On the Binomial Equation $x^p-1=0$: Quinquisection. By H. W. Lloyd Tanner, M.A., Professor of Mathematics in the University College of South Wales.

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The binomial equation has recently been discussed by Prof. Cayley in three papers published in the *Proceedings of the London Mathematical Society*; viz.,

Vol. XI. (1879), pp. 4—17 ....(𝔄),

Vol. XII. (1880), pp. 15, 16 ....(𝔅),

Vol. XVI. (1885), pp. 61—63 ....(ℭ).

A further contribution to the question is contained in a paper by Miss Charlotte A. Scott, D.Sc., published in the *American Journal of Mathematics* (1886), Vol. VIII., pp. 261—264 ....(𝔇).

If X_0, X_1, X_2, X_3, X_4 denote the five periods (when $p=5\lambda+1$), we have

$$X_0^2 = (a, b, c, d, e \between X_0, X_1, X_2, X_3, X_4),$$

$$X_0X_1 = (f, g, h, i, j \between \quad \ldots \quad \ldots \quad),$$

$$X_0X_2 = (k, l, m, n, o \between \quad \ldots \quad \ldots \quad),$$

and equations derived from these by cyclical permutations. The coefficients $a, b, c, \ldots o$ are integers, the properties of which are considered in Prof. Cayley's papers; and in (ℭ) they are expressed as linear functions of 5 integers $\theta, \alpha, \beta, f, k$, which satisfy one linear equation and two quadric equations. In this place is quoted another reduction, due to Prof. F. S. Carey, of the 15 coefficients to 7 integers connected by 3 linear and 2 quadric equations. In that part of Miss Scott's paper which relates to Quinquisection, proofs are given of some of Prof. Cayley's theorems, and Miss Scott finds the values of all but one of the coefficients of the equation whose roots are the five periods.

In the following discussion the theory is based on five integers q_0, q_1, q_2, q_3, q_4, the "coordinates" of a certain complex factor of p;

viz., $$q\omega, = q_0+q_1\omega+q_2\omega^2+q_3\omega^3+q_4\omega^4,$$

where ω is a root of the equation

$$\omega^4+\omega^3+\omega^2+\omega+1=0.$$

This complex integer is, in fact, the quotient

$$F\omega \,.\, F\omega / F\omega^2,$$

made definite as to its coordinates by the condition

$$1+q_0+q_1+q_2+q_3+q_4=0,$$

$F\omega$ being the resolvent

$$X_0+X_1\omega+X_2\omega^2+X_3\omega^3+X_4\omega^4.$$

The coefficients $a, b, c, \ldots o$ are expressed linearly in terms of the q; and the relations of these 15 coefficients follow with great ease. The coefficients of the period equation are also determined. It is found convenient here to make use of three numbers, $\sigma_1, \sigma_2, \sigma_3$, closely related to the q. Tables follow in which are given the values of the q, the σ, and the coefficients of the period equations for values of p up to 1000. I take this opportunity of expressing my thanks to my colleague, Mr. W. F. Pelton, M.A., who has kindly verified the Tables for me.

The periods, X, depend, as far as their arrangement is concerned, upon the primitive root used in their formation. The consideration of the effect of changing this primitive root leads to a set of cyclic permutations different from that mentioned above; one being represented by adding the same number to each subscript, the second by multiplying every subscript by one number; thus leaving X_0 unchanged, while X_1, X_2, X_4, X_3 are cyclically permuted. The double set of substitutions much facilitates the work.

Preliminary.

Let p be a prime number of the form $5\lambda+1$, and x any root of the equation

$$x^{p-1}+x^{p-2}+\ldots+x+1=0 \quad\ldots\ldots\ldots\ldots(1).$$

The periods X_0, X_1, X_2, X_4, X_3 are the sums

$$\left.\begin{aligned} X_0 &= x+x^{g^5}+x^{g^{10}}+\ldots+x^{g^{5\lambda-5}} \\ X_1 &= x^{g^1}+x^{g^6}+\ldots\ldots\ldots+x^{g^{5\lambda-4}} \\ X_2 &= x^{g^2}+x^{g^7}+\ldots\ldots\ldots+x^{g^{5\lambda-3}} \\ X_4 &= x^{g^4}+x^{g^9}+\ldots\ldots\ldots+x^{g^{5\lambda-1}} \\ X_3 &= x^{g^3}+x^{g^8}+\ldots\ldots\ldots+x^{g^{5\lambda-2}} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(2),$$

where g is any primitive root of p. Each of the lines is characterised by a number, the subscript of X, to which all the exponents of g in the line are congruent, to modulus 5. The last four rows are arranged in an order which will be much used in the sequel, the subscripts forming a geometrical progression 1, 2, 4, 3 ($\equiv 8$).

The functions F are defined by the equations

$$\left.\begin{aligned} F1 &= X_0+\ \ X_1+\ \ X_2+\ \ X_4+\ \ X_3, = -1 \\ F\omega &= X_0+\omega X_1+\omega^2X_2+\omega^4X_4+\omega^3X_3 \\ F\omega^2 &= X_0+\omega^2X_1+\omega^4X_2+\omega^3X_4+\omega X_3 \\ F\omega^4 &= X_0+\omega^4X_1+\omega^3X_2+\omega X_4+\omega^2X_3 \\ F\omega^3 &= X_0+\omega^3X_1+\omega X_2+\omega^2X_4+\omega^4X_3 \end{aligned}\right\} \quad \text{.........(3)},$$

wherein ω represents any root of

$$\omega^4+\omega^3+\omega^2+\omega+1 = 0 \quad \text{......................(4)}.$$

It is well known that

$$F\omega \,.\, F\omega^4 = F\omega^2 \,.\, F\omega^3 = p \quad \text{......................(5)}.$$

Also the quotient

$$F\omega^h \,.\, F\omega^k / F\omega^{h+k}$$

is a complex integer; that is to say, it is of the form

$$A_0+A_1\omega+A_2\omega^2+A_4\omega^4+A_3\omega^3,$$

where the "coordinates" A_0, A_1, A_2, A_4, A_3 are integers. In the case in which $h = k = 1$, we shall call this complex integer

$$q\omega, = q_0+q_1\omega+q_2\omega^2+q_4\omega^4+q_3\omega^3,$$

so that we have the first of the following set of equations,

$$\left.\begin{aligned} F\omega \,.\, F\omega &= q\omega \,.\, F\omega^2 \\ F\omega^2 \,.\, F\omega^2 &= q\omega^2 \,.\, F\omega^4 \\ F\omega^4 \,.\, F\omega^4 &= q\omega^4 \,.\, F\omega^3 \\ F\omega^3 \,.\, F\omega^3 &= q\omega^3 \,.\, F\omega \end{aligned}\right\} \quad \text{......................(6)},$$

the others being formed by successively doubling the exponents of ω.

These equations determine $q\omega$, but not its coordinates, since, in virtue of (4), $q\omega$ is not changed when any, the same, constant is dded to each of the coordinates. To fix these, we assume that the equations (6) hold good when ω is replaced by $\omega^5, = 1$. This gives

$$q1 = -1 \text{ or } 1+q_0+q_1+q_2+q_4+q_3 = 0 \text{..............(7)}.$$

From (5), (6), we find

$$\left.\begin{aligned} F\omega\,.\,F\omega^3 &= q\omega\,.\,F\omega^4 \\ F\omega^2\,.\,F\omega &= q\omega^2\,.\,F\omega^3 \\ F\omega^4\,.\,F\omega^2 &= q\omega^4\,.\,F\omega \\ F\omega^3\,.\,F\omega^4 &= q\omega^3\,.\,F\omega^2 \end{aligned}\right\} \text{......................(8)}.$$

The equations (5), (6), (8) enable us to express any rational integral function of the F as a linear function of the F. They give, also, the equations

$$\left.\begin{aligned} (F\omega)^5 &= p\,.\,q\omega\,.\,q\omega\,.\,q\omega^2 \\ p\,.\,F\omega^2 &= q\omega^4\,.\,(F\omega)^2 \\ p^3\,.\,F\omega^4 &= q\omega^4\,.\,q\omega^4\,.\,q\omega^3\,.\,(F\omega)^4 \\ p^2\,.\,F\omega^3 &= q\omega^4\,.\,q\omega^3\,.\,(F\omega)^3 \end{aligned}\right\} \text{.................(9)},$$

showing that $F\omega^2$, $F\omega^4$, $F\omega^3$ (and therefore also the periods) are rational functions of $F\omega$, which is determined as the fifth root of the complex integer, $p\,q\omega\,q\omega\,q\omega^2$.

From the equations (5), (6), we find

$$p = q\omega\,.\,q\omega^4 = q\omega^2\,.\,q\omega^3 \text{ ....(10)},$$

showing that each $q\omega$ is one of two conjugate factors of p. Writing the products at length,

$$\begin{aligned} p &= q_0^2+q_1^2+q_2^2+q_4^2+q_3^2 \\ &\qquad +(q_0q_1+q_1q_2+q_2q_3+q_3q_4+q_4q_0)(\omega+\omega^4) \\ &\qquad +(q_0q_2+q_2q_4+q_4q_1+q_1q_3+q_3q_0)(\omega^2+\omega^3) \\ &= q_0^2+q_1^2+q_2^2+q_4^2+q_3^2 \\ &\qquad +(q_0q_2+q_2q_4+q_4q_1+q_1q_3+q_3q_0)(\omega+\omega^4) \\ &\qquad +(q_0q_1+q_1q_2+q_2q_3+q_3q_4+q_4q_0)(\omega^2+\omega^3). \end{aligned}$$

Hence, since the equation (4) for ω is irreducible,

$$\left.\begin{aligned} q_0q_1+q_1q_2+q_2q_3+q_3q_4+q_4q_0 &= -\lambda, = -\tfrac{1}{5}(p-1) \\ q_0q_2+q_2q_4+q_4q_1+q_1q_3+q_3q_0 &= -\lambda \\ q_0^2 + q_1^2 + q_2^2 + q_4^2 + q_3^2 &= 4\lambda+1 \end{aligned}\right\} \text{.........(11)},$$

of which the first pair are found to be equivalent to Prof. Cayley's quadric relations (𝕮, p. 62).

The following method of calculating the coordinates is a slight modification of one given in Bachmann's *Lehre von der Kreistheilung* (pp. 93–95); or is easily verified from the discussion in Serret's *Cours d'Algèbre supérieure*, § 552. We set down the values of the expression ind. μ+ind. $(\mu+1)$ for all values of μ from 1 to $\frac{1}{2}(p-1)$ inclusive, or, what is sufficient for our purpose, their residues to modulus 5. Twice the number of times the residue i appears is called A_i; but the last term in the series, viz., ind. $\frac{1}{2}(p-1)$+ind. $\frac{1}{2}(p+1)$, which is $\equiv 2$ ind. $\frac{1}{2}(p-1)$, is only counted once in forming the A to which it belongs. The expression

$$A_0+A_1\omega+A_2\omega^2+A_4\omega^4+A_3\omega^3$$

is a complex factor of p, having all the properties of $q\omega$, except that the sum of the coordinates is $p-2$. If, then, from each A we subtract λ, $=\frac{1}{2}(p-1)$, we shall obtain the coordinates of $q\omega$. Since λ is even, it follows that one, and only one, of the coordinates of q is odd.

As an example, take the case $p=11$:

$$\mu = 1, 2, 3, 4, 5, 6,$$

$$\text{ind}_2\,\mu \equiv 0, 1, 3, 2, 4, 4 \pmod{5},$$

$$\text{ind}_2\,\mu+\text{ind}_2\,(\mu+1) \equiv 1, 4, 0, 1, \mid 3 \;(\;\;,,\;\;),$$

therefore $A_0=2,\; A_1=4,\; A_2=0,\; A_4=2,\; A_3=1,$

$$q_0=0,\; q_1=2,\; q_2=-2,\; q_4=0,\; q_3=-1,$$

$$q\omega = 2\omega-2\omega^2-\omega^3.$$

Change of Arbitrary Elements.

In the formation of the periods and the F, three arbitrary elements enter: g, the primitive root of p; ω, the root of (4); and x, the root of (1). We propose to consider the effect of replacing these arbitrary quantities by others.

Suppose that the periods and the functions F have been determined for a particular value of g and a particular value of ω. Let us now change the primitive root, the other elements being the same as before, and denote the new periods and F by accents. The new primitive root is $\equiv g^i \pmod{p}$, where i is prime to $p-1$, and therefore prime to 5. Hence it follows that the periods are interchanged as wholes, there is no transference of isolated terms from period to period. In fact, the period

$$X_{ik},\ = x \mid g^{ik}+\&c.,\ = x \mid (g^i)^k+\&c.,\ = X'_k.$$

To put this otherwise, we write $i \equiv 2^\iota$, $k \equiv 2^\kappa$ (mod. 5), which is permissible, as 2 is a primitive root of 5; and, to avoid complicated subscripts, we write $X_{(\kappa)}$ for X_k when we wish to put in evidence the index κ. With this notation, the fact that X'_k is X_{ik} is expressed by saying that $X'_{(\iota)}$ is $X_{(\iota+\kappa)}$, showing that the new periods consist of the X_0 unchanged and the four periods $X_1 X_2 X_4 X_3$ in the same cyclic order; subjected, in fact, to the substitution

$$(X_1 X_2 X_4 X_3)^\iota.$$

This being so, we have

$$\begin{aligned} F'\omega,\ &= X'_0+\omega X'_{(0)}+\omega^2 X'_{(1)}+\omega^4 X'_{(2)}+\omega^3 X'_{(3)} \\ &= X_0+\omega X_{(\iota)}+\omega^2 X_{(\iota+1)}+\omega^4 X_{(\iota+2)}+\omega^3 X_{(\iota+3)} \\ &= F\omega^{2^{-\iota}}; \end{aligned}$$

that is, the new F' are $F\omega$, $F\omega^2$, $F\omega^4$, $F\omega^3$ acted on by the substitution

$$(F\omega,\ F\omega^2,\ F\omega^4,\ F\omega^3)^{-\iota}.$$

If we now change the root ω, there is no effect upon the X, and this new change may be such as to undo the effect on the F of the change of primitive root. In fact, taking the new ω' to be ω^{2^ι}, we have

$$F'\omega' = F(\omega')^{2^{-\iota}} = F(\omega^{2^\iota})^{2^{-\iota}} = F\omega.$$

The $q\omega$, being derived from the F alone, will also be unaltered by the double change.

It still remains to consider the effect of a change of the primitive root upon the several coordinates q_0, q_1, q_2, q_4, q_3. The change in $q\omega$ is the same as that in $F\omega$; viz., we have

$$q'\omega = q\omega^{2^{-\iota}}.$$

Hence $$q'_0,\ q'_1,\ q'_2,\ q'_4,\ q'_3 = q_0,\ q_{(\iota)},\ q_{(\iota+1)},\ q_{(\iota+2)},\ q_{(\iota+3)}.$$

Consider, for instance, the coefficients of ω in the first equation. On the left it is q'_1; on the right it is q_x, where x is such that $(\omega^{2^{-\iota}})^x = \omega$, so that $x = 2^\iota$. Hence the coordinates are changed in the same manner and to the same extent as the periods. This also follows from the method by which q_0, &c. are calculated.

These results may be expressed in the following form:

Any relation between the periods, the coordinates, the numbers $q\omega$, the functions $F\omega$, and the ω, remains true if the subscripts of all the periods and of the coordinates and the exponents of all the free ω be multiplied by i, leaving the $q\omega$, $F\omega$ unchanged. Here $i = 0, 1, 2, 3, 4$. ... (12).

If we replace the x of equation (1) by another root x^{g^i}, we get a cyclic change in the five periods X_0, X_1, X_2, X_3, X_4 in the arithmetic order; namely,

$$X'_h = X_{h+i},$$

and, as a consequence,

$$\begin{aligned} F'\omega &= X'_0 + \omega X'_1 + \ldots + \omega^{-i} X'_{-i} + \omega^{-i+1} X'_{-i+1} + \ldots \\ &= X_i + \omega X_{i+1} + \ldots + \omega^{-i} X'_0 + \omega^{-i+1} X'_1 + \ldots \\ &= \omega^{-i} F\omega. \end{aligned}$$

Generally, $$F'\omega^h = \omega^{-hi} F\omega,$$

and $q\omega^h$ is unchanged.

The change in each coordinate is not simple, depending upon the $F\omega$ it multiplies as well as upon i.

Thus, *any relation between the periods, the numbers $q\omega$, the functions $F\omega$, and ω, remains true if the subscript of each period be increased by i and every $F\omega^h$ be multiplied by ω^{hi}, the $q\omega$ and the free ω being unaltered.* ... (13).

From this we get some useful results. Thus, if Ξ be any function of the periods, and

$$\Xi = A_0 + A_1 F\omega + A_2 F\omega^2 + A_3 F\omega^3 + A_4 F\omega^4,$$

then $$\Sigma\Xi = 5A_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

if A_0 do not involve ω, $q\omega$. In this case $\frac{1}{5}\Sigma\Xi$ is the absolute term in Ξ. Here $\Sigma\Xi$ is written for the sum of five terms formed from Ξ by adding to the subscripts.

Again, since, from (3), (4),

$$5X_0 = -1 + F\omega + F\omega^2 + F\omega^3 + F\omega^4,$$

it follows that $$\Sigma X_0 \Xi = -A_0 + p\,(A_1 + A_2 + A_3 + A_4) \ldots\ldots\ldots (15),$$

for the expression on the right is the absolute term in $5X_0\Xi$.

We may add for reference, though not related to what has preceded some other results of summation. If a be the sum of the coordinates of a complex function,

$$A\omega, = A_0+A_1\omega+A_2\omega^2+A_4\omega^4+A_3\omega^3,$$

then
$$\Sigma A\omega = 5A_0-a,$$

where the Σ means the sum of the *four* terms formed by successively doubling the exponent of ω. Hence

$$\Sigma q\omega = 5q_0+1, = 5\sigma_1+1 \quad \text{.....................(16)},$$

where σ_1 is written for q_0, in order to mark the relation of (16) to (17), (18), *post.*

$$\Sigma\omega^{-h}q\omega = 5q_h+1.$$

$$\Sigma q\omega q\omega^2 = 5\sigma_2-1 \quad \text{..........................(17)},$$

where σ_2 is the absolute part of $q\omega\,.\,q\omega^2$; or

$$\sigma_2 = q_0^2+q_1q_2+q_2q_4+q_4q_3+q_3q_1 \quad \text{..................(17*)}.$$

$$\Sigma q\omega q\omega q\omega^2 = 5\sigma_3+1 \quad \text{.......................(18)},$$

where σ_3 is the absolute part of $q\omega q\omega q\omega^2$; or

$$\sigma_3 = 3q_0^3+2q_0^2-4\lambda q_0+(q_1+q_4)\,q_1q_4+(q_2+q_3)\,q_2q_3$$
$$+2\,(q_1^2q_2+q_2^2q_4+q_4^2q_3+q_3^2q_1) \quad \text{.....................(19)}.$$

Cayley's Coefficients.

Solving the equations (3), we obtain

$$\left.\begin{aligned}5X_0 &= (1,\ 1,\ 1,\ 1,\ 1\between -1,\ F\omega,\ F\omega^2,\ F\omega^4,\ F\omega^3)\\ 5X_1 &= (1,\ \omega^4,\ \omega^3,\ \omega,\ \omega^2) \quad ,,\\ 5X_2 &= (1,\ \omega^3,\ \omega,\ \omega^2,\ \omega^4) \quad ,,\\ 5X_4 &= (1,\ \omega,\ \omega^2,\ \omega^4,\ \omega^3) \quad ,,\\ 5X_3 &= (1,\ \omega^2,\ \omega^4,\ \omega^3,\ \omega\ ) \quad ,,\end{aligned}\right\} \text{......(20)}.$$

Hence
$$\begin{aligned}25X_0^2 &= (-1+F\omega+F\omega^2+F\omega^4+F\omega^3)^2\\ &= 1+4p+(-2+2q\omega^4+q\omega^3)\,F\omega+\dots\\ &= 1+4p+\{-2+2q_0+q_0+(2q_4+q_2)\,\omega+(2q_3+q_4)\,\omega^2\\ &\qquad +(2q_1+q_3)\,\omega^4+(2q_2+q_1)\,\omega^3\}\,F\omega+\dots.\end{aligned}$$

On the other hand, using Prof. Cayley's notation (p. 214),

$$25X_0^2 = 5a \,.\, 5X_0 + 5b \,.\, 5X_1 + 5c \,.\, 5X_2 + 5e \,.\, 5X_4 + 5d \,.\, 5X_3$$

$$= -5(a+b+c+e+d)$$

$$+5(a+b\omega^4+c\omega^3+e\omega+d\omega^2)\,F\omega + \&c.$$

Comparing the absolute terms

$$a+b+c+d+e = -\tfrac{1}{5}(4p+1) = -4\lambda-1 \quad\ldots\ldots\ldots\ldots(21),$$

and expressing that the coefficients of $F\omega$ are equal, we get

$$5a = -2+2q_0+q_0+C,$$

$$5b = 2q_4+q_2+C,$$

$$5c = 2q_3+q_4+C,$$

$$5e = 2q_1+q_3+C,$$

$$5d = 2q_3+q_4+C.$$

To determine C, we add these equations, and make use of (7) and (21).

Thus, $$C = -4\lambda.$$

It is convenient to eliminate q_0 by means of (7). The equation for a then becomes

$$5(a+1) = -4\lambda - 3q_1 - 3q_2 - 3q_4 - 3q_3.$$

Again,

$$5X_0 \,.\, 5X_1 = (-1+F\omega+F\omega^2+F\omega^4+F\omega^3)$$

$$(-1+\omega^4F\omega+\omega^3F\omega^2+\omega F\omega^4+\omega^2F\omega^3)$$

$$= 1+p(\omega^2+\omega^4+\omega^3+\omega)$$

$$+\left\{-1-\omega^4+(\omega^3+\omega^2)\,q\omega^4+\omega^2q\omega^3\right\}F\omega+\&c.$$

$$= 1-p$$

$$+\left\{-1+q_3+2q_1+(q_0+q_2+q_3)\,\omega+(q_0+q_1+q_4)\,\omega^2\right.$$

$$\left.+(-1+q_2+2q_4)\,\omega^4+(q_0+q_2+q_3)\,\omega^3\right\}F\omega+\&c.$$

But $$5X_0 \,.\, 5X_1 = 5f \,.\, 5X_0+5g \,.\, 5X_1+5h \,.\, 5X_2+5j \,.\, 5X_4+5i \,.\, 5X_3$$

$$= -5(f+g+h+i+j)$$

$$+5(f+g\omega^4+h\omega^3+j\omega+i\omega^2)\,F\omega+\ldots.$$

Hence, as in the preceding case,

$$5f = \lambda+2q_1+q_3,$$
$$5g = \lambda+2q_4+q_2,$$
$$5h = \lambda- \;q_1-q_4,$$
$$5j = \lambda- \;q_1-q_4,$$
$$5i = \lambda- \;q_2-q_3.$$

The expression X_0X_2 is formed from X_0X_1 by doubling the subscripts. The coordinates must be similarly changed. The equation

$$X_0X_1 = fX_0+gX_1+hX_2+jX_4+iX_3$$

becomes

$$X_0X_2 = kX_0+mX_2+oX_4+nX_3+lX_1;$$

and the equations for f, g, &c., just written, become

$$5k = \lambda+2q_2+q_1,$$
$$5m = \lambda+2q_3+q_4,$$
$$5o = \lambda- \;q_2-q_3,$$
$$5n = \lambda- \;q_2-q_3,$$
$$5l = \lambda- \;q_1-q_4.$$

These results are combined in the following Table. The left side contains the notation used by Prof. Cayley (𝕮, p. 62); the right contains the notation quoted in the same place from a dissertation by Prof. F. S. Carey. Throughout, $\lambda = \frac{1}{5}(p-1)$.

Cayley's notation.	λ	q_1	q_2	q_4	q_3	Carey's notation.
$5(a+\lambda+1) =$	1	−3,	−3,	−3,	−3	$= 5(\alpha+1)$
$5\theta \quad =$	3	1,	1,	1,	1	
$5(c+\lambda) = 5k =$	1	1,	2,	.	.	$= 5\gamma$
$5(e+\lambda) = 5g =$	1	.	1,	2,	.	$= 5\epsilon$
$5(d+\lambda) = 5m =$	1	.	.	1,	2	$= 5\delta$
$5(b+\lambda) = 5f =$	1	2,	.	.	1	$= 5\beta$
$5a = 5h = 5j = 5l =$	1	−1,	.	−1,	.	$= 5\rho$
$5\beta = 5i = 5n = 5o =$	1	.	−1,	.	−1	$= 5\sigma$

where the first line, for example, is to be read

$$5(a+\lambda+1) = \lambda-3q_1-3q_2-3q_4-3q_3 = 5(\alpha+1).$$

From inspection of this Table, it is clear that θ, a are unaffected by any change of the primitive root; that the coefficients k, g, m, f are cyclically permuted by such change; and that (Cayley's) α, β are interchanged when the index of one primitive root with respect to other $\equiv 2$ or 3 (mod. 5), all this being an immediate consequence of the cyclic changes in the coordinates.

By substituting for α, β, θ, f, k, in Prof. Cayley's quadric equations, their values as given in the above Table, I find that they reduce to

$$(q_1+q_4)^2+q_1+q_4+q_1q_3-q_2q_3+q_2q_4 = -\lambda,$$

$$(q_2+q_3)^2+q_2+q_3+q_2q_1-q_4q_1+q_4q_3 = -\lambda,$$

and these are equivalent to the first two of (11), as appears on eliminating q_0 therefrom.

The Period Equation.

We proceed to determine the coefficients P, of the equation

$$\eta^5+\eta^4+P_2\eta^3+P_3\eta^2+P_4\eta+P_5 = 0,$$

whose roots are the periods. P_2, P_3, P_4 have been calculated by Miss Scott (𝔇, pp. 261, 262), and the forms given below are equivalent to those obtained by Miss Scott.

Two methods will be used. The first is to form the coefficients P directly from the expressions (20) for the roots; the second consists in forming an equation whose roots are $5X+1$, whence the period equation is immediately derived.

Since $$5X_1 = -1+\omega^4F\omega+\omega^3F\omega^2+\omega F\omega^4+\omega^2F\omega^3,$$

we have, by (15),

$$5\Sigma X_0X_1 = 1-p, \quad = 5\Sigma X_0X_2, \text{ by (12)},$$

therefore $$5P_2, = 5\Sigma(X_0X_1+X_0X_2), \quad = -2(p-1),$$

therefore $$P_2 = -2\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (P_2).$$

We have already found

$$25X_0X_1 = 1-p+\{-1-\omega^4+(\omega^3+\omega)q\omega^4+\omega^2q\omega^3\} F\omega+\&c.,$$

therefore, by (13),

$$25X_2X_3 = 1-p+\{-\omega^3-\omega^2+(\omega+\omega^4)\,q\omega^4+q\omega^3\}\,F\omega+\&c.\ldots(2,3),$$

therefore, by (15),

$$\begin{aligned}25\Sigma X_0X_2X_3 &= p-1+p\,\{-2\Sigma\omega+\Sigma\,(\omega^4+\omega+1)\,q\omega\}\\ &= p-1+p\,\{2+5\,(q_1+q_4+q_0)+3\}\\ &= p-1-5p\,(q_2+q_3),\end{aligned}$$

therefore $$25\Sigma X_0X_4X_1 = p-1-5p\,(q_4+q_1),$$

therefore $$\begin{aligned}25P_3, &= -25\Sigma\,(X_0X_2X_3+X_0X_4X_1)\\ &= -2\,(p-1)-5p\,(1+\sigma_1),\end{aligned}$$

therefore $$P_3 = -\tfrac{1}{5}\,\{2\lambda+p\,(1+\sigma_1)\}\ldots\ldots\ldots\ldots\ldots\ldots(P_3),$$

where σ_1 is written for q_0, as in (16).

From the expression for $25X_2X_3$, we get, by doubling the subscripts, by (12),

$$25X_4X_1 = 1-p+\{-\omega-\omega^4+(\omega^2+\omega^3)\,q\omega^4+q\omega^6\}\,F\omega+\&c.$$

The product $25X_2X_3\,.\,25X_1X_4$ gives

$$625\,X_1X_2X_3X_4 = A_0+A_1F\omega+A_2F\omega^2+A_4F\omega^4+A_3F\omega^3,$$

where $$\begin{aligned}A_0 &= (1-p)^2+p\Sigma\,\{-\omega^2-\omega^3+(\omega+\omega^4)\,q\omega^4+q\omega^3\}\\ &\qquad\times\{-\omega-\omega^4+(\omega^2+\omega^3)\,q\omega+q\omega^2\}\\ &= (1-p)^2+p\,(-4-2\Sigma q\omega-\Sigma\,q\omega q\omega^2)\\ &= (1-p)^2+p\,\{-5-10\sigma_1-5\sigma_2\}.\end{aligned}$$

Hence, by (14), $$\begin{aligned}125P_4, &= 125\Sigma X_1X_2X_3X_4,\\ &= (p-1)^2-5p\,(1+2\sigma_1+\sigma_2),\end{aligned}$$

therefore $$P_4 = \tfrac{1}{5}\,\{\lambda^2-\tfrac{1}{5}p\,(1+2\sigma_1+\sigma_2)\}\ldots\ldots\ldots\ldots\ldots\ldots(P_4).$$

Also, we have, from the expression for $625\,X_1X_2X_3X_4$,

$$\begin{aligned}3125P_5 &= -3125X_0X_1X_2X_3X_4\\ &= A_0-p\,(A_1+A_2+A_4+A_3) = A_0-p\,\Sigma A_1,\end{aligned}$$

and here A_0 is known.

A_1 consists of three parts. The first arises from the product of the absolute terms of $25X_2X_3$, $25X_4X_1$ into the terms containing $F\omega$, and

is
$$= (1-p)\left\{-\omega^3-\omega^2+(\omega+\omega^4)\,q\omega^4+q\omega^3 \right.$$
$$\left. -\omega-\omega^4+(\omega^2+\omega^3)\,q\omega^4+q\omega^3\right\}$$
$$= (1-p)\,(1-q\omega^4+2q\omega^3).$$

This contributes to ΣA_1 the portion
$$(1-p)\,(4+\Sigma q\omega).$$

The second part of A_1 is the coefficient of
$$F\omega^2\,.\,F\omega^3, = q\omega^4\,.\,F\omega,$$
in the product of $25X_2X_3$ and $25X_1X_4$. This gives to ΣA_1 the part
$$\Sigma\left[\left\{-\omega-\omega^4+(\omega^2+\omega^3)\,q\omega^3+q\omega\right\}\left\{-\omega-\omega^4+(\omega^2+\omega^3)\,q\omega+q\omega^2\right\}\right.$$
$$\left.+\left\{-\omega^2-\omega^3+(\omega+\omega^4)\,q\omega^3+q\omega\right\}\left\{-\omega^2-\omega^3+(\omega+\omega^4)\,q\omega+q\omega^2\right\}\right]q\omega^4$$
$$= 12p+3\,(p+1)\,\Sigma q\omega+3\Sigma q\omega\, q\omega^2.$$

The third part of A_1 comes from the term involving
$$(F\omega^3)^2, = q\omega^3\,.\,F\omega,$$
in the product of $25X_2X_3$, $25X_1X_4$. The corresponding part of ΣA_1 is
$$\Sigma\left\{-\omega-\omega^4+(\omega^2+\omega^3)\,q\omega^2+q\omega^4\right\}\left\{-\omega^2-\omega^3+(\omega+\omega^4)\,q\omega^2+q\omega^4\right\}q\omega^3$$
$$= -12p-(2p+1)\,\Sigma q\omega+\Sigma q\omega\, q\omega^2+\Sigma q\omega\, q\omega\, q\omega^2.$$

Hence $\qquad \Sigma A_1 = 4-4p+3\Sigma q\omega+4\Sigma q\omega\, q\omega^2+\Sigma q\omega\, q\omega\, q\omega^2,$

therefore $\qquad 3125P_5, = A_0-p\Sigma A_1$
$$= 1-10p+5p^2-5p\,\Sigma q\omega-5p\,\Sigma q\omega\, q\omega^2-p\,\Sigma q\omega\, q\omega\, q\omega^2$$
$$= 1-6p+5p^2-5p\,(1+5\sigma_1+5\sigma_2+\sigma_3),$$

therefore $\quad P_5 = \frac{1}{25}\left[\lambda^2-\frac{1}{5}\left\{p\,(\sigma_1+\sigma_2)+\frac{1}{5}\,(p\sigma_3+\lambda+1)\right\}\right] \;\ldots\ldots\ldots(P_5).$

[Miss Scott's formulæ are
$$P_2 = -2\lambda,$$
$$P_3 = \quad 2\lambda^2-p\,(\alpha+\beta),$$
$$P_4 = -\lambda^3+p\,\{\theta\,(\alpha+\beta)-\alpha\beta\},$$
where θ, α, β have the values given in the Table on p. 223. Remembering the values of σ_1, σ_2 in terms of the q, viz.,
$$\sigma_1 = q_0, \quad \sigma_2 = q_0^2 + (q_1+q_4)\,(q_2+q_3),$$
there is no difficulty in verifying that these expressions for P are equivalent to those just found.]

Second Form of Period Equation.

The first of the equations (20) may be written

$$Y_0, = 5X_0 + 1 = F\omega + F\omega^2 + F\omega^4 + F\omega^3.$$

By raising this to the 2nd, 3rd, 4th, 5th powers, we get expressions for Y_0^2, Y_0^3, Y_0^4, Y_0^5. In order to form the sums ΣY^i, by reason of (14), we only require the part independent of F in the expansion of the right-hand member.

Now, the absolute term in $(F\omega + F\omega^2 + \dots)^i$ arises from the terms involving

$$F\omega^h \,.\, F\omega^k \dots,$$

where $$h + k + \quad \equiv 0 \text{ (mod. 5)},$$

and every such product of F is replaced, by means of (5), (6), (8), by a product of some of the quantities p, $q\omega$, $q\omega^2$, $q\omega^4$, $q\omega^3$. In this way, and using (14), we obtain the equations

$$\Sigma Y = 0,$$

$$\Sigma Y^2 = 20p,$$

$$\Sigma Y^3 = 15p\, \Sigma q\omega,$$

$$\Sigma Y^4 = 180p^2 + 20p\, \Sigma q\omega\, q\omega^2,$$

$$\Sigma Y^5 = 250p^2 \Sigma q\omega + 5p\, \Sigma q\omega\, q\omega\, q\omega^2$$

From these power-sums it is easy to find the coefficients of the equation whose roots are the Y, viz.,

$$P_1' = 0,$$

$$P_2' = -10p,$$

$$P_3' = -\, 5p\, \Sigma q\omega,$$

$$P_4' = 5p^2 - 5p\, \Sigma q\omega\, q\omega^2,$$

$$P_5' = -p\, \Sigma q\omega\, q\omega\, q\omega^2.$$

Hence the period equation may be written

$$(5\eta+1)^5 - 10p\, (5\eta+1)^3 - 5p\, \Sigma q\omega\, (5\eta+1)^2$$
$$+ 5p\, (p - \Sigma q\omega\, q\omega^2)(5\eta+1) - p\, \Sigma q\omega\, q\omega\, q\omega^2 = 0.$$

If herein we replace $\Sigma q\omega$, $\Sigma q\omega q\omega^2$, $\Sigma q\omega q\omega q\omega^2$ by their equivalents

$5\sigma_1+1$, $5\sigma_2-1$, $5\sigma_3+1$, and expand the binomials, we again obtain the values of P_2, P_3, P_4, P_5 given above.

Inverse Problem.

It is a somewhat interesting problem to find q given the period equation; for the process ought to indicate the order as well as the values of the coordinates.

From the values of P_2, P_3, P_4, P_5, we at once obtain p, σ_1, σ_2, σ_3 in succession, and without ambiguity.

Next we find the two sums q_1+q_4, q_2+q_3 as the two roots of a quadratic equation

$$Q^2+(1+\sigma_1)\,Q+\sigma_2 = \sigma_1^2 \quad\text{......................}(22).$$

To complete the determination we have only the equation for σ_3, and two of the quadric relations (11). By using the latter system, we get equations such as

$$\left.\begin{aligned} 8\lambda+2\sigma_2-5\sigma_1^2-2\sigma_1+1 &= (Q_1-2q_1)^2+(Q_2-2q_2)^2 \\ -8\lambda-6\sigma_2+11\sigma_1^2+6\sigma_1+1 &= (Q_1+2q_1)^2+(Q_2+2q_2)^2 \end{aligned}\right\} \text{......}(23),$$

where $Q_1, = q_1+q_4$, and $Q_2, = q_2+q_3$, are the roots of the equation (22). If, then, the numbers on the right be expressed as the sum of two squares in every possible way, we get a number of solutions, from amongst which the proper one is selected by using the σ_3 equation. It goes without saying, that the equations such as (23) indicate relations between the σ and λ; for instance, σ_2 cannot be $\equiv 3 \pmod{4}$.

One of the properties of the coordinates deserves notice, as it suggests a simple way of arranging the table. The four coordinates q_1, q_2, q_1, q_3, so far as they appear in the annexed table, are divisible into two distinct groups. In the first and larger of these, the five numbers $\frac{1}{2}\lambda, = \frac{1}{10}(p-1)$, q_1, q_2, q_4, q_3 are all of different classes with respect to 5. If we arrange them so that $q_1 \equiv \frac{1}{2}\lambda+1$, then each coordinate $q_i \equiv \frac{1}{2}\lambda+i$ and $q_4 \equiv q_0$. This order is adopted in the table for those values of p to which the process is applicable.

The second group of coordinates comprises those belonging to $p=211$, 281, 421, 461, 521, 691, 881, 991 (marked with an asterisk in the table). In this group all the numbers $\frac{1}{2}\lambda$, q_1, q_2, q_4, q_3 have the same remainder when divided by 5. The coordinate q_0 is excepted, for $q_0 \equiv \frac{1}{2}\lambda-1$ always (for otherwise P_3 would not be integral). In this case I have not found a satisfactory mode of selecting q_1.

TABLE OF q, σ.

p	q_1	q_2	q_4	q_3	$q_0=\sigma_1$	σ_2	σ_3
11	2	− 2	0	− 1	0	− 6	− 18
31	− 1	0	2	− 4	2	0	− 82
41	0	− 4	3	2	− 2	− 2	196
61	2	3	0	− 6	0	− 6	222
71	− 2	− 1	6	0	− 4	12	20
101	6	2	− 1	− 2	− 6	36	54
131	− 6	0	7	− 4	2	0	− 802
151	− 4	2	− 6	8	− 1	− 99	119
181	− 1	−10	2	6	2	0	338
191	0	− 9	− 2	2	8	78	266
*211	− 9	6	− 4	6	0	−156	192
241	0	6	−12	2	3	− 87	− 669
251	− 4	12	− 6	− 2	− 1	− 99	− 1801
271	8	− 6	− 9	0	6	42	− 1920
*281	−12	3	− 2	8	2	−150	− 1132
311	7	8	0	− 6	−10	114	872
331	− 6	0	2	− 9	12	180	1308
401	− 9	2	14	− 2	− 6	36	294
*421	12	2	2	−13	− 4	−138	6350
431	−16	0	2	6	7	− 35	− 1937
*461	− 4	− 4	16	− 9	0	−156	− 6108
491	15	− 4	−12	2	− 2	− 2	− 1244
*521	2	2	−18	7	6	−108	3530
541	0	− 9	8	12	−12	168	2796
571	−12	14	6	0	− 9	− 3	8895
601	− 4	2	14	3	−16	306	− 3346
631	− 1	0	12	6	−18	390	− 6252
641	−20	6	8	2	3	− 87	6051
661	−13	8	−10	14	0	−506	− 1398
*691	− 6	− 1	14	−16	8	− 72	−13084
701	16	2	− 6	3	−16	306	− 2266
751	− 4	−18	− 1	8	14	246	− 76
761	12	8	0	− 1	−20	484	− 8768
811	2	−12	−10	− 1	20	504	8672
821	3	14	− 4	−20	6	42	− 1080
*881	−22	− 2	3	8	12	30	− 6402
911	2	8	−20	− 6	15	189	− 6873
941	15	6	− 2	2	−22	588	−12034
971	− 2	24	− 4	−10	− 9	− 3	− 3585
*991	24	− 1	− 6	− 6	−12	18	2106

TABLE OF COEFFICIENTS OF THE PERIOD EQUATIONS.

p	P_2	P_3	P_4	P_5
11	− 4	− 3	3	1
31	− 12	− 21	1	5
41	− 16	5	21	− 9
61	− 24	− 17	41	− 13
71	− 28	37	25	1
101	− 40	93	− 21	− 17
131	− 52	− 89	109	193
151	− 60	− 12	784	128
181	− 72	− 123	223	− 49
191	− 76	− 359	− 437	− 155
211	− 84	− 59	1661	269
241	− 96	− 212	1232	512
251	−100	− 20	1504	1024
271	−108	− 401	− 13	845
281	−112	− 191	2257	967
311	−124	535	− 413	539
331	−132	− 887	− 1843	− 1027
401	−160	369	879	− 29
421	−168	219	3853	− 3517
431	−172	− 724	1824	1728
461	−184	− 129	4551	5419
491	−196	59	2019	1377
521	−208	− 771	4143	− 2085
541	−216	1147	− 805	− 2629
571	−228	868	3056	− 7552
601	−240	1755	− 3731	2399
631	−252	2095	− 4985	5069
641	−256	− 564	5328	− 5120
661	−264	− 185	16837	4851
691	−276	−1299	5329	15581
701	−280	2047	− 3791	1699
751	−300	−2313	− 3761	− 571
761	−304	2831	− 8925	8775
811	−324	−3471	−12431	−13603
821	−328	−1215	3573	2179
881	−352	−2361	4257	9967
911	−364	−2988	− 1392	9856
941	−376	3877	−13445	15271
971	−388	1476	8304	7168
991	−396	2101	8039	− 1819

I take the opportunity of noting the following corrections to Prof. Cayley's trisection table (𝔄, p. 10),

p	
13	the coefficient of η^0 should be $+1$
37	$a, b, c = -10, -7, -8$
61	$b, c = -15, -12$
97	$b, c = -22, -23$

[*Postscript* (Added July, 1887).

Prof. Cayley remarks, "that $q\omega$ should be deducible from the product of two of the four prime factors of p as given in Reuschle's *Tafeln* (Berlin, 1875), p. 3. Thus, $p=11$, $f\omega\,.\,f\omega^2=(\omega^3+2\omega^4)(\omega^4+2\omega^3)$ which, subtracting $2(1+\omega+\omega^2+\omega^3+\omega^4)$, is $=2\omega^2-2\omega^4-\omega$, and, writing ω^3 for ω, this becomes $2\omega-2\omega^2-\omega^3$, which is the $q\omega$ of the memoir; but possibly in other cases the necessity of introducing complex units might interfere with the easy identification of the results."

The method of calculating $q\omega$ here suggested by Prof. Cayley is so easy in working, that it is important to show that it gives the desired result in all cases and without ambiguity.

Suppose $f\omega$ a prime factor of p; then we have

$$q\omega^i = u\omega\,.\,f\omega\,.\,f\omega^2,$$

where $u\omega, = u_0+u_1\omega+u_2\omega^2+u_4\omega^4+u_3\omega^3$, is a complex unit. Multiply each side of this equation by its conjugate (ω^{-1} written for ω). Then, since

$$q\omega^i\,.\,q\omega^{-i} = p = f\omega\,.\,f\omega^2\,.\,f\omega^4\,.\,f\omega^3,$$

we get

$$\begin{aligned} 1 &= u\omega\,.\,u\omega^{-1} \\ &= (u_0+u_1\omega+\ldots)(u_0+u_1\omega^{-1}+\ldots) \\ &= (u_0^2+u_1^2+u_2^2+u_4^2+u_3^2)+(u_1u_0+\ldots)\,\omega+\&\text{c.} \end{aligned}$$

Since u_0, u_1, &c. are real integers, it follows, from comparing the absolute terms, that one of $u^2=1$, and that all the rest vanish. The coefficient of each power of ω in the expansion of $u\omega\,.\,u\omega^{-1}$ then vanishes without further condition, so that

$$u\omega = \pm\,\omega^n,\ (n = 0, 1, 2, 3, 4).$$

Hence $$q\omega^i = \pm\,\omega^n\,.\,f\omega\,.\,f\omega^2+m\,(1+\omega+\omega^2+\omega^4+\omega^3)\ \ldots\ldots\ldots(24).$$

It is to be proved that m, n and the $\pm$ can always be chosen, and that in one way only, so that the expression on the right of (24) may be identical with one of the four numbers, $q\omega^i$.

Note that the sum of the coordinates of $f\omega \, . \, f\omega^2$ is a square number (the square of the sum of the coordinates of $f\omega$), and it is therefore $\equiv \pm 1$ (mod. 5). Hence we can choose m, and a sign from the $\pm$ in (24), so as to get a complex integer, the sum of whose coordinates is -1. At each stage there is only one proper choice. (24) now takes the form

$$q\omega^i = \omega^n \, (a + b\omega + c\omega^2 + e\omega^4 + d\omega^8) \quad \text{................} (25),$$

where a, b, c, d, e are known integers, such that

$$a + b + c + d + e = -1 \quad \text{.........................} (7').$$

Hence the residues of a, b, c, d, e to modulus 5 cannot be all equal nor all unequal, for in either case their sum would be $\equiv 0$. Two at least of these coordinates have, then, the same residue, say r; and all the essentially distinct sets of residues which are consistent with (7′) are shown in the following Table (omitting the first column), where each row gives a set of residues,

$\tfrac{1}{2}\lambda$,	r,	r,	r,	r,	$r+4$,
$\tfrac{1}{2}\lambda+2$,	r,	r,	r,	$r+1$,	$r+3$,
$\tfrac{1}{2}\lambda+1$,	r,	r,	r,	$r+2$,	$r+2$,
$\tfrac{1}{2}\lambda$,	r,	r,	$r+1$,	$r+1$,	$r+2$,
$\tfrac{1}{2}\lambda-1$,	r,	r,	$r+2$,	$r+3$,	$r+4$.

But, since $(a + b\omega + \dots)$ is one of two conjugate factors of p, its coordinates must satisfy the equations (*cf.* p. 217),

$$\left.\begin{aligned} ab + bc + cd + de + ea &= -\lambda \\ ac + ce + eb + bd + da &= -\lambda \\ a^2 + b^2 + c^2 + d^2 + e^2 &= 4\lambda + 1 \end{aligned}\right\} \quad \text{.................} (11').$$

The third of these equations shows that the r of the first row of the above Table must be $\equiv 3\lambda$ (which $\equiv \tfrac{1}{2}\lambda \equiv$ the tens-figure in p), and similarly for each of the other rows r is the residue of the number written in the first column.

The first set of residues in the Table comprises four congruent to $\tfrac{1}{2}\lambda$, and a fifth congruent to $\tfrac{1}{2}\lambda - 1$. Since q_0 is known to be congruent to $\tfrac{1}{2}\lambda - 1$, we must take for q_0 that one of the coordinates a, b, ... which is congruent to $\tfrac{1}{2}\lambda - 1$; and n, in (25), must be such as to make this coordinate the absolute term. The determination of $q\omega^i$ is in this case complete, The residues in the second, third, and fourth rows of Table include none that are congruent to $\frac{1}{2}\lambda-1$; so that there is no coordinate that can be taken for q_0. I find, however, that coordinates with these residues cannot occur on the right of (25), as they do not satisfy the first two equations (11′) in whatever way they are arranged.

In the case corresponding to the last row of the Table, there are two coordinates (call them a', e') congruent to $\frac{1}{2}\lambda-1$, and we have to decide which is the right one to take for q_0. Suppose it is a', and, when n is fixed accordingly, let $e'\omega^j$ be the term involving e', so that j is known. Now write $-j$ for the undetermined index i, in (25), and then change ω^{-j} into ω. Thus we get

$$q\omega = a'+b'\omega+c'\omega^2+e'\omega^4+d'\omega^8,$$

where a', b', c', d', e' are a, b, c, d, e in a different order, but still satisfying (7′), (11′); and $a'\equiv e'\equiv\frac{1}{2}\lambda-1$. The second (improper) form is easily found to be

$$\omega^4 q\omega^4 = e'+d'\omega+c'\omega^2+a'\omega^4+b'\omega^8;$$

and, to distinguish one from the other, we must have further information as to the coordinates b', c', d'. From (7′) and the second equation of (11′), we find

$$c'\equiv\tfrac{1}{2}\lambda+2,$$

but the equations become symmetrical with respect to b', d'. To complete the investigation, we use the equation

$$\sigma_2 = q^2+(q_1+q_4)(q_2+q_3),$$

which is $$\sigma_2 = a^2+(b+e)(c+d)$$

(where the accents have been omitted as unnecessary). Now,

$$a\equiv e\equiv\tfrac{1}{2}\lambda-1,\quad c\equiv\tfrac{1}{2}\lambda+2,$$

so that, supposing $$b\equiv\tfrac{1}{2}\lambda+1,\quad d\equiv\tfrac{1}{2}\lambda+3,$$

we get $$\sigma_2\equiv-\lambda+1.$$

On the alternative hypothesis, viz.,

$$b\equiv\tfrac{1}{2}\lambda+3,\quad d\equiv\tfrac{1}{2}\lambda+1,$$

we should have $$\sigma_2\equiv-\lambda+2.$$

But only the former congruence for σ_2 is consistent with P_4 being integral (see p. 226).

Hence $b\equiv\frac{1}{2}\lambda+1$, $d\equiv\frac{1}{2}\lambda+3$, and the proper form is characterised by the condition

$$d-b, = q_3-q_1, \equiv 2 \pmod{5},$$

if q_4 is the coordinate congruent to q_0.

I note that the properties recorded in the concluding paragraphs of the paper as a result of observation, have now been proved to obtain generally.

As an example of the working, take

$$p = 31, \quad f\omega \,.\, f\omega^2 = (2\omega^2-\omega^4)(2\omega^4-\omega^8) \quad \text{(Reuschle, p. 3)}$$
$$= -2+4\omega+\omega^2-2\omega^3,$$

where sum of coordinates $= 1$. Therefore

$$q\omega^i = \omega^n (2-4\omega-\omega^2+2\omega^3),$$

so that $n = 0$, or 2; therefore

$$q\omega = 2-\ \omega+2\omega^4-4\omega^3, \qquad (n=0,\ i=2)$$

or

$$2-4\omega+2\omega^4-\ \omega^3, \qquad (n=2,\ i=3).$$

Of these, the former is right, since

$$q_3-q_1 = -4+1 = -3 \equiv 2.]$$

March 10*th*, 1887.

Sir JAMES COCKLE, F.R.S., President, in the Chair.

Messrs. A. E. Hough Love, B.A., Fellow of St. John's College, Cambridge, and A. W. Cave, M.A., Oxford High School, were elected Members.

The following communications were made :—

A Metrical Property of Plane Curves: R. Lachlan.

Note on the Weierstrass Functions: A. G. Greenhill.

Second Paper on change of the Independent Variable, with applications to some Functions of the Reciprocant kind: C. Leudesdorf.

Note on Knots: the Treasurer.

The following presents were received:

"Proceedings of the Royal Society," Vol. XLI., No. 250.

"Educational Times," for March.

"Proceedings of the Physical Society," Vol. VIII., Pt. III., Jan., 1887.

"Hints for the Solution of Problems in the Third Edition of Solid Geometry," by Percival Frost, D.Sc., F.R.S.; 8vo, London, 1887.

Royal Irish Academy—Proceedings, "Science," Vol. IV., No. 5; "Polite Literature and Antiquities," Vol. II., No. 7. Transactions—"Science," Vol. XXVIII., Nos. 21 to 25; "Polite Literature and Antiquities," Vol. XXVII., Nos. 6 to 8. "Cunningham Memoirs," Nos. 2 and 3.

"Johns Hopkins University Circulars," Vol. VI., No. 55.

"Bulletin des Sciences Mathématiques," Feb., 1887.

"Annales de l'Ecole Polytechnique de Delft," Tome II., Liv. 3 and 4.

"Atti della R. Accademia dei Lincei—Rendiconti," Vol. III., Fasc. 1 and 2; Roma, 1887.

"Beiblätter zu den Annalen der Physik und Chemie," Index to B. x.

"Mitteilungen der Mathematischen Gesellschaft in Hamburg," No. 7.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," Nos. 27 and 28.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," XL. to LIII.

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Second Paper on Change of the Independent Variable; with applications to some Functions of the Reciprocant kind. By C. LEUDESDORF, M.A.

[*Read March 10th*, 1887.]

In a previous paper (*Proceedings*, Vol. XVII., p. 329) I have shown how any rational integral homogeneous isobaric function of the differential coefficients of y with respect to x can, by means of the operator $e^{-V/t}$, be expressed as a similar function of the differential coefficients of x with respect to y. The variables x and y are supposed to be connected by one equation, and V is the annihilator of pure reciprocants. The objects of the present paper are as follows. Firstly, to investigate the more general problem of the change of the independent variable from x to z in a function of the differential coefficients of y and z with respect to x; the three variables being supposed connected by two equations. Secondly, to develop some of the properties of the differential operators to which the solution of the just mentioned problem leads. Thirdly, to discuss some of the systems of functions of the reciprocant kind which arise in connection with these operators. Some of these functions have, I find, been treated of by M. Halphen in a memoir "Sur les Invariants Différentiels des Courbes Gauches," published in the *Journal de l'Ecole Polytechnique*, XLVII^{e} Cahier, 1880; but his method of approaching the subject is quite