The Prime-Square Chamber Lift:
Short-Interval Occupancy via Quadratic Boundary Windows

Jeffery Huckstead
Cerebral Graphix

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Abstract

This note gives an expository chamber-coordinate presentation of the prime-between-squares problem family. Legendre's conjecture asks for a prime in every interval $(n^2, (n+1)^2)$, Oppermann's conjecture asks for primes in both subintervals split by the pronic number $n(n+1)$, and Brocard's conjecture asks for multiple-prime occupancy between consecutive prime-square boundaries. The Chamber Lift vocabulary used here is not offered as a proof mechanism or as a new analytic method. It is an atlas: a visual and notational organization of classical short-interval prime conjectures over the multiplication-grid geometry already present in the square diagonal and its adjacent rectangular seam.

Nonclaim statement

This paper does not prove Legendre's conjecture, Oppermann's conjecture, Brocard's conjecture, Andrica's conjecture, or any new prime-gap theorem. The square grid supplies a faithful coordinate picture for the relevant intervals, but it does not force primality. All prime occupancy statements below are either definitions, conjectures, known conditional statements, verified finite computations, or cited almost-prime analogues.

The intended contribution is an expository one: the classical problem family is placed into a single chamber map,

$$\text{square diagonal} \quad \longrightarrow \quad \text{pronic seam} \quad \longrightarrow \quad \text{prime occupancy.}$$

The theorem describes the grid. The grid improves the description.

1 Classical problem family

Let $\pi(x)$ denote the prime-counting function.

1.1 Legendre occupancy

Legendre's conjecture asserts that for every integer $n \geq 1$ there exists a prime p satisfying

$$n^2 < p < (n+1)^2.$$

Equivalently,

$$L_n := \pi((n+1)^2) - \pi(n^2) \geq 1.$$

The interval

$$(n^2, (n+1)^2)$$

has length

$$(n+1)^2 - n^2 = 2n + 1.$$

Since $x = n^2$, this is the short interval scale

$$2\sqrt{x} + 1.$$

Thus Legendre's conjecture is naturally a short-interval prime occupancy problem at the square-root scale.

1.2 Oppermann occupancy

Oppermann's conjecture strengthens Legendre's conjecture by splitting the square window at the pronic number

$$R_n = n(n+1).$$

It asks for at least one prime in each open interval

$$(n^2, n(n+1))$$

and

$$(n(n+1), (n+1)^2).$$

Define

$$O_n^- := \pi(n(n+1)) - \pi(n^2),$$

and

$$O_n^+ := \pi((n+1)^2) - \pi(n(n+1)).$$

Then Oppermann's conjecture is

$$O_n^- \geq 1, \qquad O_n^+ \geq 1$$

for all relevant n, commonly stated for $n > 1$. Since

$$L_n = O_n^- + O_n^+,$$

Oppermann implies Legendre immediately.

1.3 Brocard occupancy

Let p_k be the kth prime. Brocard's conjecture asks whether

$$B_k := \pi(p_{k+1}^2) - \pi(p_k^2) \geq 4$$

for consecutive primes $p_k < p_{k+1}$, usually for $k \geq 2$. This is another prime-square occupancy problem, but its boundaries are prime-indexed squares rather than consecutive integer squares.

2 The square diagonal and pronic seam

The multiplication grid contains the chamber structure before any new terminology is introduced.

Lemma 1 (Grid geometry). *For every integer $n \geq 1$,*

$$(n+1)^2 - n^2 = 2n + 1,$$

$$n(n+1) - n^2 = n,$$

and

$$(n+1)^2 - n(n+1) = n + 1.$$

Hence

$$n^2 < n(n+1) < (n+1)^2.$$

Proof. Expanding gives

$$(n+1)^2 = n^2 + 2n + 1.$$

Also,

$$n(n+1) = n^2 + n.$$

Subtracting the stated boundary values gives the three displayed identities. Because $n > 0$, the strict inequalities follow. □

In grid coordinates, the square boundary is the diagonal map

$$(n, n) \mapsto n^2,$$

the next square boundary is

$$(n+1, n+1) \mapsto (n+1)^2,$$

and the opposed off-diagonal positions are

$$(n, n+1), (n+1, n) \mapsto n(n+1).$$

This off-diagonal value is the pronic number R_n. In this paper the phrase *rectangular seam* is used only as visual vocabulary for that classical pronic split.

3 Square-growth collar

The transition from the $n \times n$ square to the $(n+1) \times (n+1)$ square has size

$$(n+1)^2 - n^2 = 2n + 1.$$

As a lattice picture, this is

$$2n + 1 = n + n + 1,$$

which may be read as two unit-thick strips plus the final corner unit. This motivates the descriptive phrase *square-growth collar*. The phrase does not carry proof content. It names only the capacity interval

$$(n^2, (n+1)^2).$$

Prime distribution inside that interval is governed by arithmetic and sieve structure, not by the visual collar itself.

4 Occupancy coordinates

The cleanest chamber notation is just prime-counting notation with visible boundaries:

$$L_n = \pi((n+1)^2) - \pi(n^2),$$

$$O_n^- = \pi(n(n+1)) - \pi(n^2),$$
$$O_n^+ = \pi((n+1)^2) - \pi(n(n+1)),$$

and

$$B_k = \pi(p_{k+1}^2) - \pi(p_k^2).$$

The logical relations are:

$$O_n^- \geq 1 \text{ and } O_n^+ \geq 1 \quad \Longrightarrow \quad L_n \geq 2 \quad \Longrightarrow \quad L_n \geq 1.$$

Thus Oppermann is stronger than Legendre. Brocard is related, but it uses prime-square windows rather than consecutive integer-square windows. Oppermann's conjecture is known to imply Brocard's conjecture through a standard interval subdivision argument across consecutive square layers; Legendre alone is much weaker.

Remark 1 (Endpoint convention)**.** *For $n \geq 2$, the values n^2, $n(n+1)$, and $(n+1)^2$ are composite. Open or closed endpoint conventions therefore do not affect prime counts in the chambers. The small case $n = 1$ should be handled separately because $n(n+1) = 2$ is prime.*

5 Prime-gap translation

A failure of Legendre's conjecture at n would mean that no prime lies in

$$(n^2, (n+1)^2),$$

so a prime gap spans a window of length $2n + 1$. Since $x = n^2$, the threshold is approximately

$$2\sqrt{x}.$$

A failure of Oppermann's conjecture is stronger: one of the two subwindows

$$(n^2, n(n+1)) \quad \text{or} \quad (n(n+1), (n+1)^2)$$

lacks a prime. The smaller half-window has length n, so Oppermann asks for a prime-gap control near the $\sqrt{x}$ scale in both halves.

The common implication map is:

$$\text{Cramer-type gaps } O((\log x)^2) \quad \Longrightarrow \quad \text{Legendre for sufficiently large } n,$$

$$\text{Andrica's conjecture} \quad \Longrightarrow \quad \text{Legendre},$$

and

$$\text{Riemann Hypothesis alone} \quad \not\Longrightarrow \quad \text{Legendre by the standard gap bound.}$$

The last statement reflects the familiar conditional estimate

$$g(x) = O(\sqrt{x}\log x),$$

which remains longer than the square-window scale $2\sqrt{x}$ by a logarithmic factor.

6 Known results and near-results

Current unconditional short-interval theorems do not reach the square-window scale. Baker, Harman, and Pintz proved a prime in intervals of length on the order of

$$x^{21/40} = x^{0.525}$$

for sufficiently large x [7]. Since

$$0.525 > 0.5,$$

this is close to, but still wider than, the Legendre scale $x^{1/2}$. Consequently, it does not prove Legendre's conjecture.

Computationally, Sorenson and Webster verified Oppermann's conjecture, hence also Legendre's conjecture, for all

$$n \leq 3.33 \cdot 10^{13}$$

using a parallel computation [10]. This is a finite verification, not a proof for all n.

There are also recent almost-prime analogues. Dudek and Johnston proved that every interval

$$(n^2, (n+1)^2)$$

contains an integer with at most four prime factors, counted with multiplicity [11]. Campbell subsequently announced an improvement to at most three prime factors for every such interval [12]. These are not prime-occupancy results, but they show that the square window admits strong near-prime occupancy theorems.

Table 1: Selected conjectures, theorems, and verified ranges.

Name	Statement	Source	Status
Legendre	$\pi((n+1)^2) - \pi(n^2) \geq 1$ for all $n \geq 1$.	Legendre	Open
Oppermann	$O_n^- \geq 1$ and $O_n^+ \geq 1$ for all relevant n.	Oppermann	Open
Brocard	$\pi(p_{k+1}^2) - \pi(p_k^2) \geq 4$.	Brocard	Open
Andrica	$\sqrt{p_{k+1}} - \sqrt{p_k} < 1$.	Andrica	Open
Cramer	Maximal gaps expected to be $O((\log x)^2)$.	Cramer	Open
BHP short intervals	Primes occur in intervals of length $x^{21/40}$ for sufficiently large x.	Baker, Harman, Pintz	Theorem
RH gap bound	RH gives a gap estimate of order $\sqrt{x}\log x$, not enough alone for Legendre.	Cramer, RH literature	Conditional
Sorenson-Webster	Oppermann verified through $n \leq 3.33 \cdot 10^{13}$.	Sorenson, Webster	Computation
Dudek-Johnston	Every square interval contains an integer with at most four prime factors.	Dudek, Johnston	Theorem
Campbell	Every square interval contains an integer with at most three prime factors.	Campbell	Preprint theorem

7 Visual atlas

The original triangular-fractional multiplication grid is useful here because it already presents chamber order, thick chamber boundaries, and exact diagonal square cells. The best practice is not to overprint primes directly on the original grid. Instead, preserve the original figure as the structural scaffold, then give a separate minimal occupancy table. This avoids implying that the grid itself proves the prime placements.

multiplication_grid_cropped.pdf

Figure 1: Original triangular-fractional multiplication grid for $N = 1$ to 6, used as an unmodified structural reference. The diagonal highlighted cells are exact square entries, and the thick lines mark chamber boundaries. In the present note this figure is read as the square-boundary scaffold. Prime positions are not overprinted here in order to keep the scaffold distinct from prime occupancy.

Table 2: Prime occupancy in the first square windows. This table supplies the prime overlay separately from the original grid.

n	n^2	left chamber	$R_n = n(n+1)$	right chamber	$(n+1)^2$
1	1	$\varnothing$	2	3	4
2	4	5	6	7	9
3	9	11	12	13	16
4	16	17, 19	20	23	25
5	25	29	30	31	36
6	36	37, 41	42	43, 47	49

The first line is exceptional because the seam $R_1 = 2$ is itself prime. For $n \geq 2$, the boundaries and seams are composite, so primes appear only in the open chambers.

A safe captioning principle for future figures is:

> The grid shows the interval scaffold. Circled or listed primes show observed occupancy in small examples. The visual arrangement does not prove prime occupancy at scale.

8 Red-team limitations

The coordinates are useful only when their classical identity remains visible:

Chamber term	Standard object	Status
square diagonal	consecutive squares n^2	classical object
square-growth collar	interval $(n^2, (n+1)^2)$	visual vocabulary
rectangular seam	pronic number $n(n+1)$	visual vocabulary
occupancy	prime-count difference $\pi(b) - \pi(a)$	standard notation
signed chamber coordinate	recentering around $n(n+1)$	optional label only

The signed coordinate

$$C_n(M) = M - n(n+1)$$

may help label left and right chambers, but it does not alter primality or provide a sieve. It is a recentering coordinate, not an analytic invariant.

9 Conclusion

The Prime-Square Chamber Lift is a faithful visual atlas for the Legendre-Oppermann-Brocard family:

$$\text{Legendre} = \text{full square-collar occupancy},$$

$$\text{Oppermann} = \text{pronic seam-split occupancy},$$

and

$$\text{Brocard} = \text{prime-square interval occupancy}.$$

The multiplication grid gives exact boundary coordinates:

$$(n, n) \mapsto n^2,$$

$$(n, n+1), (n+1, n) \mapsto n(n+1),$$

and

$$(n+1, n+1) \mapsto (n+1)^2.$$

The atlas does not catch the prime. It draws the chamber where the prime is conjectured to appear.

References

[1] A.-M. Legendre, *Essai sur la Theorie des Nombres*, Courcier, Paris, 1808.

[2] L. Oppermann, remarks on primes between square intervals, 1877. Commonly cited as Oppermann's conjecture.

[3] H. Brocard, early statements on primes between consecutive prime squares, 1876. Commonly cited as Brocard's conjecture.

[4] D. Andrica, Note on a conjecture in prime number theory, *Studia Univ. Babes-Bolyai Math.* 31, 1986.

[5] H. Cramer, On the order of magnitude of the difference between consecutive prime numbers, *Acta Arithmetica* 2, 1936, 23-46.

[6] A. E. Ingham, On the difference between consecutive primes, *Quarterly Journal of Mathematics* 8, 1937, 255-266.

[7] R. C. Baker, G. Harman, and J. Pintz, The difference between consecutive primes, II, *Proceedings of the London Mathematical Society* 83, 2001, 532-562.

[8] K. Ford, B. Green, S. Konyagin, J. Maynard, and T. Tao, Long gaps between primes, *Journal of the American Mathematical Society* 31, 2018, 65-105.

[9] E. Carneiro, M. B. Milinovich, and K. Soundararajan, Fourier optimization and prime gaps, *Commentarii Mathematici Helvetici* 94, 2019, 533-568.

[10] J. Sorenson and J. Webster, An algorithm and computation to verify Legendre's Conjecture up to $3.33 \cdot 10^{13}$, arXiv:2401.13753, 2024.

[11] A. W. Dudek and D. R. Johnston, Almost primes between all squares, arXiv:2501.18048, 2025.

[12] P. J. Campbell, On the existence of integers with at most 3 prime factors between every pair of consecutive squares, arXiv:2603.10356, 2026.