ON THE APPLICATION OF QUATERNIONS TO THE PROBLEM OF THE INFINITESIMAL DEFORMATION OF A SURFACE

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VOLTERRA* has remarked that Weingarten's characteristic function in the deformation problem admits of a simple kinematical interpretation—viz., the normal component of the rotation which the element of the surface experiences in the deformation. This remark suggested the attempt to obtain the characteristic equation directly from the property, and thus I was led to extend the principle of moving axes dependent on two parameters† to that of moving axes dependent on three parameters.

Having obtained the equation I proceeded to apply the kinematical method to obtain the chief results in Bianchi, Kap. XI., and Darboux, Part IV., Ch. II. and III. The work occupied a considerable space, but I then saw that the application of the mere elements of quaternions would give what I wanted more directly, and add geometrical unity to the theory. This is the justification of the present paper, which does not pretend to add much to results already known, but aims rather at fuller kinematical illustration and greater simplicity of proof.

1. Consider a set of moving axes with a fixed origin whose motion is defined by the angular displacements

$$p'\,du+q'\,dv+r'\,dw, \qquad p''\,du+q''\,dv+r''\,dw, \qquad p'''\,du+q'''\,dv+r'''\,dw,$$

where the coefficients of the differential elements du, dv, dw are functions of the parameters u, v, w.

Let p, q, r respectively denote the vectors

$$p'i+p''j+p'''k, \qquad q'i+q''j+q'''k, \qquad r'i+r''j+r'''k,$$

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* The article is "Sulla deformazione delle superficie flessibile ed inestendibili," *Rendiconti della Reale Accad. dei Lincei*, Sitzung von 6 April, 1884. I have not been able to consult the article itself and only quote from Lukat's translation of Bianchi, *Vorlesungen über Differential Geometrie*, p. 289.

† Darboux, *Théorie des Surfaces*, I., p. 49, and II., p. 348.

where i, j, k are unit vectors along the axes of coordinates and therefore mutually perpendicular.

Let
$$z = z'i + z''j + z'''k,$$

where z', z'', z''' are the coordinates of a point with reference to the moving axes; and let $\bar{z}'$, $\bar{z}''$, $\bar{z}'''$ be the coordinates of the same point with reference to a set of fixed axes, and let

$$\bar{z} = \bar{z}'i + \bar{z}''j + \bar{z}'''k.$$

Let the fixed axes be so chosen that the moving ones coincide with them when
$$u = u_0, \quad v = v_0, \quad w = w_0.$$

For these values of the parameters we know that

$$\frac{\partial \bar{z}'}{\partial u} = \frac{\partial z'}{\partial u} - z''p''' + z'''p'',$$

$$\frac{\partial \bar{z}''}{\partial u} = \frac{\partial z''}{\partial u} - z'''p' + z'p''',$$

$$\frac{\partial \bar{z}'''}{\partial u} = \frac{\partial z'''}{\partial u} - z'p'' + z''p'.$$

These three equations may be replaced by the single quaternion equation

$$\frac{\partial \bar{z}}{\partial u} = \frac{\partial z}{\partial u} + Vpz = z_1, \quad \text{say.}$$

Similarly we see that
$$\frac{\partial \bar{z}}{\partial v} = \frac{\partial z}{\partial v} + Vqz = z_2,$$

$$\frac{\partial \bar{z}}{\partial w} = \frac{\partial z}{\partial w} + Vrz = z_3.$$

We also have
$$\frac{\partial^2 \bar{z}}{\partial v\, \partial u} = \frac{\partial z_1}{\partial v} + Vqz_1 = z_{21},$$

$$\frac{\partial^2 \bar{z}}{\partial u\, \partial v} = \frac{\partial z_2}{\partial u} + Vpz_2 = z_{12}.$$

Since $z_{21} = z_{12}$, we have

$$\frac{\partial z_1}{\partial v} + Vqz_1 = \frac{\partial z_2}{\partial u} + Vpz_2;$$

and therefore $\dfrac{\partial^2 z}{\partial v\,\partial u}+V\dfrac{\partial p}{\partial v}z+Vp\dfrac{\partial z}{\partial v}+Vq\dfrac{\partial z}{\partial u}+V(qVpz)$

$$=\frac{\partial^2 z}{\partial u\,\partial v}+V\frac{\partial q}{\partial u}z+Vq\frac{\partial z}{\partial u}+Vp\frac{\partial z}{\partial v}+V(pVqz).$$

It follows that $V\left(\dfrac{\partial p}{\partial v}-\dfrac{\partial q}{\partial u}\right)z+V(qVpz)-V(pVqz)=0$;

but $$V(qVpz)-V(pVqz)=V(zVpq);$$

and therefore $$V\left(\frac{\partial p}{\partial v}-\frac{\partial q}{\partial u}-Vpq\right)z=0.$$

Since z may be *any* vector we must therefore have

$$\frac{\partial p}{\partial v}-\frac{\partial q}{\partial u}=Vpq.$$

Similarly we have

$$\frac{\partial q}{\partial w}-\frac{\partial r}{v}=Vqr,\qquad \frac{\partial r}{\partial u}-\frac{\partial p}{\partial w}=Vrp.$$

The vectors p, q, r which define the motion of the axes are thus connected by the above three equations.

One set of vectors satisfying these equations are p and q zero, and r a function of w only; that gives a motion of the axes depending on w only.

Another set would be obtained by taking $r=0$, and making p and q depend on the parameters u and v only in such a way that

$$\frac{\partial p}{\partial v}-\frac{\partial q}{\partial u}=Vpq.$$

This is the motion of the axes of which Darboux makes much use in considering the properties of a surface.

The first set of vectors would be sufficient for the investigations of this paper in connection with the deformation of a known surface. I have, however, preferred to keep to the most general motion defined by the three vectorial equations, as that will allow of the application of Codazzi's formulæ to the deformed surface, and may be of use in some further investigation.

2. If a is any vector by definition

$$a_1=\frac{\partial a}{\partial u}+Vpa,\qquad a_2=\frac{\partial a}{\partial v}+Vqa,$$

$$a_{11}=\frac{\partial a_1}{\partial u}+Vpa_1,\qquad a_{22}=\frac{\partial a_2}{\partial v}+Vqa_2,$$

$$a_{12}=a_{21}=\frac{\partial a_2}{\partial u}+Vpa_2=\frac{\partial a_1}{\partial v}+Vqa_1.$$

By aid of the formulæ

$$V(aV\beta\gamma)+V(\beta V\gamma a)+V(\gamma Va\beta)=0,$$

$$Sa\beta\gamma+S\beta a\gamma=0,$$

it easily follows that

$$\left.\begin{aligned}\frac{\partial}{\partial u}Va\beta+V(pVa\beta) &= Va_1\beta+Va\beta_1\\ \frac{\partial}{\partial u}Sa\beta &= Sa_1\beta+Sa\beta_1\end{aligned}\right\}. \tag{1}$$

If z is a vector which only depends on u and v we shall write

$$dz=z_1\,du+z_2\,dv,$$

$$\partial z=z_1\,\partial u+z_2\,\partial v,$$

and shall speak of dz and ∂z as elements of the z surface.

The square of the element of arc on the z surface is then

$$-(dz)^2=z_1^2\,du^2+2Sz_1z_2\,du\,dv+z_2^2\,dv^2.$$

The vector Vz_1z_2 is parallel to the normal. We do not, however, take this particular vector, but *any* parallel vector ρ in order to obtain the condition that two elements may be conjugate.

From the definition of conjugate elements the element ∂z is conjugate to dz if, and only if, ∂z is perpendicular to the normals at z and $z+dz$.

It follows that $V\rho\,d\rho$ is parallel to ∂z, and therefore

$$V(\rho\,d\rho)\,\partial z=\partial z\,V\rho\,d\rho.$$

From this, combined with $\rho\,\partial z+\partial z\,\rho=0,$

we deduce $\rho\,S\,d\rho\,\partial z=0\,;$

and therefore for conjugate elements

$$S\,d\rho\,\partial z=0.$$

This condition is, of course, equivalent with

$$S\,\partial\rho\,dz=0,$$

and if we take ρ to be of unit length, it expresses the known fact that any element on a surface is perpendicular to the element which corresponds in the spherical representation to the conjugate element.

For self conjugate elements, that is, for asymptotic lines,

$$S\,d\rho\,dz=0,$$

or, expanded,

$$S(\rho_1z_1)du^2+(S\rho_1z_2+S\rho_2z_1)du\,dv+S(\rho_2z_2)\,dv^2=0.$$

3. Let z be a vector which for a given value w_0 of the parameter w is known in terms of u and v, and let p and q also be known for the same value of w.

Let w_0 be changed into w_0+dw_0, and let

$$\zeta = (z_3)_{w=w_0} = \left(\frac{\partial z}{\partial w}\right)_{w=w_0} + Vrz_{w=w_0}.$$

The surface traced out by $\quad z_{w=w_0}+\zeta\, dw_0$

will have for the square of its element of arc

$$-\left[\left(z_1+\frac{\partial z_1}{\partial w}\zeta\, dw_0\right) du + \left(z_2+\frac{\partial z_2}{\partial w}\zeta\, dw_0\right) dv\right]^2,$$

where w_0 is supposed to be substituted for w after differentiations have been carried out.

If, then,* $$\frac{\partial z_1}{\partial w} = \frac{\partial z_2}{\partial w} = 0,$$

the surface $z+\zeta dw_0$ will have the same element of arc as the surface traced out by z; that is, the surface will be infinitesimally deformed.

Now $$\frac{\partial z_1}{\partial w} = z_{31}+Vz_1r = z_{13}+Vz_1r = \zeta_1+Vz_1r,$$

since $$z_3 = \zeta;$$

it follows that the equations which define the infinitesimal transformation are

$$\zeta_1+Vz_1r = 0, \quad \zeta_2+Vz_2r = 0,$$

or

$$d\zeta+Vdz\,r = 0.$$

In this equation z is a known function of u and v, and ζ, r are functions of u and v which depend on the solution of a partial differential equation of the second order whose form has now to be found.

4. Let $\bar{\zeta}$ bear the same relation to ζ that $\bar{z}$ bore to z in § 1, then, as we suppose the position of the moving axes known for all values of u and v when $w = w_0$, we know ζ in terms of u and v when we know $\bar{\zeta}$, and conversely.

Suppose now that r has been found, then since ζ_1 and ζ_2 are known in terms of u and v, we know $\partial\bar{\zeta}/\partial u$ and $\partial\bar{\zeta}/\partial v$ with reference to the fixed axes through which the moving ones are passing for any given values of u and v. We therefore know $\partial\bar{\zeta}/\partial u$ and $\partial\bar{\zeta}/\partial v$ with reference to any axes fixed once for all; and as $\partial\bar{\zeta}/\partial w = 0$, we can thus obtain $\bar{\zeta}$ by quadratures,

* It might seem as though these equations only define a particular class of deformations, but as the vector r is undetermined we can choose it so as to make one of these equations hold when the other must hold necessarily.

and therefore we can obtain ζ. The constant of integration which is thus introduced is immaterial to the real problem of deformation as it merely denotes an infinitesimal translation of the surface.

From the fundamental formulæ

$$\zeta_1+Vz_1r=0, \qquad \zeta_2+Vz_2r=0,$$

we deduce
$$\zeta_{12}+Vz_1r_2+Vz_{12}r=0,$$

by aid of (1), and similarly, $\zeta_{21}+Vz_2r_1+Vz_{21}r$.

It follows that
$$Vz_1r_2=Vz_2r_1;$$

and therefore, as the vectors z_1, z_2, r_1, r_2 are thus shown to be all parallel to the same plane, the surface traced out by the vector r—the r surface we shall call it—and the z surface have their normals parallel at corresponding points.

5. We have shown that the deformation is given when r is known, for then ζ is obtained by quadratures, and we shall now show how r depends on the solution of a partial differential equation of the second order.

Let λ be a unit vector drawn through the origin parallel to the normal (outwards) to the z surface. The λ surface will be the spherical representation of the z surface and also of the r surface.

Let
$$R+S\lambda r=0,$$

so that R is the perpendicular from the origin on the r surface.

The z surface being known, λ is also known in terms of u and v, and, of course, λ_1 and λ_2 are known; we shall first express r, r_1 and r_2 in terms of R, λ, λ_1 and λ_2. From
$$R+S\lambda r=0$$

we deduce, since
$$S\lambda r_1=S\lambda r_2=0,$$

that
$$\frac{\partial R}{\partial u}+S\lambda_1 r=0, \qquad \frac{\partial R}{\partial v}+S\lambda_2 r=0.$$

It follows that

$$rS\lambda\lambda_1\lambda_2+RV\lambda_1\lambda_2+\frac{\partial R}{\partial u}V\lambda_2\lambda+\frac{\partial R}{\partial v}V\lambda\lambda_1=0.$$

Since
$$\lambda^2=-1 \quad \text{and} \quad S\lambda\lambda_1=S\lambda\lambda_2=0,$$

we have
$$\lambda S\lambda\lambda_1\lambda_2+V\lambda_1\lambda_2=0;$$

and therefore
$$r=R\lambda+V\lambda\mu,$$

where $$\mu S\lambda\lambda_1\lambda_2 = \frac{\partial R}{\partial u}\lambda_2 - \frac{\partial R}{\partial v}\lambda_1;$$

thus r is expressed in terms of R and known vectors.

Again, $$\lambda\frac{\partial R}{\partial u} + V\lambda_1\mu = 0,$$

$$\lambda\frac{\partial R}{\partial v} + V\lambda_2\mu = 0;$$

and therefore, by (1), $$r_1 = R\lambda_1 + V\lambda\mu_1,$$

$$r_2 = R\lambda_2 + V\lambda\mu_2.$$

The function R is called the characteristic function because on it depends, as we have now proved, the infinitesimal deformation.

Before proceeding to obtain the differential equation which R satisfies, we shall deduce some formulæ required for that equation. From the expression for μ we deduce

$$S\lambda\lambda_1\mu_1 = S\lambda_{11}\lambda\mu + \frac{\partial^2 R}{\partial u^2},$$

$$S\lambda\lambda_2\mu_1 = S\lambda\lambda_1\mu_2 = S\lambda_{12}\lambda\mu + \frac{\partial^2 R}{\partial u\,\partial v},$$

$$S\lambda\lambda_2\mu_2 = S\lambda_{22}\lambda\mu + \frac{\partial^2 R}{\partial v^2};$$

and therefore $$S\lambda_1 r_1 = R\lambda_1^2 - \frac{\partial^2 R}{\partial u^2} - S\lambda_{11}\lambda\mu,$$

$$S\lambda_1 r_2 = S\lambda_2 r_1 = RS\lambda_1\lambda_2 - \frac{\partial^2 R}{\partial u\,\partial v} - S\lambda_{12}\lambda\mu,$$

$$S\lambda_2 r_2 = R\lambda_2^2 - \frac{\partial^2 R}{\partial v^2} - S\lambda_{22}\lambda\mu.$$

Similarly, if $$Z + S\lambda z = 0,$$

so that Z is the perpendicular on the z surface, and if

$$\nu S\lambda\lambda_1\lambda_2 = \frac{\partial Z}{\partial u}\lambda_2 - \frac{\partial Z}{\partial v}\lambda_1,$$

then $$z = Z\lambda + V\lambda\nu,$$

$$z_1 = Z\lambda_1 + V\lambda\nu_1,$$

$$z_2 = Z\lambda_2 + V\lambda\nu_2,$$

and
$$S\lambda_1 z_1 = Z\lambda_1^2 - \frac{\partial^2 Z}{\partial u^2} - S\lambda_{11}\lambda\nu,$$
$$S\lambda_1 z_2 = S\lambda_2 z_1 = ZS\lambda_1\lambda_2 - \frac{\partial^2 Z}{\partial u\, \partial v} - S\lambda_{12}\lambda\nu,$$
$$S\lambda_2 z_2 = Z\lambda_2^2 - \frac{\partial^2 Z}{\partial v^2} - S\lambda_{22}\lambda\nu.$$

We can now obtain the characteristic equation satisfied by R.

We saw that $$Vr_1 z_2 = Vr_2 z_1;$$
and therefore $$SV\lambda_1\lambda_2 . Vr_1 z_2 = SV\lambda_1\lambda_2 . Vr_2 z_1.$$
Applying the formula
$$SV\alpha\beta . V\gamma\delta = S\alpha\delta . S\beta\gamma - S\alpha\gamma . S\beta\delta,$$
and remembering that
$$S\lambda_1 z_2 = S\lambda_2 z_1 \quad \text{and} \quad S\lambda_1 r_2 = S\lambda_2 r_1,$$
we obtain the equation
$$S\lambda_1 z_1 . S\lambda_2 r_2 + S\lambda_2 z_2 . S\lambda_1 r_1 = 2S\lambda_1 z_2 . S\lambda_2 r_1.$$
On substituting for $S\lambda_1 z_1$, $S\lambda_1 r_1$, ..., the expressions just found, we have a differential equation of the second order in R which involves Z—a known function—symmetrically with R and also λ and its derivatives.

6. Darboux has shown that there are twelve surfaces, such that when the deformation problem is solved for one of the surfaces it is solved for all the others, and he has pointed out many interesting relations between these surfaces.

To express these surfaces in vector notation we consider the vector defined by
$$\rho = -\lambda/R,$$
and can now express them all in terms of
$$r, \quad \rho, \quad z, \quad \text{and} \quad \zeta.$$
From the definition of R we see that
$$Sr\rho = 1.$$
The vectors of the twelve surfaces are
$$z, \quad \rho, \quad r/Sr\zeta, \quad \zeta/Sz\zeta, \quad (z+V\zeta\rho)/Sz\rho, \quad \zeta+Vzr,$$
which we shall denote respectively by
$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad a_5, \quad a_6,$$

where the suffixes now have no longer any meaning of differentiation, but are merely a convenient notation; and

$$\zeta, \quad r, \quad \rho/Sz\rho, \quad z/Sz\rho, \quad (\zeta+Vzr)/S\zeta r, \quad z+V\zeta\rho,$$

which will be denoted by

$$\alpha_1, \quad \alpha_2, \quad \alpha_3, \quad \alpha_4, \quad \alpha_5, \quad \alpha_6.$$

It will be found that by the transformation

$$z'=\zeta, \quad r'=\rho, \quad \zeta'=z, \quad \rho'=r,$$

the twelve vectors are merely transformed amongst themselves, each a into the corresponding α with the same suffix. Let this transformation be symbolized by A.

From the fundamental equation for the deformation of the z surface

$$d\zeta+Vdzr=0,$$

we at once obtain by aid of the formula

$$Va\,V\beta\gamma=\gamma Sa\beta-\beta Sa\gamma,$$
$$Vd\zeta\rho=rS\rho dz-dzS\rho r;$$

but from the definition of ρ,

$$S\rho dz=0 \quad \text{and} \quad S\rho r=1;$$

and therefore $$dz+Vd\zeta\rho=0;$$

that is $$d\zeta'+Vdz'r'=0.$$

The transformation A therefore transforms z into ζ, a surface for which the deformation problem is also solved.

From $$d\zeta+Vdzr=0 \quad \text{and} \quad dz+Vd\zeta\rho=0,$$

we deduce $$d(\zeta+Vzr)+Vdrz=0,$$

Consider next the transformation B defined by

$$z'=r, \quad r'=z, \quad \zeta'=\zeta+Vzr, \quad \rho'=\rho/Sz\rho.$$

It will be found that this transforms a_i into α_{i+1} and α_{i+1} into a_i, where we make the conventions

$$a_{i+j}=a_k, \quad \alpha_{i+j}=\alpha_k, \quad \text{if} \quad i+j=k \pmod 6,$$

and leaves unaltered the fundamental equation

$$dz+Vd\zeta\rho=0.$$

By continued applications of the transformations A and B we thus obtain Darboux's twelve surfaces for each of which the deformation problem is solved when it is solved for one.

7. For the z surface ζdw_0 is the linear, and $r dw_0$ the angular displacement; we say then that ζ is the linear, and r the angular velocity. But ζ is the linear velocity, not of the origin of coordinates which is fixed, but of the extremity of the vector z.

These velocities, linear and angular, may therefore be replaced by the motion defined by a linear velocity $\zeta + Vzr$ of the origin and an angular velocity r at it.

The central axis of the velocity of displacement of the z surface has its direction parallel to the vector r, and it passes through the extremity of the vector

$$V(\zeta + Vzr)/r.$$

The angular velocity along this axis is the tensor of r, and the pitch of the screw-motion is

$$S(\zeta + Vzr)/r = S\zeta/r.$$

8. The geometrical relations between the twelve surfaces in the notation of this paper may be expressed as follows:—

The surfaces a_i and a_k

correspond orthogonally,	if $i+k = 2 \pmod 6$;
have their radii vectores and linear displacements parallel,	,, $= 5 \pmod 6$;
have their normals parallel,	,, $= 3 \pmod 6$;
are polar reciprocal,	,, $= 4 \pmod 6$;
have their central axes parallel and the pitch of either screw $= 1/Sa_{k+2}a_{i+2}$,	,, $= 1 \pmod 6$;
have their polar reciprocals corresponding orthogonally,	,, $= 0 \pmod 6$.

The asymptotic lines correspond on

$$a_1, \quad a_6, \quad a_4, \quad a_3,$$

and to these correspond conjugate lines with equal point invariants on

$$a_2, \quad a_4, \quad a_5, \quad a_1,$$

and conjugate lines with equal tangential invariants on

$$a_3, \quad a_5, \quad a_6, \quad a_2.$$

These geometrical relations may all be proved by the quaternion method of this paper. Thus suppose we wish to prove that the asymptotic lines on the z surface correspond to conjugate lines with equal point invariants on the ρ surface.

Take the parametric lines to be asymptotic on the z surface, and let suffixes have their earlier meaning of differentiation. We have

$$S\rho_1 z_1 = S\rho_2 z_2 = 0,$$

since the coefficients of du^2 and dv^2 in the equation of the asymptotic lines must be zero, and from the definition of ρ,

$$S\rho z_1 = S\rho z_2 = 0.$$

It follows that $$z_1 = aV\rho\rho_1, \quad z_2 = bV\rho\rho_2,$$

where a and b are some scalars. From

$$z_1 + V\rho\zeta_1 = 0, \quad z_2 + V\rho\zeta_2 = 0,$$

we deduce $$\zeta_1 = a\rho_1 + c\rho, \quad \zeta_2 = b\rho_2 + d\rho,$$

where c and d are scalars. From

$$S\rho r = 1, \quad Sr\zeta_1 = Sr\zeta_2 = 0, \quad V\zeta_1\rho_2 = V\zeta_2\rho_1,$$

we deduce that c and d are zero and $a+b = 0$. The equations

$$\zeta_1 = a\rho_1, \quad \zeta_2 = -a\rho_2,$$

now give that the parametric lines are conjugate lines with equal point invariants both on the ρ and on the ζ surface.