ON THE DISTRIBUTION OF THE POINTS OF UNIFORM CONVERGENCE OF A SERIES OF FUNCTIONS

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1. In my paper on "Non-uniform Convergence," *supra*, p. 89, I showed that, in the general case when the functions $f_1(x), f_2(x), \ldots$ are pointwise discontinuous functions, the points of non-uniform convergence of the series $F(x) = f_1(x) + f_2(x) + \ldots$ can be divided into two classes, which, for reasons there given, I called *visible* and *invisible* points of non-uniform convergence, and that, while the visible points (which cannot present themselves in the case treated by Prof. Osgood in the *American Journal of Mathematics*, Vol. XIX., when all the functions concerned are continuous) may fill up whole intervals, the invisible points obey Osgood's law of distribution, viz., they form a set which is the limit G of a series of closed sets $G_1, G_2, \ldots$, each of which is dense nowhere, and contains all the preceding sets: it was further shown that, when only a finite number of the functions are discontinuous, the whole set of points of non-uniform convergence, visible and invisible, form such a limiting set G.

From the investigation of the paper first referred to it was not evident whether *any* restriction could be laid on the distribution of the points of non-uniform convergence in the general case, or, which comes to the same thing, on the distribution of the points of uniform convergence. In the present paper I give the restriction in question, and show that *the most general distribution of the points of uniform convergence of a series of pointwise discontinuous functions is when they form what I have elsewhere called an* inner limiting set.

2. An *inner limiting set* was defined as the set of points consisting of all the points internal to intervals of every set of a series of sets of intervals, and it was shown* that, any inner limiting set being given, the defining intervals could be so chosen that (i.) the intervals in each individual set did not overlap, and (ii.) each interval of any set was contained

* "Zur Lehre der nicht abgeschlossenen Punktmengen," *Leipziger Bericht*, Aug. 1, 1903.

in an interval of the preceding set; such defining intervals I called *normal intervals.*

It is at once evident that Osgood's set G is a special case of the complementary set of an inner limiting set, viz., when the normal intervals of each set are dense everywhere; these normal intervals may in fact be taken to be the black intervals * of the closed sets $G_1, G_2, \ldots$.

3. THEOREM 1.—*The points of uniform convergence of a series of pointwise discontinuous functions form an inner limiting set.*

By definition, a series of functions of x is said to be *uniformly convergent* at a point ξ if, given any positive quantity A, however small, an interval δ can be described, having ξ as internal point, so that, for all points x within the interval δ, the residue $R_n(x) < A$, for all values of $n \geqslant m$, where m is an integer, independent of x, which can always be determined.†

Hence, assuming any sequence of small positive quantities, having zero as limit, $A_1, A_2, \ldots$, and describing intervals corresponding, $\delta_{\xi,1}, \delta_{\xi,2}, \ldots$, for each point ξ of uniform convergence of a given series of functions, we get a series of sets of intervals (in general overlapping intervals) whose inner limiting set certainly contains all the points ξ of uniform convergence, Further, if x be any point of this inner limiting set, we can assign one interval from each successive set of intervals, such that x is internal to that interval, so that, corresponding to any assigned A, however small, we have an interval such as is required in order that, by the definition, x should be a point of uniform convergence.

Thus the inner limiting set of the series of sets of intervals so constructed consists of all the points of uniform convergence of the series of functions. Q. E. D.

4. Before proceeding to prove the converse of Theorem 1, a few preliminary remarks will not be out of place.

Let E be any given inner limiting set, and let the sets of defining intervals be denoted by $I_1, I_2, \ldots$.

If the intervals in each of these sets are dense everywhere, Osgood has shown ‡ how to form a series of *continuous* functions whose points of

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* That is, the intervals which have no points of the closed set internal to them, but whose end points are points of the closed set.

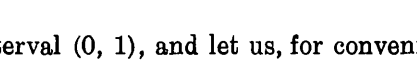
† W. H. Y., *loc. cit.*

‡ *Loc. cit.*, p. 171, Theorem 3. A ξ-point is Osgood's notation for a point of non-unif convergence.

uniform convergence form the set E. If, on the other hand, the above is not the case, we can assign a definite set (and we may without any loss of generality take it to be I_1), in which the intervals are not dense everywhere, so that we can find an interval which is entirely external to every one of the intervals of I_1. In this interval, so found, there are then no points of E; it must therefore lie inside one of the black intervals of the set got by closing E. Thus we see that, if we add to each set I_n the set of black intervals of the set got by closing E, the intervals of each of the new sets, which we may denote by $J_1, J_2, \ldots$, are dense everywhere. We can therefore find an Osgood series having as points of uniform convergence the points of the inner limiting set of $J_1, J_2, \ldots$. It is at once evident that this inner limiting set will consist of the set E together with all the internal points of the black intervals of the set got by closing E. In this way we have succeeded in constructing a series of continuous functions uniformly convergent at all points of E, and at all points internal to the black intervals of the set got by closing E, and non-uniformly convergent at all the remaining points (which are, of course, limiting points of E).

5. From the definition of non-uniform convergence at a point it is evident that, if we add together two series each of which is uniformly convergent at a point P, the resulting series will be uniformly convergent at P; and that, if one of the series is non-uniformly convergent at P and the other uniformly convergent, the resulting series will be non-uniformly convergent at P.

From this we see that, if we can solve the problem of constructing a series non-uniformly convergent at every internal point of a set of non-overlapping intervals and uniformly convergent at all the remaining points of the segment under discussion, this, in conjunction with § 4, will enable us to prove the converse of Theorem 1. As a preliminary we require a solution of the following problem :—



Consider the interval (0, 1), and let us, for convenience, use the binary notation.

Let all the functions $f_1, f_2, \ldots$ be zero always, except at the points to be specified; further, let f_{i+1} have values equal and opposite to those of f_i at all points where f_i is positive; finally, let

f_1	have the value	·1	at the point	·1,
f_2	,,	·01	,,	·01,
		1	,,	·11,
f_3	,,	$\cdot 0^2 1$	,,	·001,
		·01	,,	·011,
		1	,,	·101 and ·111,
f_4	,,	$\cdot 0^3 1$	,,	$\cdot 0^3 1$,
		$\cdot 0^2 1$	,,	$\cdot 0^2 11$,
		·01	,,	·0101 and ·0111,
		1	,,	·1001, ·1011, and ·1111,

and so on, the general rule being as follows :—Let $\cdot N$ denote any binary fraction with n figures; then f_n has at the point $\cdot 0^{n-1}1$ the value $\cdot 0^{n-1}1$, and at any other point $\cdot N$ to the left of the point ·1 the value of f_n is to be that of f_{n-1} at the nearest point on the left of $\cdot N$ where f_{n-1} was positive; while, if N lie to the right of ·1, f_n is to have there the value 1.

To show that this function is uniformly convergent at the origin only, and non-uniformly convergent at every other point of the segment (0, 1), we have to consider that between P and the origin there is a definite point $\cdot N$ with least number n of figures; in any neighbourhood of P on the left there are then, by construction, discontinuities equal to $\cdot N$ of an infinite number of the functions $s_i(x)$, and therefore, since F is continuous, of $R_i(x)$; thus P is a point of non-uniform convergence, whose measure of non-uniformity is $\cdot N$. That the origin is a point of right-handed uniform convergence is evident, since the above mentioned measure of non-uniformity decreases without limit as we approach the origin.

If instead of the segment (0, 1) we have any segment (P, M), we proceed precisely on the same lines. We divide this segment by continued bisection as before, and denote the successive points of division by P_1, P_{01}, P_{11}, P_{001}, ... (as we formerly denoted them by ·1, ·01, ·11, ·001, ...). The value of each function f_i at P_N is then taken to be the same as formerly at the point $\cdot N$.

6. The preceding construction enables us to solve the required problem:—*To construct a series, non-uniformly convergent at the internal points of a set of non-overlapping intervals, and uniformly convergent at all the remaining points.*

If (P, Q) be any one of the given intervals, we bisect it at M, and apply the construction just given to the segments (P, M) and (Q, M) separately, P and Q being the points of uniform convergence, and we multiply all the functions f_i by the length of (P, Q).

Doing this with all the intervals, and making all the functions f_i zero at all the remaining points, we have the desired series. It is, in fact, evident that at all the limiting points of the intervals the series is uniformly convergent, since the measure of non-uniformity in the neighbourhood of such a point decreases without limit, as the lengths of the intervals do so.

7. THEOREM 2.—*Given any inner limiting set E a series of pointwise discontinuous functions can be constructed having the points of E for its points of uniform convergence, and non-uniformly convergent at every other point of the segment under consideration.*

We have, in fact, only to take as our set of intervals the black intervals of the set got by closing E, and apply the construction of the preceding article to form a series of pointwise discontinuous functions non-uniformly convergent throughout those intervals, and uniformly convergent at all their end points and external points. This series we add term by term to the series formed as in § 4. By the reasoning at the beginning of § 5 the series so formed from the two series has the required properties.