On In-and-Circumscribed Polyhedra. *By* Prof. A. R. Forsyth.

[*Read Nov. 9th*, 1882.]

This paper is intended to solve for quadric surfaces a problem which corresponds to the porism of polygons in-and-circumscribed to two conics, the case considered here being that of polyhedra which are circumscribed to one quadric and have pairs of opposite edges lying along the surface of another. It will be seen that the method here adopted is the natural generalisation of that used by Prof. Cayley in the discussion of the porism of polygons in-and-circumscribed to two conics;* the limited case when the conics are both circles had been previously discussed by Füss, Steiner, Jacobi, Richelot, and Minding.†

Consider two quadrics and refer them to their common self-conjugate tetrahedron, then their equations may be taken as

$$U \equiv ax^2+by^2+cz^2+dw^2 = 0,$$

$$V \equiv x^2 + y^2 + z^2 + w^2 = 0,$$

and any quadric through their curves of intersection will be given by

$$U+\xi V = 0 \quad \text{.................................(1)},$$

in which ξ may be looked upon as a parameter defining the surface. Any plane will touch three of the surfaces, say those given by ξ, η, ζ; and η and ζ may be looked upon as parameters of the positions of points lying on the surface ξ, these being the points of contact of tangent planes to ξ. It is not difficult to prove that the equation to such a plane is

$$x\,(\mathrm{a}\,.\,a+\xi\,.\,a+\eta\,.\,a+\zeta)^{\frac{1}{2}}+y\,(\mathrm{b}\,.\,b+\xi\,.\,b+\eta\,.\,b+\zeta)^{\frac{1}{2}}$$
$$+z\,(\mathrm{c}\,.\,c+\xi\,.\,c+\eta\,.\,c+\zeta)^{\frac{1}{2}}+z\,(\mathrm{d}\,.\,d+\xi\,.\,d+\eta\,.\,d+\zeta)^{\frac{1}{2}} = 0,$$

where a, b, c, d are given by the equations,

$$\left.\begin{array}{r} \mathrm{a}+\ \mathrm{b}+\ \mathrm{c}+\ \mathrm{d} = 0 \\ a\mathrm{a}+b\mathrm{b}+c\mathrm{c}+d\mathrm{d} = 0 \\ a^2\mathrm{a}+b^2\mathrm{b}+c^2\mathrm{c}+d^2\mathrm{d} = 0 \end{array}\right\} \quad \text{......................(2)},$$

* *Phil. Mag.*, Ser. 4, Vols. 5, 6, 7, 1853 and 1854; *Phil. Trans.*, 1861.
† For particular references, *cf.* Cayley's papers just cited.

say,
$$\mathrm{a} = -\,\overline{b-c}\;\overline{c-d}\;\overline{d-b},$$
$$\mathrm{b} = +\,\overline{c-d}\;\overline{d-a}\;\overline{a-c},$$
$$\mathrm{c} = -\,\overline{d-a}\;\overline{a-b}\;\overline{b-d},$$
$$\mathrm{d} = \;\;\overline{a-b}\;\overline{b-c}\;\overline{c-a},$$
or, if
$$b-c = A,\quad c-a = B,\quad a-b = C,$$
$$a-d = F,\quad b-d = G,\quad c-d = H,$$
then
$$\mathrm{a} = AHG,\quad \mathrm{b} = HBF,\quad \mathrm{c} = GFC,\quad \mathrm{d} = ABC,$$

The parameter of the quadric $V = 0$ is ∞; let θ, ϕ be the parameters of a point on it; then the coordinates x', y', z', w' are given by
$$x' : y' : z' : w'$$
$$= (\mathrm{a}\,.\,a+\theta\,.\,a+\phi)^{\frac12} : (\mathrm{b}\,.\,b+\theta\,.\,b+\phi)^{\frac12} : (\mathrm{c}\,.\,c+\theta\,.\,c+\phi)^{\frac12} : (\mathrm{d}\,.\,d+\theta\,.\,d+\phi)^{\frac12}.$$

Suppose now that θ', ϕ' is some other point of V; the coordinates of any point on the straight line joining them may be taken as
$$\lambda\,(\mathrm{a}\,.\,a+\theta\,.\,a+\phi)^{\frac12} + \mu\,(\mathrm{a}\,.\,a+\theta'\,.\,a+\phi'),$$
$$\lambda\,(\mathrm{b}\,.\,b+\theta\,.\,b+\phi)^{\frac12} + \mu\,(\mathrm{b}\,.\,b+\theta'\,.\,b+\phi'),$$
$$\lambda\,(\mathrm{c}\,.\,c+\theta\,.\,c+\phi)^{\frac12} + \mu\,(\mathrm{c}\,.\,c+\theta'\,.\,c+\phi'),$$
$$\lambda\,(\mathrm{d}\,.\,d+\theta\,.\,d+\phi)^{\frac12} + \mu\,(\mathrm{d}\,.\,d+\theta'\,.\,d+\phi'),$$
with the relation $\lambda + \mu = 1$. If such a point lie on the quadric V for values of λ and μ other than zero or unity, then the straight line is obviously a generator. Substituting in the equation, the condition for this is
$$\mathrm{a}\,(a+\theta\,.\,a+\phi\,.\,a+\theta'\,.\,a+\phi')^{\frac12}$$
$$+ \ldots + \mathrm{d}\,(d+\theta\,.\,d+\phi\,.\,d+\theta'\,.\,d+\phi')^{\frac12} = 0.$$
To obtain a rationalised equivalent of this, we may either write, by means of equations (2),
$$(x+\theta\,.\,x+\phi\,.\,x+\theta'\,.\,x+\phi')^{\frac12} = \alpha + \beta x + \gamma x^2,$$
for $x = a$, $x = b$, $x = c$, $x = d$, and then eliminate α, β, and γ from four relations such as
$$\theta + \phi + \theta' + \phi' + (1-\gamma^2)(a+b+c+d) = 2\beta\gamma\,;$$
or we may proceed as follows:—Take θ', ϕ' to be a point on the surface consecutive to θ, ϕ, so that we may write
$$\theta' = \theta + \vartheta,\quad \phi' = \phi + \chi,$$
where ϑ and χ are both small. Then
$$(a+\theta\,.\,a+\phi\,.\,a+\theta'\,.\,a+\phi')^{\frac12} = a+\theta\,.\,a+\phi\left(1+\frac{\vartheta}{a+\theta}\right)^{\frac12}\left(1+\frac{\chi}{a+\phi}\right)^{\frac12}$$
$$= a+\theta\,.\,a+\phi\left[1 + \tfrac12\,\frac{\vartheta}{a+\theta} + \tfrac12\,\frac{\chi}{a+\phi} - \tfrac18\left(\frac{\vartheta}{a+\theta} - \frac{\chi}{a+\phi}\right)^2\right],$$
on expanding and retaining terms up to the second order inclusive;

and the equation to the generator becomes

$$\Sigma \mathrm{a} \,.\, a+\theta \,.\, a+\phi$$
$$+\tfrac{1}{2}\vartheta \Sigma \mathrm{a} \,.\, a+\phi + \tfrac{1}{2}\chi \Sigma \mathrm{a} \,.\, a+\theta + \tfrac{1}{4}\vartheta\chi \Sigma \mathrm{a}$$
$$-\tfrac{1}{8}\vartheta^2 \Sigma \mathrm{a} \frac{a+\phi}{a+\theta} - \tfrac{1}{8}\chi^2 \Sigma \mathrm{a} \frac{a+\theta}{a+\phi} + \text{terms of higher order} = 0,$$

where Σ implies summation for the four letters a, b, c, d. Obviously, by our equations (2), the first term, the terms of the first order, and the term involving $\vartheta\chi$ all disappear; and hence, when we make ϑ, χ small, we have, as the relation connecting them,

$$\vartheta^2 \Sigma \mathrm{a} \frac{a+\phi}{a+\theta} + \chi^2 \Sigma \mathrm{a} \frac{a+\theta}{a+\phi} = 0,$$

or $$\vartheta^2 \Phi \Sigma \mathrm{a} \,.\, a+\phi \,.\, b+\theta \,.\, c+\theta \,.\, d+\theta$$
$$+\chi^2 \Theta \Sigma \mathrm{a} \,.\, a+\theta \,.\, b+\phi \,.\, c+\phi \,.\, d+\phi = 0,$$

where $$\Phi = a+\phi \,.\, b+\phi \,.\, c+\phi \,.\, d+\phi,$$
$$\Theta = a+\theta \,.\, b+\theta \,.\, c+\theta \,.\, d+\theta.$$

Let $$\Delta_1 = abcd, \quad \Delta = ab+ac+ad+bc+bd+cd,$$
$$\Delta_2 = bcd+cda+dab+abc, \quad \Delta_3 = a+b+c+d;$$

then $$bc+cd+db = \Delta - a\Delta_3 + a^2,$$
$$ab+ac+ad = a\Delta_3 - a^2,$$
$$abc+acd+adb = \Delta_2 - bcd.$$

Now $$\Sigma \mathrm{a} \,.\, a+\phi \,.\, b+\theta \,.\, c+\theta \,.\, d+\theta$$
$$= \phi\Sigma\left[\mathrm{a}\left\{bcd+\theta\,(bc+cd+db)+(b+c+d)\,\theta^2+\theta^3\right\}\right]$$
$$+\Sigma\left[\mathrm{a}\left\{abcd+\theta\,(abc+acd+adb)+\theta^2\,(ab+ac+ad)+a\theta^3\right\}\right].$$

Now, in virtue of the identities just written down, and the equations giving the quantities a, the first line on the right-hand side reduces to the term $\phi\,(\mathrm{a}bcd+\mathrm{b}cda+\mathrm{c}dab+\mathrm{d}abc)$,

say to $\phi\,\Sigma\,(\mathrm{a}bcd)$,

and the second line gives $-\theta\,\Sigma\,(\mathrm{a}bcd)$;

so that the equation to the generator is

$$[\vartheta^2\Phi\,(\phi-\theta)+\chi^2\Theta\,(\theta-\phi)] \,.\, \Sigma\,(\mathrm{a}bcd) = 0.$$

Now $\Sigma\,(\mathrm{a}bcd)$ does not vanish, and θ and ϕ, being parameters of different quadrics, are unequal; and hence we have

$$\vartheta^2\Phi - \chi^2\Theta = 0,$$

or, writing $$\vartheta = d\theta,$$
$$\chi = d\phi,$$

this is equivalent to $\left(\frac{d\theta}{\sqrt{\Theta}}\right)^2 - \left(\frac{d\phi}{\sqrt{\Phi}}\right)^2 = 0,$

which is, of course, equivalent to the two

$$\frac{d\theta}{\sqrt{\Theta}} - \frac{d\phi}{\sqrt{\Phi}} = 0,$$

$$\frac{d\theta}{\sqrt{\Theta}} + \frac{d\phi}{\sqrt{\Phi}} = 0,$$

and these are the differential equations to the generators on the surface $V = 0$.* There are, as it is obvious there should be, two distinct equations corresponding to the two distinct systems.

Consider now a tetrahedron, two pairs of opposite edges of which are formed by a pair of generators from each system, and the remaining edges by lines joining the intersections of generators of different systems. Let (Fig. 1) A, B, C, D be respectively the points $\theta_1\phi_1$, $\theta_2\phi_2$, $\theta_3\phi_3$, $\theta_4\phi_4$; and suppose AB and CD belong to the first equation, and BC and DA to the second. Then writing

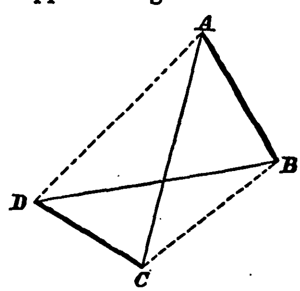

Fig. 1.

$$\int \frac{d\theta}{\sqrt{\Theta}} = u, \quad \int \frac{d\phi}{\sqrt{\Phi}} = v,$$

we have $u-v$ constant along AB and CD,

$u+v$ " " AD and BC,

so that

$$u_1 - v_1 = u_2 - v_2,$$
$$u_3 - v_3 = u_4 - v_4,$$
$$u_3 + v_3 = u_2 + v_2,$$
$$u_1 + v_1 = u_4 + v_4;$$

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* See Cayley "On Geodesic Lines, in particular those of a Quadric Surface," *Proc. Lond. Math. Soc.*, t. iv., 1872, p. 199: viz., equation of surface being

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1,$$

and expressions of coordinates in terms of parameters p and q being

$$-\beta\gamma\, x^2 = a \,.\, a+p \,.\, a+q,$$
$$-\gamma a\, y^2 = b \,.\, b+p \,.\, b+q,$$
$$-a\beta\, z^2 = c \,.\, c+p \,.\, c+q,$$

then differential equation of right lines is

$$\frac{dp}{\sqrt{a+p \,.\, b+p \,.\, c+p}} + \frac{dq}{\sqrt{a+q \,.\, b+q \,.\, c+q}} = 0.$$

from which we at once obtain

$$u_1+u_3=u_2+u_4 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A),$$

$$v_1+v_3=v_2+v_4 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(B).$$

Now, by Abel's theorem, the algebraical equivalent of (A) is known to be

$$\begin{vmatrix} 1, & \theta_1, & \theta_1^2, & \sqrt{\Theta_1} \\ 1, & \theta_2, & \theta_2^2, & \sqrt{\Theta_2} \\ 1, & \theta_3, & \theta_3^2, & \sqrt{\Theta_3} \\ 1, & \theta_4, & \theta_4^2, & \sqrt{\Theta_4} \end{vmatrix}=0.$$

The plane face DAB is the plane containing two generators, and is therefore the tangent plane at A to V, and by the parameters of A it is known to touch the quadric θ_1. Moreover Θ_1, being the discriminant of

$$U+\theta_1 V=0,$$

is an invariant, and hence we need no longer suppose the quadrics referred to their common self-conjugate tetrahedron. If we write

$$\Delta'=abcd,\quad \Phi=bc+ca+ab+ad+bd+cd,\quad \Delta=1,$$

$$\Theta'=abc+bcd+cda+dab,\quad \Theta=a+b+c+d,$$

quantities which are invariants, Θ_1 being the discriminant of $U+\theta_1 V$ will, in the most general case, be

$$\Delta'+\Theta'\theta_1+\Phi\theta_1^2+\Theta\theta_1+\Delta\theta_1^4.$$

Writing this $\square\,\theta_1$, we have the theorem :—

If a tetrahedron be described having two pairs of opposite edges lying on the surface of the quadric $V=0$, and its plane faces touching the respective quadrics

$$U+\theta_1 V=0,$$

$$U+\theta_2 V=0,$$

$$U+\theta_3 V=0,$$

$$U+\theta_4 V=0,$$

then the necessary and sufficient relation between the parameters θ is

$$\begin{vmatrix} 1, & \theta_1, & \theta_1^2, & \sqrt{\square\theta_1} \\ 1, & \theta_2, & \theta_2^2, & \sqrt{\square\theta_2} \\ 1, & \theta_3, & \theta_3^2, & \sqrt{\square\theta_3} \\ 1, & \theta_4, & \theta_4^2, & \sqrt{\square\theta_4} \end{vmatrix}=0,$$

in which $\square\theta$ denotes the discriminant of the quadric

$$U+\theta V=0.$$

To find what this theorem becomes when $\theta_1, \theta_2, \theta_3, \theta_4$ are all zero, so that the faces of the tetrahedron all touch the quadric $U=0$, we must expand $\sqrt{\Box\theta_1}$ in powers of θ_1; then, if

$$\sqrt{\Box\theta_1} = A_0 + A_1\theta_1 + \ldots + A_n\theta_1^n + \ldots,$$

the foregoing determinant becomes divisible by

$$\begin{vmatrix} 1, & \theta_1, & \theta_1^2, & \theta_1^3 \\ 1, & \theta_2, & \theta_2^2, & \theta_2^3 \\ 1, & \theta_3, & \theta_3^2, & \theta_3^3 \\ 1, & \theta_4, & \theta_4^2, & \theta_4^3 \end{vmatrix},$$

and leaves as the required condition

$$A_3 = 0.$$

We shall afterwards return to this condition.

A tetrahedron is a solid proper, *i.e.*, one which has all its solid angles contained by the same number of plane angles, and all its faces bounded by the same number of straight lines, each of these numbers in the case already considered being three. In order that such a solid may have edges lying along the generators of a quadric, it must have triangles for its faces, since there are only two generators of a quadric passing through any point on it; and therefore the only other solids which can be so described are the octahedron and the icosahedron.

We proceed to consider first the octahedron. Let the angular points be denoted by 1, 2, 3, 4, 5, 6 (Fig. 2), and let the edges 12, 23, 34, 45, 56, 61 lie along generators of the quadric V; then any point, as 1, may be denoted by ∞, θ_1, ϕ_1. Let the thick lines belong to one system, the dotted to the other; then, by joining 15, 53, 31, 42, 26, 64 by the thin lines, we have an octahedron which has three central planes 1643, 3265, 5421.

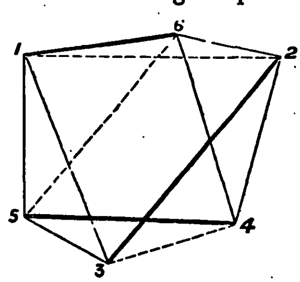

Fig. 2.

Then, adopting the same notation as in the case of the tetrahedron, we have

$$\begin{aligned} u_1 - v_1 &= u_6 - v_6 \text{ along } 16, \\ u_3 - v_3 &= u_2 - v_2 \;\ldots\ldots\; 23, \\ u_5 - v_5 &= u_4 - v_4 \;\ldots\ldots\; 45, \\ u_1 + v_1 &= u_2 + v_2 \;\ldots\ldots\; 12, \\ u_3 + v_3 &= u_4 + v_4 \;\ldots\ldots\; 34, \\ u_5 + v_5 &= u_6 + v_6 \;\ldots\ldots\; 56; \end{aligned}$$

and these equations at once give

$$u_1+u_3+u_5 = u_2+u_4+u_6,$$

$$v_1+v_3+v_5 = v_2+v_4+v_6.$$

Taking either of these equations, say the first, its algebraical equivalent is, by Abel's theorem,

$$\{1,\ \theta,\ \theta^2,\ \theta^3,\ \sqrt{\Box\theta},\ \theta\sqrt{\Box\theta}\} = 0,$$

where the expression { } denotes the determinant of six lines formed by substituting for θ the values $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$. Moreover, $\Box\theta$ is the discriminant of $U+\theta V$ in the forms assumed for their equations, but, being an invariant, it may be replaced by the same value as before, viz.,

$$\Delta'+\Theta'\theta+\Phi\theta^2+\Theta\theta^3+\Delta\theta^4,$$

and U, V may have the most general forms possible. Hence we have the theorem :—

If an octahedron be described having three pairs of opposite edges lying on the surface of a quadric V, and, of its plane faces formed by these edges, five touch the respective quadrics

$$U+\theta_\mu V = 0$$

($\mu = 1, 2, 3, 4, 5$), then the sixth will touch the quadric

$$U+\theta_6 V = 0,$$

where θ_6 is given by the equation

$$\{1,\ \theta,\ \theta^2,\ \theta^3,\ \sqrt{\Box\theta},\ \theta\sqrt{\Box\theta}\} = 0$$

(the determinant of six rows obtained by substituting for θ the values $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$), in which $\Box\theta$ denotes the discriminant of

$$U+\theta V = 0.$$

To find the condition that all these six faces may touch the same quadric $U = 0$, so that $\theta_1 = \ldots\ldots = \theta_6 = 0$, we must expand $\sqrt{\Box\theta}$; taking the same value as before, the determinant divides by

$$\{1,\ \theta,\ \theta^2,\ \theta^3,\ \theta^4,\ \theta^5\},$$

and leaves as the required condition

$$\begin{vmatrix} A_3, & A_4 \\ A_4, & A_5 \end{vmatrix} = 0,$$

or

$$A_3 A_5 = A_4^2.$$

Now, consider the icosahedron which has 12 angular points and 30 edges. Since each edge joins two angular points, and through any point on the quadric V only two generators can be drawn, it follows that there will be twelve edges of the solid lying on the surface, six of them being generators of one system, and six of the other. Of the accompanying figures, Figure 3 shows these on the surface of the

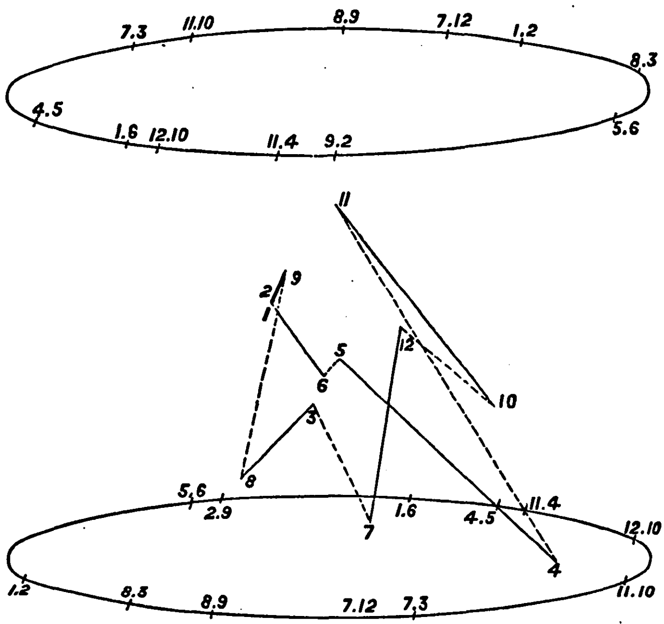

Fig. 3.

quadric (assumed a hyperboloid of one sheet), the two sections being equidistant from and parallel to the principal section, the numbers at each generator being the numbers of the angular points which lie on it; and Figure 4 is an icosahedron, more regular in form, with the angular points numbered exactly as in Figure 3, and the remaining edges formed by the thin lines joining the angular points. The dark and dotted lines indicate, as before, generators belonging to the respective systems.

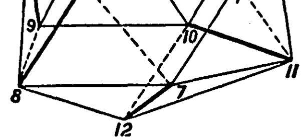

Fig. 4.

Denoting the point μ by ∞, θ_r, ϕ_r, we have along

$$\begin{aligned}
1.2 &\;\ldots\ldots\; u_1+v_1=u_2+v_2,\\
5.6 &\;\ldots\ldots\; u_5+v_5=u_6+v_6,\\
8.9 &\;\ldots\ldots\; u_9+v_9=u_8+v_8,\\
3.7 &\;\ldots\ldots\; u_3+v_3=u_7+v_7,\\
10.12 &\;\ldots\ldots\; u_{12}+v_{12}=u_{10}+v_{10},\\
11.4 &\;\ldots\ldots\; u_{11}+v_{11}=u_4+v_4,\\
1.6 &\;\ldots\ldots\; u_1-v_1=u_6-v_6,\\
2.9 &\;\ldots\ldots\; u_9-v_9=u_2-v_2,\\
4.5 &\;\ldots\ldots\; u_5-v_5=u_4-v_4,\\
3.8 &\;\ldots\ldots\; u_3-v_3=u_8-v_8,\\
12.7 &\;\ldots\ldots\; u_{12}-v_{12}=u_7-v_7,\\
11.10 &\;\ldots\ldots\; u_{11}-v_{11}=u_{10}-v_{10};
\end{aligned}$$

from which we have

$$u_1+u_3+u_5+u_9+u_{11}+u_{12}=u_2+u_4+u_6+u_7+u_8+u_{10},$$

and the same equation in the v's. Again applying Abel's theorem, the algebraical equivalent of this is

$$\{1,\theta,\ldots,\theta^6,\sqrt{\Box\theta},\ldots,\theta^4\sqrt{\Box\theta}\}=0,$$

where the expression $\{\ \}$ denotes the determinant of 12 rows obtained by substituting for θ the values $\theta_1, \theta_2, \ldots, \theta_{12}$. Replacing, as before, the particular form of $\Box\theta$ by the discriminant of $U+\theta V=0$ when U and V are the general equations, we have the above as the condition that an icosahedron, having six pairs of opposite edges lying on the surface of the quadric $V=0$, should have its twelve faces formed by these edges respectively touching the quadrics

$$U+\theta_\mu V=0,$$

where $\mu=1, 2, \ldots, 12$.

When these faces all touch $U=0$, the condition becomes

$$\begin{vmatrix} A_3, & A_4, & A_5, & A_6, & A_7 \\ A_4, & A_5, & A_6, & A_7, & A_8 \\ A_5, & A_6, & A_7, & A_8, & A_9 \\ A_6, & A_7, & A_8, & A_9, & A_{10} \\ A_7, & A_8, & A_9, & A_{10}, & A_{11} \end{vmatrix}=0,$$

the determinant having been divided by

$$\{1,\theta,\theta^2,\ldots,\theta^{11}\}.$$

We have written

$$(\Delta'+\Theta'\theta+\Phi\theta^2+\Theta\theta^3+\Delta\theta^4)^{\frac{1}{2}}=A_0+A_1\theta+\ldots+A_n\theta^n+\ldots,$$

and the conditions requisite for each case have been expressed in terms of the coefficients on the right-hand side. The values of these, as far as A_{11}, are as follow :—

$$A_0\Delta'^{-\frac{1}{2}} = 1,$$

$$A_1\Delta'^{\frac{1}{2}} = \frac{1}{2}\Theta',$$

$$A_2\Delta'^{\frac{3}{2}} = \frac{1}{2}\Phi\Delta' - \frac{1}{8}\Theta'^2,$$

$$A_3\Delta'^{\frac{5}{2}} = \frac{1}{2}\Theta\Delta'^2 - \frac{1}{4}\Theta'\Delta'\Phi + \frac{1}{16}\Theta'^3,$$

$$A_4\Delta'^{\frac{7}{2}} = \frac{1}{2}\Delta\Delta'^3 - \frac{1}{8}\Phi^2\Delta'^2 - \frac{1}{4}\Theta\Theta'\Delta'^2 + \frac{3}{16}\Theta'^2\Phi\Delta' - \frac{5}{128}\Theta'^4,$$

$$A_5\Delta'^{\frac{9}{2}} = -\frac{1}{4}\Theta'\Delta\Delta'^3 - \frac{1}{4}\Phi\Theta\Delta'^3 + \frac{3}{16}\Theta'^2\Theta\Delta'^2 + \frac{3}{16}\Phi^2\Theta'\Delta'^2$$
$$- \frac{5}{32}\Theta'^3\Phi\Delta' + \frac{7}{256}\Theta'^5,$$

$$A_6\Delta'^{\frac{11}{2}} = -\frac{1}{8}\Theta^2\Delta'^4 - \frac{1}{4}\Delta\Phi\Delta'^4 + \frac{1}{16}\Phi^3\Delta'^3 + \frac{3}{16}\Theta'^2\Delta\Delta'^3 + \frac{3}{8}\Theta\Theta'\Phi\Delta'^3$$
$$- \frac{5}{32}\Theta\Theta'^3\Delta'^2 - \frac{15}{64}\Theta'^2\Phi^2\Delta'^2 + \frac{35}{256}\Theta'^4\Phi\Delta' - \frac{21}{1024}\Theta'^6,$$

$$A_7\Delta'^{\frac{13}{2}} = -\frac{1}{4}\Theta\Delta\Delta'^5 + \frac{3}{16}\Phi^2\Theta\Delta'^4 + \frac{3}{16}\Theta^2\Theta'\Delta'^4 + \frac{3}{8}\Phi\Theta'\Delta\Delta'^4$$
$$- \frac{5}{32}\Theta'^3\Delta\Delta'^3 - \frac{5}{32}\Theta'\Phi^3\Delta'^3 - \frac{15}{32}\Theta'^2\Phi\Theta\Delta'^3 + \frac{35}{256}\Theta'^4\Theta\Delta'^2$$
$$+ \frac{35}{128}\Theta'^3\Phi^2\Delta'^2 - \frac{63}{512}\Theta'^5\Phi\Delta' + \frac{33}{2048}\Theta'^7,$$

$$A_8\Delta'^{\frac{15}{2}} = -\frac{1}{8}\Delta^2\Delta'^6 + \frac{3}{16}\Phi^2\Delta\Delta'^5 + \frac{3}{16}\Theta^2\Phi\Delta'^5 + \frac{3}{8}\Theta\Theta'\Delta\Delta'^5 - \frac{15}{32}\Theta'^2\Phi\Delta\Delta'^4$$
$$- \frac{15}{64}\Theta^2\Theta'^2\Delta'^4 - \frac{15}{32}\Theta\Theta'\Phi^2\Delta'^4 - \frac{5}{128}\Phi^4\Delta'^4 + \frac{35}{256}\Theta'^4\Delta\Delta'^3$$
$$+ \frac{35}{64}\Theta'^3\Phi\Theta\Delta'^3 + \frac{35}{128}\Theta'^2\Phi^3\Delta'^3 - \frac{63}{512}\Theta'^5\Theta\Delta'^2 - \frac{315}{1024}\Theta'^4\Phi^2\Delta'^2$$
$$+ \frac{231}{2048}\Theta'^6\Phi\Delta' - \frac{429}{32768}\Theta'^8,$$

$$A_9\Delta'^{\frac{17}{2}} = \frac{1}{16}\Theta^3\Delta'^6 + \frac{3}{16}\Delta^2\Theta'\Delta'^6 + \frac{3}{8}\Phi\Theta\Delta\Delta'^6 - \frac{15}{32}\Theta'^2\Theta\Delta\Delta'^5$$
$$- \frac{15}{32}\Theta'\Phi^2\Delta\Delta'^5 - \frac{15}{32}\Theta'\Phi\Theta^2\Delta'^5 - \frac{5}{32}\Phi^3\Theta\Delta'^5 + \frac{35}{228}\Theta'^3\Theta^2\Delta'^4$$
$$+ \frac{35}{64}\Theta'^3\Phi\Delta\Delta'^4 + \frac{105}{128}\Theta'^2\Phi^2\Theta\Delta'^4 + \frac{35}{256}\Theta'\Phi^4\Delta'^4 - \frac{63}{512}\Theta'^5\Delta\Delta'^3$$
$$- \frac{315}{512}\Theta'^4\Phi\Theta\Delta'^3 - \frac{105}{256}\Theta'^3\Phi^3\Delta'^3 + \frac{231}{2048}\Theta'^6\Theta\Delta'^2 + \frac{693}{2048}\Theta'^5\Phi^2\Delta'^2$$
$$- \frac{429}{4096}\Theta'^7\Phi\Delta' - \frac{715}{65536}\Theta'^9,$$

$$A_{10}\Delta'^{\frac{19}{2}} = \frac{3}{16}\Theta^2\Delta\Delta'^7+\frac{3}{16}\Phi\Delta^2\Delta'^7-\frac{15}{64}\Theta'^2\Delta^2\Delta'^6-\frac{15}{64}\Theta'\Phi\Theta\Delta\Delta'^6$$

$$-\frac{5}{32}\Theta'\Theta^3\Delta'^6-\frac{5}{32}\Phi^3\Delta\Delta'^6-\frac{15}{64}\Phi^2\Theta^2\Delta'^6+\frac{35}{64}\Theta'^3\Theta\Delta\Delta'^5$$

$$+\frac{105}{128}\Theta'^2\Phi^2\Delta\Delta'^5+\frac{105}{128}\Theta^2\Theta'^2\Phi\Delta'^5+\frac{35}{64}\Theta'\Phi^3\Theta\Delta'^5+\frac{7}{256}\Phi^5\Delta'^5$$

$$-\frac{315}{1024}\Theta'^4\Theta^2\Delta'^4-\frac{315}{512}\Theta'^4\Phi\Delta\Delta'^4-\frac{315}{256}\Theta'^3\Phi^2\Theta\Delta'^4$$

$$-\frac{315}{1024}\Theta'^2\Phi^4\Delta'^4+\frac{231}{2048}\Theta'^6\Delta\Delta'^3+\frac{693}{1024}\Theta'^5\Theta\Phi\Delta'^3$$

$$+\frac{1155}{2048}\Theta'^4\Phi^3\Delta'^3-\frac{429}{4096}\Theta^7\Theta\Delta'^2-\frac{3003}{8192}\Theta'^6\Phi^2\Delta'^2$$

$$-\frac{6435}{65536}\Theta'^8\Phi\Delta'+\frac{2431}{262144}\Theta'^{10},$$

$$A_{11}\Delta'^{\frac{21}{2}} = \frac{3}{16}\Theta\Delta^2\Delta'^8-\frac{15}{32}\Theta'\Phi\Delta^2\Delta'^7-\frac{15}{32}\Theta'\Theta^2\Delta\Delta'^7-\frac{15}{32}\Phi^2\Theta\Delta\Delta'^7$$

$$-\frac{5}{32}\Phi\Theta^3\Delta'^7+\frac{35}{128}\Theta'^3\Delta^2\Delta'^6+\frac{35}{128}\Theta'^2\Theta^3\Delta'^6+\frac{105}{64}\Theta'^2\Phi\Theta\Delta\Delta'^6$$

$$+\frac{35}{64}\Theta'\Phi^3\Delta\Delta'^6+\frac{105}{128}\Theta'\Phi^2\Theta^2\Delta'^6+\frac{35}{256}\Phi^4\Theta\Delta'^6-\frac{315}{1024}\Theta'^4\Theta\Delta\Delta'^5$$

$$-\frac{315}{256}\Theta'^3\Phi^2\Delta\Delta'^5-\frac{315}{256}\Theta'^3\Phi\Theta^2\Delta'^5-\frac{315}{256}\Theta'^2\Phi^3\Theta\Delta'^5-\frac{63}{256}\Theta'\Phi^5\Delta'^5$$

$$+\frac{693}{1024}\Theta'^5\Phi\Delta\Delta'^4+\frac{693}{2048}\Theta'^5\Theta^2\Delta'^4+\frac{3465}{2048}\Theta'^4\Phi^2\Theta\Delta'^4$$

$$+\frac{1155}{2048}\Theta'^3\Phi^4\Delta'^4-\frac{429}{4096}\Theta^7\Delta\Delta'^3-\frac{3003}{4096}\Theta'^6\Phi\Theta\Delta'^3$$

$$-\frac{3003}{4096}\Theta'^5\Phi^3\Delta'^3-\frac{6435}{65536}\Theta'^8\Theta\Delta'^2-\frac{6435}{16384}\Theta^7\Phi^2\Delta'^2$$

$$+\frac{12155}{131072}\Theta'^9\Phi\Delta'-\frac{4199}{524288}\Theta'^{11}.$$

The condition that a tetrahedron could have two pairs of opposite edges lying on a quadric V and its four faces touching another quadric U was

$$A_3 = 0,$$

or, from the above, $$8\Theta\Delta'^2-4\Theta'\Delta'\Phi+\Theta'^3 = 0,$$

which agrees with the result given by Salmon (*Geometry of Three Dimensions*, § 207).