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Cabinet likenesses of Rev. R. Harley, F.R.S., and Rev. Dr. Logan.

*On Certain Results obtained by means of the Arguments of Points
on a Plane Curve.* BY R. A. ROBERTS, M.A.

[Read Nov. 8th, 1883.]

1. We know that there are certain relations connecting the arguments of collinear points on a plane curve, the argument of a point on the curve being

$$\int \frac{dx}{\frac{dF}{dy}} \quad \text{or} \quad \int \frac{ds}{\sqrt{\left\{ \left(\frac{dF}{dx} \right)^2 + \left(\frac{dF}{dy} \right)^2 \right\}}},$$

where $F(x, y) = 0$ is the equation of the curve in rectangular Cartesian coordinates. But I am not aware whether it has been noticed that, if the line containing the points touch an envelope, the ratios of the differentials of the arguments can be expressed in terms of the distances of the points on the curve from the point of contact with the envelope. By means of this result several geometrical theorems can be arrived at with respect to triangles, or figures of more sides, inscribed in one curve and circumscribed about another.

2. For the cubic, the ratios of the differentials of the arguments of three collinear points on the curve are the anharmonic ratios of the three points and the point where the line touches its envelope. I give a proof of this theorem which can be applied to give similar results for curves of higher degree.

Suppose the cubic, a point A on the curve being origin, to be written in the form

$$U \equiv v_3 + v_2 + v_1 = 0 \dots\dots\dots(1)$$

where V_n denotes the terms of the n^{th} degree in x and y ; then, transforming to polar coordinates, we see that, if a line which makes an angle ϑ with the axes of x meet the curve again in B, C , we must

$$\text{have} \quad AB \cdot AC = \frac{g \cos \vartheta + f \sin \vartheta}{M} \dots\dots\dots(2),$$

where we have put $v_1 = gx + fy$, and M is a function depending only on the direction of the line ABC . But $gx + fy = 0$ is the tangent to

the curve at A ; hence, if ϕ is the angle which ABC makes with this line, we have $g \cos \phi + f \sin \phi = \sqrt{(g^2 + f^2)} \sin \phi$. Now $g^2 + f^2$ is evidently the value of $\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2$ at the origin, and this function, as is well known, is independent of any system of rectangular axes to which the curve is referred. Thus (2) gives

$$AB \cdot AC = \frac{\sin \phi}{M} \sqrt{\left\{ \left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2 \right\}} \dots\dots\dots (3).$$

Hence, if ϕ_1, ϕ_2, ϕ_3 are the angles which the line makes with the curve at A, B, C , respectively, and P_1, P_2, P_3 are the values of

$$\sqrt{\left\{ \left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2 \right\}}$$

corresponding to the same points, we have, from (3),

$$AB \cdot AC = \frac{P_1 \sin \phi_1}{M}, \quad BA \cdot BC = \frac{P_2 \sin \phi_2}{M}, \quad CA \cdot CB = \frac{P_3 \sin \phi_3}{M} \dots\dots (4).$$

Now, if the line ABC touch its envelope at O , it is easy to see, by infinitesimals, that we have

$$OA d\omega = \sin \phi_1 ds_1, \quad OB d\omega = \sin \phi_2 ds_2, \quad OC d\omega = \sin \phi_3 ds_3 \dots\dots (5),$$

where ds_1, ds_2, ds_3 are the elements of the arcs of the curve at A, B, C , respectively, and $d\omega$ is the angle between two consecutive positions of the line.

Hence, if we put $ds_1 = P_1 du_1, ds_2 = P_2 du_2, ds_3 = P_3 du_3$, we obtain, from (4) and (5),

$$\frac{du_1}{OA \cdot BC} = \frac{du_2}{OB \cdot CA} = \frac{du_3}{OC \cdot AB} \dots\dots\dots (6),$$

which gives the result I have stated above.

For curves of the fourth order, we find, in a similar manner,

$$\begin{aligned} & du_1 : du_2 : du_3 : du_4 \\ &= \frac{OA}{AB \cdot AC \cdot AD} : \frac{OB}{BA \cdot BC \cdot BD} : \frac{OC}{CA \cdot CB \cdot CD} : \frac{OD}{DA \cdot DB \cdot DC} \dots\dots (7) \end{aligned}$$

and similar expressions for curves of higher degrees.

3. I proceed to deduce some results from (6). If we take two corresponding points A, B on a cubic, we have

$$u_1 - u_2 = \frac{1}{2} \omega \dots\dots\dots (8),$$

where $\omega \equiv 2mK + 2m'K'$; for, since the tangents at these points intersect at a point u on the curve, we get, from the relation connecting three collinear points, $2u_1 + u = a$, a constant, and $2u_2 + u = a$; hence we get (8). We have, therefore, $du_1 = du_2$, and, from (6), $OA \cdot BC = OB \cdot CA$. We thus obtain the known theorem that the

tangent to the Cayleyan is divided harmonically at the pair of corresponding points on the Hessian, the point of contact, and the point where it meets the Hessian again.

4. Suppose a triangle inscribed in a cubic so that the tangents at the vertices pass through the points where the opposite sides meet the curve again; then, if u_1, u_2, u_3 are the arguments of the vertices, and v_1, v_2, v_3 those of the points where the sides meet the curve again, we have $2u_1 + v_1 = \alpha$, and $u_2 + u_3 + v_1 = \alpha$; therefore $2u_1 - u_2 - u_3 = 0$, or ω , and similarly $2u_2 - u_1 - u_3$ and $2u_3 - u_1 - u_2$ are respectively equal to a multiple of ω . Hence we may take $u_2 - u_3 = \frac{1}{3}\omega$, $u_3 - u_1 = \frac{1}{3}\omega$, $u_1 - u_2 = -\frac{2}{3}\omega$.

Thus we see that these three conditions are not independent, but are only equivalent to two. Hence there are an infinite number of such triangles, and, since from $u_1 - u_2 = \frac{1}{3}\omega$ we have $du_1 = du_2$, a side of one of these triangles is divided harmonically at its point of contact and the three points where it meets the curve.

5. If we take pairs of points P, P' on a cubic so that P, Q are collinear with a fixed point A , and P', Q with a fixed point B , the points A and B being on the curve, then we have $u_1 + v + a = \alpha$, $u_2 + v + b = \alpha$, and therefore $u_1 - u_2 = b - a$, a constant. Hence, as before, we see that the line PP' is divided harmonically at P, P' , the point of contact with its envelope, and the point where it meets the curve again. We can show from this that the envelope, which is of the sixth class, touches the given curve eighteen times. This result includes the two preceding as particular cases.

6. Suppose a curve of the m^{th} degree to have p -pointic contact with a cubic at the point u_1 and to meet the cubic again at a variable point u_2 and $3m - p - 1$ fixed points, then, by Clebsch's theory of the arguments of points on a cubic, we have $pu_1 + u_2 = \alpha$ a constant.

Hence we see, from (6), that the line joining the points u_1, u_2 is divided in a constant anharmonic ratio at these points, the point of contact with its envelope, and the point where it meets the curve again. It may be shown that the class of the envelope is $2(p^2 - p + 1)$.

7. In a paper in No. 189, Vol. xiii., p. 150, of the *Proceedings*, I have remarked that, if we take collinear points on the cuspidal cubic $U \equiv y^3 - x^2z = 0$, whose parameters are subject to the relations

$$\Sigma \mathfrak{g}_1 = 0, (\Sigma \mathfrak{g}_1 \mathfrak{g}_2)^2 + 27k (\mathfrak{g}_1 \mathfrak{g}_2 \mathfrak{g}_3)^3 = 0 \dots\dots\dots (9),$$

then the tangents at these points form a triangle inscribed in the cubic $V \equiv y^3 - kx^2z = 0$. Since from (9) we see that the parameters are in a constant ratio to one another, it follows that the line which touches $ky^3 - x^2z = 0$ is divided in a constant anharmonic ratio at the three points on U and the point of contact. Also, by recipro-

cation, the tangents to V at the vertices pass through a point on the cubic $y^3 - k^3 x^2 z$, and these tangents and the tangent to the latter cubic have a constant anharmonic ratio.

8. If we put $x = \mathfrak{J}y$ in the equation of the cuspidal cubic $U \equiv y^3 - x^2 z = 0$, the coordinates of the intersection of the tangents $\mathfrak{J}_1, \mathfrak{J}_2$ are

$$x = \frac{1}{2} \mathfrak{J}_1 \mathfrak{J}_2 (\mathfrak{J}_1 + \mathfrak{J}_2), \quad y = \frac{1}{2} (\mathfrak{J}_1^2 + \mathfrak{J}_2^2 + \mathfrak{J}_1 \mathfrak{J}_2), \quad z = 1.$$

Hence if $\mathfrak{J}_1 - \mathfrak{J}_2 = c$, a constant (9), we have

$$x = \frac{1}{2} \mathfrak{J}_1 \mathfrak{J}_2 (\mathfrak{J}_1 + \mathfrak{J}_2), \quad cy = \frac{1}{2} (\mathfrak{J}_1^2 - \mathfrak{J}_2^2), \quad c^3 z = (\mathfrak{J}_1 - \mathfrak{J}_2)^3;$$

whence, eliminating $\frac{\mathfrak{J}_1}{\mathfrak{J}_2}$, we obtain

$$(y + kz) (y + 4kz)^2 - x^2 z = 0 \dots\dots\dots(10),$$

where $k = -12c^3$, which represents a nodal cubic. But from $d\mathfrak{J}_1 = d\mathfrak{J}_2$ we see that the tangent to (10) is a fourth harmonic to the three tangents drawn to U . Again, it is evident that the tangents to the three curves of the system (10) which pass through a point P are the three fourth harmonics to the tangents drawn from P to U .

9. Suppose a variable triangle A, B, C , inscribed in a cubic; and let O_1, O_2, O_3 be the points of contact of the sides with their envelopes, and a, b, c the points in which the sides meet the curve again. Then, if u_1, u_2, u_3 are the arguments of A, B, C , we have, from (6),

$$\begin{aligned} \frac{du_1}{du_2} &= -\frac{AO_3 \cdot Bc}{BO_3 \cdot Ac}, & \frac{du_2}{du_3} &= -\frac{BO_1 \cdot Ca}{CO_1 \cdot Ba}, \\ \frac{du_3}{du_1} &= -\frac{CO_2 \cdot Ab}{AO_2 \cdot Cb}; \end{aligned}$$

$$\text{therefore} \quad \frac{AO_3 \cdot BO_1 \cdot CO_2}{BO_3 \cdot CO_1 \cdot AO_2} + \frac{Bc \cdot Ca \cdot Ab}{Ac \cdot Ba \cdot Cb} = 0 \dots\dots\dots(11).$$

From this result it follows at once, by a known geometrical theorem, that if the lines Aa, Bb, Cc pass through a point, then the points O_1, O_2, O_3 will lie on a line, and *vice versa*. Now, if the lines Aa, Bb, Cc pass through a point, it can be shown that the tangents to the curve at A, B, C also pass through a point. For, referring the cubic to the triangle ABC , it may be written in the form

$$x(b_1 y^2 + c_1 z^2) + y(a_2 x^2 + c_2 z^2) + z(a_3 x^2 + b_3 y^2) + 2mxyz = 0.$$

The lines joining the vertices to the points where the opposite sides meet the curve again are then

$$b_1 y + a_2 x = 0, \quad c_1 z + a_3 x = 0, \quad c_2 z + b_3 y = 0 \dots\dots\dots(12),$$

and the tangents at the vertices are

$$c_1 x + c_2 y = 0, \quad b_1 x + b_3 z = 0, \quad a_2 y + a_3 z = 0 \dots\dots\dots(13).$$

But if the lines (12) pass through a point, we have

$$a_2c_1b_3 + a_3b_1c_2 = 0;$$

which is also the condition that the lines (12) should pass through a point. Now, if tangents be drawn to the cubic from a point P lying on a given locus, the lines joining their points of contact will have a definite envelope, and, from what we have shown above, if the curve is non-singular, these fifteen lines will touch their envelope in points lying by threes on twenty right lines.

10. In the paper referred to above, I have shown that an infinite number of triangles can be circumscribed about the cuspidal cubic $U \equiv y^4 - x^2z = 0$, so as to be inscribed in a cuspidal cubic V , this curve being connected with U by certain relations. Now, if two further relations are satisfied, the points of contact of the sides of the triangle will always lie on a line; in fact, in this case, the parameters $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$ are connected by the equations

$$\mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_3 = 0, \quad a(\mathfrak{g}_1\mathfrak{g}_2 + \mathfrak{g}_2\mathfrak{g}_3 + \mathfrak{g}_3\mathfrak{g}_1) + b\mathfrak{g}_1\mathfrak{g}_2\mathfrak{g}_3 = 0,$$

where a and b are constants. Hence, from what we have shown above, we see that in this case the tangents to V at the vertices of the triangle always pass through a point.

11. Suppose the points a, b, c to lie on a line, then from (11) the lines AO_1, BO_2, CO_3 pass through a point. Hence we see that, if a series of quadrilaterals A, B, C, a, b, c have their six intersections of sides on a cubic, and if the points of contact of two of the sides with their envelopes be given, then the points of contact of the remaining two sides can be constructed at once.

12. If it be possible to inscribe a series of triangles in a cubic which shall be circumscribed about a fixed conic, we know that the lines AO_1, BO_2, CO_3 pass through a point, and, therefore, as we have seen, the points a, b, c lie on a line. But, when a quadrilateral has its six intersections of sides on a cubic, the extremities of the diagonals are corresponding points on the curve, and therefore from (8) the differentials of the arguments of these points are equal. Hence, from (6), we see that the anharmonic ratio of the three points on a side and the point of contact of that side with its envelope has the same value for each of the four sides of the quadrilateral. But this is evidently the result which we should obtain by supposing a line divided in a constant anharmonic ratio by four fixed lines, to coincide with each of the fixed lines in turn. Thus we see that the fourth side abc must touch the conic. Now this result coincides with what we have known before—namely, that there exist conics about which an infinite number of quadrilaterals can be circumscribed so as to have their six inter-

sections of sides on a cubic; but, by the method which I have employed, it appears that these conics are the *only* system about which an infinite number of triangles can be circumscribed so as to be inscribed in the cubic.

13. Let us suppose a triangle A, B, C inscribed in a curve of the fourth order, and let a, b, c, a', b', c' be the points where the sides meet the curve again, and O_1, O_2, O_3 the points of contact of the sides with their envelopes, then, from (7), we find, as at (11),

$$\frac{AO_3 \cdot BO_1 \cdot CO_2}{BO_3 \cdot CO_1 \cdot AO_2} - \frac{Bc \cdot Bc' \cdot Ca \cdot Ca' \cdot Ab \cdot Ab'}{Ac \cdot Ac' \cdot Ba \cdot Ba' \cdot Cb \cdot Cb'} = 0 \dots\dots\dots(14).$$

Hence we see, by Carnot's theorem, that if the lines AO_1, BO_2, CO_3 pass through a point, then the points a, b, c, a', b', c' will lie on a conic. This will be the case if the triangle is always circumscribed about a fixed conic. I am not aware whether such a conic exists for the general quartic, but it can be found for certain special quartics. If the quartic has a double point, such a conic exists; for in the mode of generation of a general quartic as the envelope of a series of conics having a common Jacobian (Salmon's *Higher Plane Curves*, Art. 251), if we suppose the conics to have a common point, this point will evidently be a node of the quartic. Now the Cayleyan is the envelope of the lines joining the points of contact of the variable system of conics (*Curves*, Art. 253), and, in this case, it breaks up into a point and a conic. We thus have a series of triangles inscribed in the quartic and circumscribed about a fixed conic. We see then, from what we have shown above, that the points where the sides of one of these triangles meet the curve again lie on a conic.

14. We can verify the result (14) in a certain case. If the invariant B of a quartic vanish, it can be expressed linearly in an infinite number of ways in terms of the fourth powers of five lines. These lines all touch a fixed conic; namely, a conic such that, if we substitute differential symbols in its tangential equation and operate on the quartic, the result will vanish. If we take the covariant S (Salmon's *Higher Plane Curves*, Art. 297) of this quartic, we get another quartic which passes through all the intersections of the five lines. We thus have an infinite number of triangles circumscribed about a conic, and inscribed in the latter quartic. In this case, then, the conic passing through the points where the sides meet the curve again breaks up into two lines.

15. We can find, by the method employed above, for triangles inscribed in a curve of the n^{th} degree, a relation connecting the points of contact of the sides with their envelopes and the points where the sides meet the curve again. As a particular case, we find that, if it be

possible to circumscribe about a conic a series of triangles which shall be inscribed in a curve of the n^{th} degree, then the $3(n-2)$ points where the sides meet the curve again will lie on a curve of the $(n-2)^{\text{th}}$ degree. This result may, however, be obtained directly by means of Carnot's theorem. For, since two triangles circumscribed about a conic have their vertices on another conic, it follows that, if we consider two consecutive triangles, the vertices must be the points of contact of a conic touching the curve three times. But, if we make use of Carnot's theorem for a triangle inscribed in a curve of the n^{th} degree, we obtain a relation connecting the intercepts on the sides and the sines of the angles which the sides make with the tangents to the curve at the vertices (Salmon's *Higher Plane Curves*, Art. 126). Now the factors containing the sines of the angles will disappear in consequence of the fact that the tangents to the curve at the vertices are tangents to a conic passing through these points, and the remaining relation connecting the segments expresses the result I have stated above.

16. If an infinite number of polygons of m sides are inscribed in a curve of the n^{th} degree, we can find a relation connecting the points where the sides touch their envelopes and the points where they meet the curve again.

This relation is, in fact,

$$\frac{AO_1 \cdot BO_2 \cdot \&c.}{AO_m \cdot BO_1 \cdot \&c.} \times \frac{(A)'(B)'\&c.}{(A)'(B)'\&c.} = (-1)^{mn+1} \dots\dots\dots(15),$$

where $A, B, C, \&c.$ are the vertices of the polygon, $O_1, O_2, \&c.$ the points of contact of $AB, BC, \&c.$, and $(B)', (B)$ denote the [continued products of the $n-2$ segments made on the sides BC and BA , respectively, between B and the curve. This theorem can be applied to several particular cases, but does not appear to give any geometrical results of interest.

17. By means of the results obtained in § 2, we can write down a system of relations connecting the points on a curve which lie on a line touching a given conic. For, if we take points on a tangent to a conic S , their distances from the point of contact are proportional to the square roots of the results of substituting their coordinates in the equation of S . In the case of the cubic, we have then, from (6),

$$\frac{du_1}{\sqrt{S_1}} : \frac{du_2}{\sqrt{S_2}} : \frac{du_3}{\sqrt{S_3}} = BC : CA : AB;$$

$$\text{therefore } \sum \frac{du}{\sqrt{S}} = 0, \quad \sum \frac{Ldu}{\sqrt{S}} = 0,$$

where L is an arbitrary line.

Similarly, for the curve of the n^{th} degree, we have the $n-1$ equations

$$\Sigma \frac{du}{\sqrt{S}} = 0, \quad \Sigma U_1 \frac{du}{\sqrt{S}} = 0 \dots \Sigma U_{n-2} \frac{du}{\sqrt{S}} = 0 \dots \dots (16),$$

where U_r is an arbitrary curve of the r^{th} degree.

For curves of any deficiency it is evident that the expression under the radical \sqrt{S} cannot be rationalized; but for unicursal curves the expressions $\int \frac{du}{\sqrt{S}}$, &c. will be hyperelliptic integrals: for then S can be expressed as a polynomial of the $(2n)^{\text{th}}$ degree in a parameter. For non-unicursal curves, however, in certain particular cases, some of the expressions (16) will admit of simplification; as, for instance, let us consider a conic S having double contact with a cubic. If a conic S have double contact with a cubic, the cubic can be written in the form $AC^2 - BS = 0$, where A, B, C are lines; hence, for any point on the cubic,

$$\sqrt{S} = C \sqrt{\left(\frac{A}{B}\right)}, \text{ and } \frac{Ldu}{\sqrt{S}} \text{ becomes } \frac{L}{C} \sqrt{\left(\frac{B}{A}\right)} du = \sqrt{\left(\frac{B}{A}\right)} du,$$

if the line L is taken so as to coincide with C . Now, if we take any point O on the curve as origin, it is easy to see that for any point P on the curve

$$du \propto \frac{dt}{\sqrt{\{(t-\alpha)(t-\beta)(t-\gamma)(t-\delta)\}}},$$

where t is the tangent of the angle which OP makes with a fixed line, and $\alpha, \beta, \gamma, \delta$ are the values of t corresponding to the four tangents drawn from O to the curve. Now, suppose O to be the point AB , then, since B is one of the tangents from AB , we may put

$$\frac{B}{A} = \frac{t-\delta}{t-\epsilon},$$

where ϵ is the value of t corresponding to the line A .

$$\text{Hence } \sqrt{\left(\frac{B}{A}\right)} du = \frac{dt}{\sqrt{\{(t-\alpha)(t-\beta)(t-\gamma)(t-\epsilon)\}}} = d\nu, \text{ say,}$$

where ν is an elliptic integral; thus we have

$$\nu_1 \pm \nu_2 \pm \nu_3 = \text{a constant.}$$

If the conic S have triple contact with the curve, A is a tangent, and we may take then $\epsilon = \gamma$, in which case ν becomes either a logarithm or a circular function.

We can obtain a very simple relation in the case of a conic S touching a bicircular quartic four times. The quartic may then be written in the form $\Sigma^2 - S = 0$, where Σ is a circle; hence

$$\frac{U_2 du}{\sqrt{S}} \text{ becomes } \pm \frac{U_2 du}{\Sigma} = \pm du,$$

if we take U_3 so as to coincide with Σ . Thus we have $\Sigma \pm du = 0$; but we have $\Sigma du = 0$; hence we obtain

$$\begin{aligned} u_1 + u_2 &= \sigma, \text{ a constant,} \\ u_3 + u_4 &= \omega - \sigma \dots\dots\dots(17), \end{aligned}$$

where $\omega = \Sigma u$. This is evidently true of three such relations corresponding to the three conics of the system which can be described to touch a given line.

Third Paper on Multiple Frullanian Integrals.

By E. B. ELLIOTT.

[Read November 8th, 1883.]

The two previous papers, implied in the title of this one, are to be found in the volume of the Society's *Proceedings* for the Session 1876—77. Their main subject was the evaluation, when possible, of the multiple definite integral

$$\int_0^\infty \int_0^\infty \dots \int_0^\infty \left\{ S(a_1 x_1, a_2 x_2, \dots, a_n x_n) - S(b_1 x_1, b_2 x_2, \dots, b_n x_n) \right\} \frac{dx_1 dx_2 \dots dx_n}{x_1 x_2 \dots x_n},$$

the function S being a symmetric one in its arguments. The results arrived at in them were rendered at once simpler and more complete in a subsequent paper by Mr. Leudesdorf, which paper, together with a method of arrival at the same conclusions obtained quite independently and given me by Mr. Alfred Lodge, has materially aided me in the following more general investigation.

The object before me now is to replace in the denominator of the expression under the signs of integration the first power of the product by any power whatever which will make the result finite, be that power positive, negative, or vanishing. Thus in any particular subclass of results, such, for instance, as the one below specially considered, in which $S(x_1, x_2, \dots, x_n) \equiv f(x_1) f(x_2) \dots f(x_n)$, the few isolated forms of function f , for which the result with the first power is finite, may be expected to have corresponding to them a like number of isolated series of forms, each particular form giving a finite result with some other power, integral or fractional, positive or negative, in place of that first power.

The theorem as to single integrals which has to be made fundamental, the first case of a more general one which I gave in the *Messenger of Mathematics* for January last, is readily obtained as follows. Let r be such a real constant and $f(x)$ such a function of x that $\int_0^\infty f(x) \frac{dx}{x^r}$ is finite. This being so, $\frac{f'(x)}{x^{r-1}}$ must vanish both for $x = 0$ and $x = \infty$; and consequently, by the theorem known as