order must have a correction applied to it if it is required to determine the aberration at the centre of the stop.

The values of the constants in \mathcal{E} for which freedom from aberrations is secured for the two magnifications G, S obviously follows by substituting from (12) in

$$\mathcal{E} = \mathcal{E}_G + \frac{1}{\kappa} \left(\frac{\sqrt{1 - (1)}}{G} + \sqrt{1 - (3)} G \right),$$

and writing for I its value

$$\frac{(1) - 2G(2) + G^2(3)}{(S - G)^2}.$$

The denominators introduced by this substitution are most simply absorbed into the e 's, changing e_n say to e'_n where

$$e'_n = \frac{S^{2n+1} - G}{(S - G)^{2n+1}}.$$

Thus the terms in \mathcal{E} of the first three orders, when aberrations are absent are

$$\mathcal{E}\kappa = - (2) - \frac{1}{8} \left[\frac{(1)^2}{G} + (3)^2 G - \frac{e'_1}{G} \{(1) - 2G(2) + G^2(3)\}^2 \right] - \frac{1}{16} \left[\frac{(1)^3}{G} + (3)^3 G + \frac{e'_2 - 2e_1'^2}{G} \{(1) - 2G(2) + G^2(3)\}^3 \right] \dots\dots(13).$$

Such a formula is in effect a particular case of the transformations it is desired to establish, and shows how the aberrations for G and S may be eliminated by giving the correct values to the various terms in the standard function \mathcal{E} .

The general transformation expressions.

For the general case it is desired to exhibit the coefficients for G', S' in terms of those for G and S . In place of I, II, III the aberrations for G' and S' are defined in terms of the variables I', II', III' where by analogy with (5)

$$\left. \begin{aligned} \text{I}' (S' - G')^2 &= (1) - 2G'(2) + G'^2(3) \\ \text{II}' (S' - G')^2 &= (1) - (S' + G')(2) + S'G'(3) \\ \text{III}' (S' - G')^2 &= (1) - 2S'(2) + S'^2(3) \end{aligned} \right\} \dots\dots(14),$$

and thus by the solution corresponding to (6)

$$\begin{aligned} \text{I} (S - G)^2 &= \text{I}' (S' - G')^2 - 2\text{II}' (S' - G) (G' - G) + \text{III}' (G' - G)^2 \\ \text{II} (S - G)^2 &= \text{I}' (S' - G) (S' - S) \\ &\quad - \text{II}' \{(S' - G) (G' - S) + (G' - G) (S' - S)\} \\ &\quad + \text{III}' (G' - G) (G' - S), \\ \text{III} (S - G)^2 &= \text{I}' (S' - S)^2 - 2\text{II}' (S' - S) (G' - S) + \text{III}' (G' - S)^2. \end{aligned}$$

These relations may be expressed in a more concise way by a slight extension of the well-known notation in which crossed brackets are used, the factors after the

intersecting brackets being expanded before the coefficients in the preceding bracket are applied. For instance

$$(a, b, c \int x, x') (y, y')$$

is used to indicate

$$axy + b(x'y + xy') + cx'y'.$$

In this notation the equations become

$$\left. \begin{aligned} \text{I} &= (\text{I}', \text{II}', \text{III}') \int \text{I} + s, -g)^2 \\ \text{II} &= (\text{I}', \text{II}', \text{III}') \int \text{I} + s, -g) (s, \text{I} - g) \\ \text{III} &= (\text{I}', \text{II}', \text{III}') \int s, \text{I} - g)^2 \end{aligned} \right\} \dots\dots(15),$$

where

$$s = \frac{S' - S}{S - G}, \quad g = \frac{G' - G}{S - G} \dots\dots(16),$$

so that s and g represent the displacements of the stop image and of the object image respectively as a fraction of the original separation between these images.

The values of I, II, III given by (15) if substituted in the expression for \mathcal{E}_G in the equation

$$\mathcal{E}_{G'} = \mathcal{E}_G + \frac{G' - G}{\kappa} \left(\frac{L}{GG'} - L' \right) \dots\dots(17)$$

reduce this to an identity when L and L' are also expressed in terms of $\text{I}', \text{II}', \text{III}'$. To obtain the desired relations it is necessary to assume that \mathcal{E}_G and $\mathcal{E}_{G'}$ are expanded as infinite series, and the relations follow on regarding (17) as an identity. The most natural form to assume for the expansion is perhaps

$$\frac{1}{\kappa} \sum c_{p+q+r} \delta_{p,q,r} \frac{(p+q+r)!}{p!q!r!} \text{I}^p (-2\text{II})^q \text{III}^r \dots\dots(18),$$

where δ is the measure of the aberration described by the three integers p, q, r and c_{p+q+r} is a numerical coefficient which it may be found convenient to apply to all aberrations of the same order. The order of the aberration is $n - 1$ where $n = p + q + r$, and the number of coefficients in the terms of order n , and therefore the number of independent aberrations of order $n - 1$, is $\frac{1}{2}(n+1)(n+2)$. Of these one relates to the stop only, viz. the term with $p = n, q = r = 0$, and those which relate to the image proper thus number $\frac{1}{2}n(n+3)$. The latter expression gives the well-known value 5 for the first order aberrations, 9 for the second order, 14 for the third, and so on. The transformation formulae which are to be derived, however, relate of necessity to the full number of aberrations, though the aberration at the centre of the stop may not appear in particular cases.

If an expression such as (18) were taken as the basis of the transformation relations the results would appear of needless complexity. To avoid this the aberrations of any order will be arranged in a number of series chosen so that all the terms which appear in the coefficient of $a_n b^n$ in the sum

$$\begin{aligned} a_0 (\text{I} - 2\text{II}b + \text{III}b^2)^n + a_1 (\text{I} - 2\text{II}b + \text{III}b^2)^{n-2} (\text{I III} - \text{II}^2) + \dots\dots \\ + a_r (\text{I} - 2\text{II}b + \text{III}b^2)^{n-2r} (\text{I III} - \text{II}^2)^r + \dots\dots \end{aligned}$$

may be grouped together and considered as an independent aberration. If the power to which $(\text{I III} - \text{II}^2)$ appears is taken as the number of the series, the number

of independent aberrations in each group is shown in the following table, from which it is clear that the rearrangement provides the correct number of independent constants to specify the total number of aberrations which in general are present.

Order of aberration	Total number of aberrations	Number of aberrations of series				
		0	I	2	3	4
1	6	5	1	0	0	0
2	10	7	3	0	0	0
3	15	9	5	1	0	0
4	21	11	7	3	0	0
5	28	13	9	5	1	0
6	36	15	11	7	3	0
$n - 1$	$\frac{1}{2} (n + 1) (n + 2)$	$2n + 1$	$2n - 3$	$2n - 7$	$2n - 11$	$2n - 15$

Since all the series with the exception of series 0 contain $I' III' - II'^2$ as a factor, and as for freedom from aberrations every term involving II or III must vanish, in a corrected lens each coefficient of every series but series 0 must vanish. Moreover, when the series are transformed by changes in S and G , the relations which exist between new and old aberrations necessarily connect only aberrations of the same series. For from (5) and (14)

$$(I' III' - II'^2) (S' - G')^2 = (1) (3) - (2)^2 = (I' III' - II'^2) (S' - G')^2 = (MN' - M'N')^2 \dots (19),$$

which shows that the power of $I' III' - II'^2$ after transformation is equal to the power of $I' III' - II'^2$ before transformation. A special consequence of great importance is that apart from the terms of series 0 the conditions for freedom from aberrations do not change with the position of the object or stop, but require the corresponding terms of \mathcal{E} to vanish. On the other hand, if the terms do not vanish the aberrations present will as a rule vary with S and G .

Since for any ray in an axial plane

$$\frac{M'}{M} = \frac{N'}{N},$$

(19) shows that it is only the terms of series 0 which affect the paths of rays in an axial plane. Thus tracing rays in this plane by the usual methods will afford no information about any of the conditions for correction which are independent of S and G . The most advantageous methods of tracing rays in an axial plane will however give all series 1 terms in addition to series 0.

These conclusions may be illustrated by the well-known case of the first order aberrations. Here five of the conditions require adjustment according to the positions of the object and stop, and the state of correction as regards these may be determined by considering rays in an axial plane. There is only one further condition to be satisfied to remove all first order defects. This is the Petzval con-

dition, which, as is well known, is the condition that rays not in an axial plane should conform to the laws which hold for rays in that plane, and which is also quite independent of the position of the stop or of the image. This example also illustrates another property of the constants of higher orders which is related to the facts that have been mentioned. Just as the Petzval condition is particularly simple in form in comparison with the conditions of the lower series, so in the higher order aberrations, as the number of the series increases, the expressions for the evaluation of the coefficients in terms of the constructional data of the system become of greater simplicity.

Assuming that the form given is justified, the terms of order n in \mathcal{C}_{ti} may be written, apart from an arbitrary common factor which need only be considered in connection with the terms arising from L and L' in the transformation expressions,

$$\begin{aligned} & (D''_0, D''_1, D''_2, \dots, D''_{2n}) \mathfrak{I} \mathfrak{I}, -2\mathfrak{II}, \mathfrak{III})^n \\ & + (\mathfrak{I} \mathfrak{III} - \mathfrak{II}^2) (S - G)^2 (D''_2, D''_3, \dots, D''_{2n-2}) \mathfrak{I}, -2\mathfrak{II}, \mathfrak{III})^{n-2} \\ & + (\mathfrak{I} \mathfrak{III} - \mathfrak{II}^2)^2 (S - G)^4 (D''_4, D''_5, \dots, D''_{2n-4}) \mathfrak{I}, -2\mathfrak{II}, \mathfrak{III})^{n-4} \\ & \dots \dots \dots \\ & + (\mathfrak{I} \mathfrak{III} - \mathfrak{II}^2)^w (S - G)^{2w} (D''_{2w}, D''_{2w+1}, \dots, D''_{2n-2w}) \mathfrak{I}, -2\mathfrak{II}, \mathfrak{III})^{n-2w} \dots \dots (20), \end{aligned}$$

where the D 's are applied in turn to the coefficients of the successive powers of b in the expansion of the proper power of $(\mathfrak{I} - 2\mathfrak{II}b + \mathfrak{III}b^2)$. It will be noted that if the δ 's are expressed in terms of the D 's, all the D 's which appear in connection with any given δ will have the same suffix.

From equations (15)

$$(\mathfrak{I}, \mathfrak{II}, \mathfrak{III}) \mathfrak{I} - b)^2 = (\mathfrak{I}', \mathfrak{II}', \mathfrak{III}') \mathfrak{I} + s - sb, -g - (\mathfrak{I} - g)b)^2,$$

and therefore from (17), (19) and (20) if $w \neq 0$

$$D'_{2w+v} = (D''_{2w}, D''_{2w+1}, \dots, D''_{2n-2w}) \mathfrak{I} + s, -s)^{2n-4w-v} (g, \mathfrak{I} - g)^v \dots \dots (21),$$

where the D with the accent on the left is the aberration coefficient for the new values of S and G , those on the right being of course the corresponding coefficients for the old values.

This result requires modification for the special case $w = 0$ owing to the presence of the terms in L and L' in equation (17). The additional terms of order n are evidently

$$- \frac{G' - G}{\kappa} \frac{(2n-2)!}{2^{2n-1} (n-1)! n!} \left\{ \frac{(S'^2 \mathfrak{I}' - 2S'G' \mathfrak{II}' + G'^2 \mathfrak{III}')^n}{GG'} - (\mathfrak{I}' - 2\mathfrak{II}' + \mathfrak{III}')^n \right\}.$$

It is convenient to adopt for the value of the constant c the numerical coefficient occurring in this term. Thus

$$c_n = - \frac{(2n-2)!}{2^{2n-1} (n-1)! n!} \dots \dots (22).$$

The quantities expressing the aberrations are also preferably pure numbers, so that the factor $1/\kappa$ which introduces the unit of length must also be excluded. The relations for the aberrations of the zero series may then be written

$$D''_v = (D''_0, D''_1, \dots, D''_{2n}) \mathfrak{I} + s, -s)^{2n-v} (g, \mathfrak{I} - g)^v + (G' - G) (S'^{2n-v} G'^{v-1} G^{-1} - \mathfrak{I}) \dots \dots (23).$$

It will be observed that the aberration relating to the stop in this notation bears the suffix o , and the successive aberrations as the suffixes increase proceed in the direction opposite to that adopted by Seidel for first order aberrations. The order adopted is obviously quite arbitrary.

The notation that has been used in this discussion requires a slight alteration to avoid ambiguity in dealing with general aberration problems. To this end the notation employed is to write $D_{n,w,v}$ for the quantity that has hitherto been denoted by D_v^w . The first suffix n gives the order of the terms, so that the aberration is of order $n - 1$; the second indicates the series to which the aberration belongs; and the third determines the particular member of that series which is required. The earlier notation may be advantageously employed when the value of n is not in doubt.

The arrangement of aberrations in series.

In obtaining relations (21) and (23) it has been assumed that (18) and (20) are equivalent expansions for C_v . Justification for this assumption beyond the equality in the number of coefficients appears desirable, and the method followed is to express both in terms of a corresponding number of arbitrary coefficients. The proof of these relations is tedious rather than difficult, and is omitted here*. The result is given by the two equations

$$\delta_{n-m+p, m-2p, p} = \sigma_{n, o, m} + p\sigma_{n, 1, m} (S - G)^2 + \dots + \frac{p!}{u!(p-u)!} \sigma_{n, u, m} (S - G)^{2u} + \dots \quad (24)$$

$$\begin{aligned} \text{and } D_{n, t, m} = & \frac{2^t n!}{t! (n-2t)!} \frac{1}{\prod_{v=1}^t (2n-2t-2v+3)} \left[\sigma_{n, t, m} \right. \\ & + \frac{(m-2t)(m-2t-1)}{2 \cdot 1! (2n-4t-1)} \sigma_{n, t+1, m} (S - G)^2 \\ & + \dots + \frac{(m-2t)! \sigma_{n, t+u, m} (S - G)^{2u}}{2^u \cdot u! (m-2t-2u)! \prod_{v=1}^u (2n-4t-2v+1)} + \dots \left. \right] \dots \dots (25), \end{aligned}$$

where the σ 's are arbitrary. It will be observed that in (25) the factor outside the bracket is common to all aberrations of the same series. In applying these formulae it is to be borne in mind that in (24) it is assumed that $m \geq n$. When this condition is not fulfilled it is obvious from symmetry that p should be replaced on the right side of the equation by $n - m + p$. Equations (24) and (25) suffice to establish the general connection between the alternative methods of specifying the aberrations. It is evident that if desired the δ 's may be expressed directly in terms of the D 's by eliminating the σ 's. This elimination is most easily effected by starting

* A copy of a demonstration of the results quoted has been filed by the Hon. Secretary, and can be consulted by arrangement with him.

from the series of highest index number. The converse relation may be written down at once from the reversal of (24)

$$(S - G)^{2t} \sigma_{n, t, m} = (\delta_{n-m, m, 0}, \delta_{n-m+1, m-2, 1}, \dots \delta_{n-m+t, m-2t, t}) \check{I}, -1)^t,$$

and enables the D 's to be evaluated when all the δ 's have arbitrary values assigned to them.

The aberrations in terms of the coefficients of \mathcal{E} .

Frequently it is of value to be able to express the coefficients in terms of the corresponding coefficients of the standard function \mathcal{E} in which the principal foci are the reference points; and conversely to express the latter coefficients in terms of the D 's. The formulae involved are easily found. For if the form assumed for \mathcal{E} is a function of (1), (2), (3) corresponding to that for \mathcal{E}_{ti} in terms of the D 's when I, II, III are the variables, the coefficients for \mathcal{E} being indicated by substituting E for D , the relation

$$(S - G)^2 (I, II, III) \check{I}, -b)^2 = (I, (2), (3) \check{I} - b, -G + Sb)^2,$$

and the converse

$$((I), (2), (3) \check{I}, -b)^2 = (I, II, III) \check{S} - b, -G + b)^2,$$

in conjunction with (19) at once yield by (3)

$$E_{2w+v}^{wv} = (D_{2w}^{wv}, D_{2w+1}^{wv}, \dots D_{2w-2v}^{wv} \check{I}, -1)^{2n-4w-r} (G, -S)^v (S - G)^{2w-2n} \dots (26),$$

$$\text{and } D_{2w+v}^{wv} = (E_{2w}^{wv}, E_{2w+1}^{wv}, \dots E_{2w-2v}^{wv} \check{S}, -1)^{2n-4w-v} (G, -1)^v \dots (27),$$

if $w \neq 0$. When $w = 0$ additional terms arise from the L and L' terms. In the earlier formula evidently only E_0^0 and E_{2n}^0 are affected, and the additional terms on the right are $1/G$ in the case of E_0^0 and G in the case of E_{2n}^0 . The modified form of (27) is

$$D_0^0 = (E_0^0, E_1^0, \dots E_{2n}^0 \check{S}, -1)^{2n-v} (G, -1)^v - G - S^{2n-v} G^{v-1} \dots (28).$$

The parallel with previous expressions can readily be extended further. For example, if in conformity with (25) the E 's are expressed by coefficients $\epsilon_{n, t, m}$ corresponding to the σ 's, the coefficients of \mathcal{E} will be of the form corresponding to (24) but with S and G absent, and the σ 's of (24) and (25) may be expressed as symmetrical functions of the ϵ 's of the same order and series.

Special application.

The formulae that have been found may be used to determine the effect of small changes in a system, such as the variation of a thickness or separation, or the alteration of a radius, the work involved being much less than if rays are traced through the modified system. The only requirement is that the aberrations of the original system should be known. Suppose, for example, it is desired to find the effect of changing the curvature of the first radius of a system by a specified amount R , which is assumed to be small. As a result of this change the image will be slightly moved by an amount proportional to R , while the position of the stop will be unaffected. Thus regarding the new system as a combination of a thin meniscus

lens in which the curvatures differ by a small amount R together with the original system, the change in spherical aberration is a known multiple (depending on G) of the aberration arising from the meniscus lens plus the change due to giving g a small value proportional to R , s remaining zero. The effect under the latter heading depends only on the original spherical aberration and the coma, equation (23) giving on putting $v = 2n$

new spherical aberration coefficient

$$\begin{aligned} &= (1 - 2ng) \times \text{old spherical aberration coefficient} \\ &\quad + 2ng \times \text{old coma coefficient} \\ &\quad + g(S - G)(G^{2n-2} - 1), \end{aligned}$$

higher powers of g than the first being neglected.

If an intermediate surface is to be changed the equations are applied to the following part of the system, the coefficients of which are known if a suitable method of calculation of the original system has been followed. Numerical results on this basis can be reached with less labour than by the system adopted by Mr Wilfred Taylor in his recent paper read before this Society*.

Many of the special cases of these formulae are of much importance to the designer. Apart from previous references by the present author in dealing with first order aberrations, the only instance where they appear to have been used is in a paper by Conrady†, who limits himself to the special case $g = 0$, and discusses only first order aberrations. Two results which are physically obvious may be deduced from (21) and (23) by inspection. The first is that if the object is not moved and is free from aberrations except for the first member of each series (D_{2ic}^{iv}), no aberration of a different type can be introduced by moving the stop. The second is the complementary case when the stop is fixed and the object moved, and only the last aberration of each series is present (D_{2i-2iv}^{iv}). The same aberration is the only one present in this case also after the object is moved, with the important exception of the aberrations of series 0, which are subject to the influence of the special additional terms involving the change in the position of the object.

* *Trans. Opt. Soc.* 23 (1921-22), 241.

† "The Five Aberrations of Lens-Systems," *Monthly Notices R.A.S.*, 80 (1919), 78.