munication, we can deduce certain relations that are of importance in the said theory. Thus, consider the theorem

$$[12x_1y_1 \dots x_ny_n][34x_1y_1 \dots x_ny_n]-[13x_1y_1 \dots x_ny_n][24x_1y_1 \dots x_ny_n]$$
$$+[14x_1y_1 \dots x_ny_n][23x_1y_1 \dots x_ny_n]$$
$$= [x_1y_1 \dots x_ny_n][1234x_1y_1 \dots x_ny_n].$$

As particular cases of this we have a set of theorems of the type

$$[1256 \dots (2m)][3456 \dots (2m)]-[1356 \dots (2m)][2456 \dots (2m)]$$
$$+[1456 \dots (2m)][2356 \dots (2m)] = [56 \dots (2m)][123 \dots (2m)], \quad (14)$$

which are of importance in the theory referred to.

## *On Burmann's Theorem.* By A. C. Dixon.

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The following method gives a proof of Burmann's form of Lagrange's theorem, and also an extension of it which is curious and may possibly be useful.

Let $Fx$, $fx$ be two functions of the complex variable $x$, and $C$ a simple contour in the $x$-plane such that $Fx$, $fx$ are analytical within and on $C$, and that $|fx| = k$, a constant, along $C$.

Let $a_1, a_2, a_3, \dots$ be the points within $C$ at which $fx$ vanishes, and $b_1, b_2, b_3, \dots$ those at which $fx$ has a value $c$ such that $|c| < k$. Denote $fx \div (x-a_r)$ by $f_r x$ $(r = 1, 2, \dots)$, and suppose $f_1a_1, f_2a_2, \dots$ not to vanish.

The value of
$$\int_{(C)} \frac{Fxf'x\,dx}{fx-c}$$
is
$$2\iota\pi \{Fb_1+Fb_2+Fb_3+\dots\}.$$

But the subject of integration may be expanded in ascending powers of $c$, since $|c| < |f|$ along $C$. Hence

$$2\iota\pi \{Fb_1+Fb_2+\dots\}$$
$$= \int_{(C)} \frac{Fxf'x}{fx}\,dx + c\int_{(C)} \frac{Fxf'x}{(fx)^2}\,dx+\dots+c^n\int_{(C)} \frac{Fxf'x}{(fx)^{n+1}}\,dx+\dots.$$

The coefficient of $c^n$ may be put in the form

$$\frac{1}{n}\int_{(C)}\frac{F'x}{(fx)^n}\,dx$$

by integrating by parts.

The subject of integration in this coefficient becomes infinite at $a_1, a_2, a_3, \ldots$. Let $C_1, C_2, \ldots$ be small contours described about these points respectively; then

$$\frac{1}{n}\int_{(C_1)}\frac{F'x}{(fx)^n}\,dx=\frac{1}{n}\int_{(C_1)}\frac{F'x}{(x-a_1)^n\,(f_1x)^n}\,dx=\frac{2\iota\pi}{n!}\left(\frac{d}{da_1}\right)^{n-1}\frac{F'a_1}{(f_1a_1)^n}.$$

The coefficient of $c^n$ is therefore

$$\frac{2\iota\pi}{n!}\sum_r\left(\frac{d}{da_r}\right)^{n-1}\frac{F'a_r}{(f_ra_r)^n}.$$

The first term in the expansion is $2\iota\pi\sum_r Fa_r$. Hence

$$\sum_r Fb_r=\sum_r Fa_r+c\sum_r\frac{F'a_r}{f_ra_r}+\ldots+\frac{c^n}{n!}\sum_r\left(\frac{d}{da_r}\right)^{n-1}\frac{F'a_r}{(f_ra_r)^n}+\ldots.$$

Putting 1 for $Fx$ in this result, we find that there are as many points $b$ as points $a$. If there is only one point $a$, we have Burmann's theorem. If there are more, the expansion on the right is the sum of a number of series each of the Burmann form; in general these series would not converge separately, but the sum converges absolutely under the conditions stated.

By putting instead of $F$ its square, cube, ..., we may find series for $\sum(Fb_r)^2$, $\sum(Fb_r)^3$, ..., and such expansions might be used to calculate the coefficients in an equation with $Fb_1, Fb_2, \ldots$ for roots.

The region of validity of the series can be readily assigned. It is bounded by a curve $|fx|=k$, where $k$ is a quantity suitably chosen. Suppose for the moment that $Fx$ has no singularity that affects the question. For small values of $k$ the curve $|fx|=k$ will consist of small ovals enclosing $a_1, a_2, \ldots$ respectively. Within each of these the corresponding Burmann series is valid. As $k$ increases the series still hold good until two of the ovals coalesce—say those about $a_1, a_2$. There will be an intermediate nodal form of the curve $|fx|=k$. Outside this the Burmann series are not valid singly, but their sum is still a valid representation of the sum of the two values of $Fx$, until a value of $k$ is reached for which a new oval coalesces with that about $a_1, a_2$; after this the sum of the two series is not valid, but the sum of three, or possibly more, will still represent the sum of the corresponding values of $Fx$. This process may be carried on until the curve $|fx| = k$ reaches a singularity of $Fx$.

For instance, let $fx \equiv (x-a_1)(x-a_2)(x-a_3)$,

and let $x_1$, $x_2$, $x_3$ be the roots of the equation

$$(x-a_1)(x-a_2)(x-a_3) = c.$$

Then the above method gives expansions for

$$\sin x_1 + \sin x_2 + \sin x_3,$$

$$\cos x_1 + \cos x_2 + \cos x_3,$$

$$e^{kx_1} + e^{kx_2} + e^{kx_3},$$

and so on, in powers of $c$, and these expansions hold for all finite values of $c$.

If $fx$ vanishes to a higher order than the first at one of the $a$ points —say $a_1$—the result must be somewhat modified. We may then put

$$fx = (x-a_1)^n f_1 x,$$

where $$f_1 a_1 \neq 0,$$

and the corresponding partial series is the result of putting $a_1$ for $x$ in

$$a\left[Fx + \frac{c}{a!}\left(\frac{d}{dx}\right)^{a-1}\frac{F'x}{f_1 x} + \ldots + \frac{c^n}{(na)!}\left(\frac{d}{dx}\right)^{na-1}\frac{F'x}{(f_1 x)^n} + \ldots\right].$$

The remainder after $n$ terms is in any case

$$\frac{1}{2\iota\pi}c^n \int_{(C)} \frac{Fx f'x}{(fx)^n (fx-c)}\,dx.$$

There is a similar extension of Taylor's or Maclaurin's theorem, when the function expanded is not uniform, but has a finite number of branches in the region considered.

Suppose, for instance, that $fx$ is a two-valued function within a circle, centre $O$, radius $k$, with no singularity within this circle except a branch-point at $e$. Let $f_1 x$, $f_2 x$ be the two branches of $fx$. Then Maclaurin's series for either of these holds good within a concentric circle of radius $|e|$. In the ring between the two circles the two series are not available separately, but their sum is still a valid representation of $f_1 x + f_2 x$. Similarly in other cases.