

On the finite Groups of linear Transformations of a Variable.

By

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In the paper „Ueber endliche Gruppen linearer Transformationen einer Veränderlichen“, *Annalen* t. XII. (1877) pp. 23—46, Prof. Gordan gave in a very elegant form the groups of 12, 24 and 60 homographic transformations $\frac{ax+b}{cx+d}$. The groups of 12 and 24 are in the like form, the group of 24 thus containing as part of itself the group of 12; but the group of 60 is in a different form, not containing as part of itself the group of 12. It is I think desirable to present the group of 60 in the form in which it contains as part of itself Gordan's group of 12: and moreover to identify the group of 60 with the group of the 60 positive permutations of 5 letters: or (writing abc for the cyclical permutation a into b , b into c , c into a , and so in other cases) say with the group of the 60 positive permutations 1, abc , $ab \cdot cd$ and $abcde$.

Any two forms of a group are, it is well known, connected as follows, viz. if $1, \alpha, \beta, \dots$ are the functional symbols of the one form, then those of the other form are $1, \vartheta \alpha \vartheta^{-1}, \vartheta \beta \vartheta^{-1}, \dots$ (where in the case in question ϑ is a functional symbol of the like homographic form, $\vartheta x = \frac{Ax+B}{Cx+D}$): but instead of obtaining the new form in this manner, I found it easier to use the values of the rotation-symbol $\cos \frac{\pi}{q} + \sin \frac{\pi}{q} (i \cos X + j \cos Y + k \cos Z)$ for the axes of the icosahedron or dodecahedron, given in my paper „Notes on polyhedra“ *Quart. Math. Jour.* t. VII (1866) pp. 304—316; viz. if for any axes, λ, μ, ν denote the parameters of rotation $\tan \frac{\pi}{q} \cos X$, $\tan \frac{\pi}{q} \cos Y$, $\tan \frac{\pi}{q} \cos Z$; then (by a formula which is in fact equivalent to that given in my note „On the correspondence of homographies and Rotations“ *Math. Annalen* t. XV, (1879) pp. 238—240) the corresponding homographic function of x is

$$\frac{(-\nu-i)x+\lambda+i\mu}{(\lambda-i\mu)x+\nu-i}$$

where i denotes $\sqrt{-1}$ as usual.

The new formulae for the group of 60, or icosahedron group, of homographic functions $\frac{\alpha x + \beta}{\gamma x + \delta}$ are contained in the following table, where the four columns show the values of the coefficients $\alpha, \beta, \gamma, \delta$ respectively: and where in the outside column, the substitution is represented as a permutation-symbol on the five letters $abcde$: moreover for shortness Θ is written to denote $\sqrt{5}$.

The Group of 60

	α	β	γ	δ	
1	1	0	0	1	1
2	-1	0	0	1	$ab \cdot cd$
3	0	1	1	0	$ac \cdot bd$
4	0	-1	1	0	$ad \cdot bc$
5	2	$-3 + \Theta + i(1 - \Theta)$	$-3 + \Theta + i(-1 + \Theta)$	-2	$bc \cdot de$
6	2	$-3 + \Theta + i(-1 + \Theta)$	$-3 + \Theta + i(1 - \Theta)$	-2	$ae \cdot bc$
7	2	$3 - \Theta + i(-1 + \Theta)$	$3 - \Theta + i(1 - \Theta)$	-2	$ad \cdot ce$
8	2	$3 - \Theta + i(1 - \Theta)$	$3 - \Theta + i(-1 + \Theta)$	-2	$ad \cdot be$
9	2	$-1 - \Theta + i(1 - \Theta)$	$-1 - \Theta + i(-1 + \Theta)$	-2	$ae \cdot cd$
10	2	$-1 - \Theta + i(-1 + \Theta)$	$-1 - \Theta + i(1 - \Theta)$	-2	$ab \cdot de$
11	2	$1 + \Theta + i(-1 + \Theta)$	$1 + \Theta + i(1 - \Theta)$	-2	$be \cdot cd$
12	2	$1 + \Theta + i(1 - \Theta)$	$1 + \Theta + i(-1 + \Theta)$	-2	$ab \cdot ce$
13	2	$-1 - \Theta + i(-3 - \Theta)$	$-1 - \Theta + i(3 + \Theta)$	-2	$ac \cdot be$
14	2	$-1 - \Theta + i(3 + \Theta)$	$-1 - \Theta + i(-3 - \Theta)$	-2	$bd \cdot ce$
15	2	$1 + \Theta + i(3 + \Theta)$	$1 + \Theta + i(-3 - \Theta)$	-2	$ae \cdot bd$
16	2	$1 + \Theta + i(-3 - \Theta)$	$1 + \Theta + i(3 + \Theta)$	-2	$ac \cdot de$
17	$-i$	i	1	1	abc
18	-1	i	1	i	acb
19	1	$-i$	1	i	adc
20	$-i$	$-i$	1	-1	acd
21	i	i	1	-1	adb
22	1	i	1	$-i$	abd
23	-1	$-i$	1	$-i$	bcd
24	i	$-i$	1	1	bdc
25	$-1 - \Theta + i(3 + \Theta)$	2	-2	$-1 - \Theta + i(-3 - \Theta)$	acc
26	$1 + \Theta + i(3 + \Theta)$	2	-2	$1 + \Theta + i(-3 - \Theta)$	ace
27	$1 + \Theta + i(-3 - \Theta)$	2	-2	$1 + \Theta + i(3 + \Theta)$	bed
28	$-1 - \Theta + i(-3 - \Theta)$	2	-2	$-1 - \Theta + i(3 + \Theta)$	bde
29	$-3 + \Theta + i(1 - \Theta)$	2	2	$3 - \Theta + i(1 - \Theta)$	bec
30	$-3 + \Theta + i(-1 + \Theta)$	2	2	$3 - \Theta + i(-1 + \Theta)$	bc

	α	β	γ	δ	
31	$3 - \theta + i(-1 + \theta)$	2	2	$-3 + \theta + i(-1 + \theta)$	aed
32	$3 - \theta + i(1 - \theta)$	2	2	$-3 + \theta + i(1 - \theta)$	ade
33	2	$-1 - \theta + i(-1 + \theta)$	$1 + \theta + i(-1 + \theta)$		cde
34	2	$1 + \theta + i(1 - \theta)$	$-1 - \theta + i(1 - \theta)$		ced
35	2	$-1 - \theta + i(1 - \theta)$	$1 + \theta + i(1 - \theta)$		acb
36	2	$1 + \theta + i(-1 + \theta)$	$-1 - \theta + i(-1 + \theta)$		abe
37	$-1 - \theta + i(-3 - \theta)$	2	2	$1 + \theta + i(-3 - \theta)$	$abcde$
38	$-1 - \theta + i(1 - \theta)$	2	2	$1 + \theta + i(1 - \theta)$	$acebd$
39	$-1 - \theta + i(-1 + \theta)$	2	2	$1 + \theta + i(-1 + \theta)$	$adbce$
40	$-1 - \theta + i(3 + \theta)$	2	2	$1 + \theta + i(3 + \theta)$	$acdeb$
41	$1 + \theta + i(3 + \theta)$	2	2	$-1 - \theta + i(3 + \theta)$	$adceb$
42	$1 + \theta + i(-1 + \theta)$	2	2	$-1 - \theta + i(-1 + \theta)$	$acbde$
43	$1 + \theta + i(1 - \theta)$	2	2	$-1 - \theta + i(1 - \theta)$	$aedbc$
44	$1 + \theta + i(-3 - \theta)$	2	2	$-1 - \theta + i(-3 - \theta)$	$abecd$
45	$-1 - \theta + i(-1 + \theta)$	2	-2	$-1 - \theta + i(1 - \theta)$	$acbed$
46	$-3 + \theta + i(-1 + \theta)$	2	-2	$-3 + \theta + i(1 - \theta)$	$abdce$
47	$3 - \theta + i(-1 + \theta)$	2	-2	$3 - \theta + i(1 - \theta)$	$aecdb$
48	$1 + \theta + i(-1 + \theta)$	2	-2	$1 + \theta + i(1 - \theta)$	$adebc$
49	$1 + \theta + i(1 - \theta)$	2	-2	$1 + \theta + i(-1 + \theta)$	$aecbd$
50	$3 - \theta + i(1 - \theta)$	2	-2	$3 - \theta + i(-1 + \theta)$	$acdeb$
51	$-3 + \theta + i(1 - \theta)$	2	-2	$-3 + \theta + i(-1 + \theta)$	$abedc$
52	$-1 - \theta + i(1 - \theta)$	2	-2	$-1 - \theta + i(-1 + \theta)$	$adbce$
53	2	$-3 + \theta + i(-1 + \theta)$	$3 - \theta + i(-1 + \theta)$	2	$aebdc$
54	2	$-1 - \theta + i(3 + \theta)$	$1 + \theta + i(3 + \theta)$	2	$abced$
55	2	$1 + \theta + i(-3 - \theta)$	$-1 - \theta + i(-3 - \theta)$	2	$adecb$
56	2	$3 - \theta + i(1 - \theta)$	$-3 + \theta + i(1 - \theta)$	2	$acdbe$
57	2	$-3 + \theta + i(1 - \theta)$	$3 - \theta + i(1 - \theta)$	2	$abdec$
58	2	$-1 - \theta + i(-3 - \theta)$	$1 + \theta + i(-3 - \theta)$	2	$adebe$
59	2	$1 + \theta + i(3 + \theta)$	$-1 - \theta + i(3 + \theta)$	2	$aebcd$
60	2	$3 - \theta + i(-1 + \theta)$	$-3 + \theta + i(-1 + \theta)$	2	$acedb$

This contains (as one of five groups of 12) the group of the positive permutations of $abcd$; and completing this into a group of 24 we have

Groups of 12 and 24

	α	β	γ	δ	
1	1	0	0	1	1
2	-1	0	0	1	$ab \cdot cd$
3	0	1	1	0	$ac \cdot bd$
4	0	-1	1	0	$ad \cdot bc$

	α	β	γ	δ	
5	$-i$	i	1	1	abc
6	-1	i	1	i	acb
7	1	$-i$	1	i	adc
8	$-i$	$-i$	1	-1	acd
9	i	i	1	-1	adb
10	1	i	1	$-i$	abd
11	-1	$-i$	1	$-i$	bcd
12	i	$-i$	1	1	bdc
13	i	0	0	1	$adcb$
14	$-i$	0	0	1	bd
15	0	i	1	0	$abcd$
16	0	i	-1	0	ac
17	1	-1	1	1	cd
18	$-i$	-1	1	i	$abdc$
19	i	1	1	i	$acdb$
20	1	1	1	-1	ab
21	-1	-1	1	-1	$acbd$
22	i	-1	1	$-i$	bc
23	$-i$	1	1	$-i$	ad
24	-1	1	1	1	$adb c$

The groups of 60 and 24 thus each of them contain the group of 12,

$$\pm x, \pm \frac{1}{x}, \pm i \frac{1-x}{1+x}, \pm i \frac{1+x}{1-x}, \pm \frac{x+i}{x-i}, \pm \frac{x-i}{x+i}.$$

It may be remarked that, to verify the periodicities of the forms contained in the group of 60, we have as the conditions that

$\frac{\alpha x + \beta}{\gamma x + \delta}$ may be periodic of the order 2, $\frac{(\alpha + \delta)^2}{\alpha \delta - \beta \gamma} = 0$, that is $\alpha + \delta = 0$

„ „ „ 3, „ = 1,

„ „ „ 5, „ = $\frac{1}{2}(3 + \sqrt{5})$.

For instance in the form

$$\frac{[-1 - \Theta + i(-3 - \Theta)]x + 2}{2x + [1 + \Theta + i(-3 - \Theta)]}, \quad \alpha \delta = -(1 + \Theta)^2 - (3 + \Theta)^2 = -20 - 8\Theta, \quad \beta \gamma = 4$$

$$a + \delta = -2i(3 + \Theta),$$

and therefore $\frac{(\alpha + \delta)^2}{\alpha \delta - \beta \gamma} = \frac{-4(3 + \Theta)^2}{-8(3 + \Theta)} = \frac{3 + \Theta}{2} = \frac{1}{2}(3 + \sqrt{5})$, as it should be.

Cambridge, 11. Nov. 1879.