ABEL'S THEOREM AND ITS CONVERSE

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1.

INTRODUCTION AND SUMMARY.

1 . 1. Our main object in this paper is to obtain as far-reaching a generalisation as possible of "Tauber's Theorem", the well-known converse of Abel's famous theorem concerning power-series. It will be necessary to give a rapid summary of the results already known; and we can do this most shortly and clearly if we begin with a few definitions of a verbal character.

We shall always denote the power-series in question by $\Sigma a_n x^n$ or by S, and its sum by $f(x)$. We suppose that the radius of convergence of S is unity, and that the point on the circle of convergence which is in question is the point $x = 1$. The series Σa_n we shall call A, and we shall also use A to denote its sum, when it is convergent.

We shall use (K), (L), (O), and (o) as abbreviations for the propositions:

(K) *A is convergent*,

(L) $f(x) \to A$,

(O) $a_n = O\left(\frac{1}{n}\right)$,

(o) $a_n = o\left(\frac{1}{n}\right)$.

We shall be concerned in the sequel with certain classes of curves C along which x may approach the limit 1.* We shall call C a *path* if it is a simple Jordan curve which does not pass outside the circle: that

* We are, of course, concerned with the nature of C only in the neighbourhood of $x = 1$. It is therefore presupposed, in the definitions which follow, that their conditions need only be satisfied for values of x near enough to unity.

is to say if it is defined by equations

$$x = \xi + i\eta, \quad \xi = \phi(t), \quad \eta = \psi(t) \quad (t_0 \leqslant t \leqslant T);$$

where ϕ and ψ are continuous for $t_0 \leqslant t \leqslant T$, $\phi(T) = 1$, $\psi(T) = 0$, $\phi^2+\psi^2 \leqslant 1$, and $\phi(t_1) = \phi(t_2)$, $\psi(t_1) = \psi(t_2)$ are not both true unless $t_1 = t_2$.

If $\phi^2+\psi^2 < 1$ except for $t = T$, we shall call C an *internal path*. If it lies entirely between two chords of the unit circle, meeting at $x = 1$, we shall call it a *Stolz-path*.

If C possesses a continuously turning tangent at every point except $x = 1$, and approaches $x = 1$ with a definite direction, so that am $(1-x)$ tends to a limit when $x \to 1$, we shall call it a *regular path*. If the limit of am $(1-x)$ is neither $\frac{1}{2}\pi$ nor $-\frac{1}{2}\pi$, C is a *regular Stolz-path*.

Thus a chord of the unit circle, or a segment of a circle which passes through $x = 0$ and $x = 1$, and contains an angle greater than a right angle, is a regular Stolz-path. An arc of a circle which touches the unit circle internally, or an arc of the unit circle itself, is regular, but not a Stolz-path. The curve

$$\eta = (1-\xi) \sin \frac{1}{1-\xi}$$

is a Stolz-path, but not regular. The curve

$$\eta = \sqrt{(1-\xi^2)} \sin \frac{1}{\sqrt{(1-\xi^2)}}$$

is a path, but neither regular nor a Stolz-path.

1.2. Abel's Theorem is

A. (K) *implies* (L) *when* C *is the radius* (0, 1).*

Stolz's generalisation is

B. (K) *implies* (L) *when* C *is any Stolz-path.*†

Proofs of **A** and **B** will be found in Bromwich's *Infinite Series.*‡ To

* N. H. Abel, "Untersuchungen über die Reihe $1+\frac{m}{1}x+\frac{m(m-1)}{1.2}x^2+\ldots$", *Journal für Math.*, Vol. 1, 1826, pp. 311-339 (*Œuvres*, Vol. 1, pp. 219-250).

† O. Stolz, "Beweis einiger Sätze über Potenzreihen", *Zeitschrift für Math.*, Jahrgang 20, 1875, pp. 369-376; "Nachtrag ...", *ibid.*, Jahrgang 29, 1884, pp. 127-128.

‡ See pp. 130, 210-212.

these theorems should be added

C. *It is not true that (K) implies (L) whenever C is a regular path.**

Thus, for example, the series

$$\Sigma n^{-b} e^{Ain^a} \quad (A > 0,\ 0 < a < 1) \tag{1.21}$$

is convergent whenever $b > 1-a$; but the associated power-series $f(x)$ does not tend to a limit if $b < 1-\frac{1}{2}a$ and C is an arc of a circle touching the unit circle.

Tauber's Theorem is

D. *(L) and (o) imply (K) when C is the radius* (0, 1).†

This theorem has been generalised in several directions. The generalisations with which we shall be most directly concerned are

E. *(L) and (o) imply (K) when C is any Stolz-path.*‡

F. *(L) and (O) imply (K) when C is the radius* (0, 1).§

But we must also mention

G. *In either* **D** *or* **E**, *(o) may be replaced by the more general condition*

$$a_1+2a_2+\ldots+na_n = o(n).$$

This condition is also necessary for the truth of (K).‖

H. *In* **F**, *(O) may be replaced by the condition that a_n is real and na_n bounded above or below.*¶

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* G. H. Hardy and J. E. Littlewood, **1** (see the list of papers in 1.5), p. 475 (Theorem 47). The proof is not given, but the materials necessary for one will be found in a paper by Hardy, "A theorem concerning Taylor's series", *Quarterly Journal*, Vol. 44, 1913, pp. 147–160 (pp. 150 *et seq.*).

† A. Tauber, "Ein Satz aus der Theorie der unendlichen Reihen", *Monatshefte für Math.*, Vol. 8, 1897, pp. 273–277. See also Bromwich, *Infinite Series*, p. 251, or Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, Berlin, 1916, p. 40.

‡ E. Landau, "Über die Konvergenz einiger Klassen von unendlichen Reihen am Rande Konvergenzgebietes", *Monatshefte für Math.*, Vol. 18, 1907, pp. 8–28. See also Bromwich and Landau, *l.c. supra*.

§ J. E. Littlewood, "The converse of Abel's theorem on power-series", *Proc. London Math. Soc.*, Ser. 2, Vol. 9, 1911, 434–448.

‖ Tauber (*l.c. supra*) proves this when C is the radius. We cannot refer to an explicit proof for the case in which C is an arbitrary Stolz-path; but the result is an immediate consequence of the arguments used by Tauber and by Landau.

¶ That is to say, $na_n < H$ or $na_n > -H$, where H is a constant. G. H. Hardy and J. E. Littlewood, **2**; see also Landau's book referred to above, pp. 45–56. The condition is plainly more general than (O) when a_n is real.

These theorems can only be appreciated if we bear in mind other results of a negative character. The trivial example

$$1-1+1-\ldots,$$

is enough to show that the existence of Abel's limit does not involve the convergence of the series; thus "(L) implies (K)", the straightforward converse of Abel's Theorem, is false. It is not quite so easy to find a similar example in which the terms of the series tend to zero. This was first done by Pringsheim*; but a more natural example is provided by the series (1 . 21) when $0 < b < 1-a$. Here $a_n = O(n^{-b})$, and a may be as small, and so b as nearly equal to 1, as we please. Thus no condition of this type, with $b < 1$, is sufficient to ensure the convergence of the series whenever Abel's limit exist. This suggests that **F** is really a "best possible" theorem of its kind; and this is shown by the theorem

K. *There is no function* $\phi(n)$, *such that* $\phi(n) \to \infty$ *and*

$$a_n = O\left\{\frac{\phi(n)}{n}\right\},$$

together with (L), *implies* (K).†

1 . 3. No extension of **F** to paths other than the radius has yet been published. The extension of theorems of the "o" character to paths other than Stolz-paths was first considered seriously in our paper **1**.‡ In this paper we confined ourselves to *regular* paths, and we found that, in order to obtain satisfactory results, it was essential to replace (L) by a different condition. This condition is

$$(\mathrm{A}) \qquad \Phi(x) = \frac{1}{1-x}\sum\frac{a_n}{n+1}(1-x^{n+1}) \to A.$$

It is to be observed that, so long as C is regular, and

$$\sum\frac{a_n}{n+1}$$

is absolutely convergent, we have

$$(1\,.\,31) \qquad \Phi(x) = \frac{1}{1-x}\int_x^1 f(t)\,dt,$$

* A. Pringsheim, "Über die Divergenz gewisser Potenzreihen an der Konvergenzgrenze", *Münchener Sitzungsberichte*, Vol. 31, 1901, pp. 505-524.

† J. E. Littlewood, *l.c.*, p. 444 (Theorem C).

‡ Pp. 475-477.

the integration being effected along C. It is an easy deduction that $\Phi(x) \to A$ in all cases in which $f(x) \to A$, whereas the converse is untrue. Thus (L) implies (Λ), and (Λ) is a generalisation of (L), at any rate in all such cases as we were considering before. This being so, we proved

L. (Λ) *and* (o) *imply* (K) *whenever* C *is a regular path.*

As a corollary we have

M. (L) *and* (o) *imply* (K) *whenever* C *is a regular path.*

This theorem includes **D** and **E** as special cases. We also proved* the direct converse of **L**, viz.

N. (K) *and* (o) *imply* (Λ) *whenever* C *is a regular path.*

This theorem becomes untrue if (Λ) is replaced by (L). It is an "Abelian" theorem, but differs fundamentally from the ordinary Abelian theorems **A** and **B** in that its truth depends upon a condition such as occurs in the "Tauberian" theorems. It is also unlike all the theorems which precede in being reversible: and, on combining it with **L**, we obtain

O. *If* A *satisfies* (o), *then the necessary and sufficient condition for its convergence is that* (Λ) *should be true when* x *tends to* 1, *either along any regular path, or along all.*

1.4. We begin our new investigations by a direct extension of **F** to a regular Stolz-path, viz.

P. (L) *and* (O) *imply* (K) *when* C *is a regular Stolz-path.*

This theorem is included in others which come after and are proved in a quite different way. But the method we use (in 2.1) seems to us of considerable interest in itself.

In 2.2 and the succeeding paragraphs we attack our main problem. Our object is to generalise **L**, (i) by replacing (o) by (O), and (ii) by getting rid of the restriction that C is a regular path; and the result is

Q. (Λ) *and* (O) *imply* (K), *for any path* C.

When C is regular, we can deduce as a corollary

R. (L) *and* (O) *imply* (K) *whenever* C *is a regular path.*

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* The proof of this theorem (Theorem 50) is not stated explicitly, but is virtually contained in that of the preceding Theorem 49 (**L** of this paper).

We can also prove

S. *(L) and (O) imply (K) whenever C is a Stolz-path.*

But we cannot here get rid of *all* restrictions upon C.

In 3 we proceed to the corresponding Abelian theorems, and generalise **N** by proving

T. *(K) and (O) imply* (Λ), *for any path C.*

And by combining **Q** and **T**, we obtain

U. *If A satisfies (O), then the necessary and sufficient condition for its convergence is that* (Λ) *should be true when x tends to* 1, *either along any path C, or along all.*

This theorem affords the complete generalisation of **O** in each of the desired directions, and is far more comprehensive than any known theorem of its kind.

There are but few questions which remain to be answered. There is one to which we have already alluded and are unable to answer, namely whether **R** (or **S**) is true without any restriction on C. The others are connected with the question whether **T** is the "best possible" theorem of its kind. We prove

V. *(K) and (O) do not necessarily imply (L) for all paths C:*

so that the (Λ) of **T** certainly cannot be replaced by (L). And almost the same example suffices to prove

W. *(K) does not necessarily imply* (Λ) *for all paths C.*

Thus T certainly becomes untrue if the condition (O) is simply omitted. But it is desirable to prove more, viz. that (O) cannot be replaced by any less restrictive condition of the type which occurs in **K**, and we have not yet succeeded in establishing this by means of an example. If we could do this, and also remove the restriction on C in **R** (or **S**), we could fairly claim that our problem had been completely solved.

1.5. If, in Theorems **Q**–**U**, we suppose that C is an arc of the unit circle itself, we obtain a number of theorems concerning the convergence of a series

$$\Sigma(a_n+i\beta_n)\,e^{ni\theta} = \Sigma(a_n \cos n\theta+\beta_n \sin n\theta)+i\Sigma\,(a_n \sin n\theta-\beta_n \cos n\theta),$$

where $$a_n = O\left(\frac{1}{n}\right), \quad \beta_n = O\left(\frac{1}{n}\right);$$

theorems, that is to say, concerning the simultaneous convergence of a Fourier series and its *conjugate* or *allied* series. It is, however, more natural and more interesting to consider the two series independently. We prove first*

X. *If a_n and b_n satisfy (O), so that*

$$\Sigma(a_n \cos n\theta + b_n \sin n\theta)$$

is certainly the Fourier series of a summable function $f(\theta)$, then the necessary and sufficient condition that the series should converge to the sum A is that

$$\frac{1}{2\alpha}\int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt \to A$$

when $\alpha \to 0$.

When (O) is replaced by (o), Theorem **X** reduces to a theorem of Fatou.† These theorems correspond to **O** and **U**. The remaining theorems of the paper are of a somewhat different character: the most interesting of them are **Y** and **Z**, which are concerned with the Fourier series of bounded functions, and do not depend upon conditions such as (o) or (O).

We conclude these introductory remarks by giving a list of the papers of our own to which we shall have to refer. They are :—

1. "Contributions to the arithmetic theory of series", *Proc. London Math. Soc.*, Ser. 2, Vol. 11, 1913, pp. 411–477.

2. "Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive", *ibid.*, Vol. 13, 1914, pp. 174–191.

3. "Some theorems concerning Dirichlet's series", *Messenger of Math.*, Vol. 43, 1914, pp. 134–147.

4. "Theorems concerning the summability of series by Borel's exponential method", *Rendiconti del Circ. Mat. di Palermo*, Vol. 41, 1916, pp. 36–53.

5. "Sur la convergence des séries de Fourier et des séries de Taylor", *Comptes Rendus*, December 24, 1917.

* We revert to the notation usual in the theory of Fourier series.

† P. Fatou, "Séries trigonométriques et séries de Taylor", *Acta Mathematica*, Vol. 30, 1906, pp. 335–400 (p. 385).

2.

The Tauberian theorems.

Proof of Theorem **P**.

2.1. Theorem **P**.—*If*

(O) $$a_n = O\left(\frac{1}{n}\right),$$

and (L) $$f(x) = \Sigma a_n x^n \to A,$$

when $x \to 1$ *along a regular Stolz-path* C, *then* Σa_n *converges to the sum* A.

We suppose, as obviously we may without loss of generality, that $|na_n| < 1$ and $A = 0$. We write

$$x = re^{i\theta} = e^{-\rho+i\theta},$$

and we suppose first that C is the particular curve

(2.11) $$\theta = k\rho.$$

Then, if $$F_k(\rho) = \Sigma a_n e^{-(1-ki)n\rho},$$

we have $$F_k(\rho) = o(1)$$

when $\rho \to 0$, and

$$\rho^p F_k^{(p)}(\rho) = (-1)^p (1-ki)^p \rho^p \Sigma n^p a_n e^{-(1-ki)n\rho}$$
$$= O(\rho^p \Sigma n^{p-1} e^{-n\rho}) = O(1),$$

for every positive integral value of p. It follows, from our fundamental theorems on derivatives*, that

(2.12) $$\rho^p F_k^{(p)}(\rho) = o(1),$$

for every such value of p.

We shall now prove that

(2.13) $$F_l(\rho) = o(1),$$

for any value of l such that $k-1 \leqslant l \leqslant k+1$.

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* The particular theorem required is obtained by supposing $\phi = \psi = 1$ in Case (b) of Theorems 6 and 8 of our paper **1**.

If $l = k+\delta$, so that $|\delta| \leqslant 1$, we have

$$F_l(\rho) = \Sigma\, a_n e^{-(1-ki)\,n\rho}\, e^{\delta i n\rho}$$

$$= \sum_{(n)} a_n e^{-(1-ki)\,n\rho} \left\{ \sum_{p=0}^{P-1} \frac{(\delta i n\rho)^p}{p!} + \Delta_P \right\}$$

$$= \sum_{p=0}^{P-1} \frac{(\delta\rho)^p}{p!} \sum_{(n)} n^p a_n e^{-(1-ki)\,n\rho} + R_P,$$

where

$$|\Delta_P| < \frac{2(\delta n\rho)^P}{P!},$$

$$|R_P| < \frac{2(\delta\rho)^P}{P!}\, \Sigma n^P\, |a_n|\, e^{-n\rho}.$$

Now $|na_n| < 1$, and

$$\Sigma n^{P-1} e^{-n\rho} < \int_0^\infty \xi^{P-1} e^{-\xi\rho}\, d\xi + 2\,\mathrm{Max}\,(\xi^{P-1} e^{-\xi\rho})$$

$$= \frac{\Gamma(P)}{\rho^P} + 2\left(\frac{P-1}{\rho}\right)^{P-1} e^{-(P-1)}$$

$$< \frac{2\Gamma(P)}{\rho^P},$$

for $0 < \rho < \rho_0$, where ρ_0 is a number independent of P. Hence

(2 . 14)
$$|R_P| < \frac{4\delta^P}{P} \leqslant \frac{4}{P}.$$

And therefore

$$F_l(\rho) = \sum_{(p)} \frac{(\delta i\rho)^p}{p!} \sum_{(n)} n^p a_n e^{-(1-ki)\,n\rho}$$

$$= \sum_{(p)} \frac{1}{p!} \left(-\frac{\delta i}{1-ki}\right)^p \rho^p F_k^{(p)}(\rho);$$

and this series is, in virtue of (2 . 14), *uniformly convergent for* $|\delta| \leqslant 1$ *and* $\rho > 0$. But, by (2 . 12), every term of the series tends to zero. Hence $F_l(\rho)$ tends to zero* : that is to say, we have proved (2 . 13).

It follows, by the repeated application of this argument, that if a_n satisfies (O) and $f(x)$ tends to zero along any Stolz-path of the type (2.11),

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* Our argument here is the same in principle as that which we used in the proof of the "general Borel-Tauber" theorem : see our paper **4**, p. 44.

then it tends to zero along any other Stolz-path of the same type. In particular it tends to zero along the real axis, to which the path reduces when $k = 0$. And hence, by Theorem **F**, the series Σa_n converges to zero.

The theorem is thus proved for paths of the special type (2.11). In general, the equation of a regular Stolz-path may be written in the form

$$\theta = k\rho + o(\rho). \tag{2.15}$$

It is easy to see that, if $f(x)$ tends to zero along (2.15), it also tends to zero along (2.11). For, if (ρ, θ) and (ρ, θ') are corresponding points on the paths (2.11) and (2.15), we have

$$|e^{ni\theta} - e^{ni\theta'}| = |e^{ni(\theta-\theta')} - 1| = o(n\rho)$$

when $\rho \to 0$, and so

$$f(e^{-\rho+i\theta}) - f(e^{-\rho+i\theta'}) = o(\rho)\Sigma n\,|a_n|\,e^{-n\rho} = o(1).$$

The truth of the theorem in its general form now follows from the argument which precedes.

Proof of Theorem **Q**.

2.2. Theorem **Q**.—*If*

$$a_n = O\left(\frac{1}{n}\right), \tag{O}$$

and

$$\Phi(x) = \frac{1}{1-x} \Sigma \frac{a_n}{n+1}(1 - x^{n+1}) \to A, \tag{Λ}$$

when $x \to 1$ along some path C, then Σa_n converges to the sum A.

We may suppose that $A = 0$, $a_0 = 0$, and $|na_n| < 1$. And we shall begin by proving

Lemma α.—*(O) being satisfied, the necessary and sufficient condition that*

$$s_n = a_1 + a_2 + \ldots + a_n$$

should be bounded is that $\Phi(x)$ should be bounded for $|x| \leqslant 1$, $x \neq 1$; and this condition is satisfied if $\Phi(x)$ is bounded when $x \to 1$ along any particular path C.

We take
$$\nu = \left[\frac{1}{|1-x|}\right],$$
and we may suppose $|1-x| < 1$, so that
$$\tfrac{1}{2} < \nu|1-x| \leqslant 1.^*$$

Then

$$(2.21)\qquad \Phi(x) = \left(\sum_0^{\nu-1} + \sum_\nu^\infty\right) \frac{a_n}{n+1}\,\frac{1-x^{n+1}}{1-x} = \phi_1+\phi_2,$$

say. In the first place we have

$$(2.22)\qquad |\phi_2| < \frac{2}{|1-x|}\sum_\nu^\infty \frac{1}{n(n+1)} = \frac{2}{\nu|1-x|} < 4.$$

Secondly,
$$1-\frac{1-x^{n+1}}{(n+1)(1-x)} = \frac{1}{n+1}\{(1-x)+(1-x^2)+\ldots+(1-x^n)\},$$
$$\left|1-\frac{1-x^{n+1}}{(n+1)(1-x)}\right| < \frac{|1-x|}{n+1}(1+2+\ldots+n) = \tfrac{1}{2}n|1-x|,$$
$$(2.23)\qquad |s_{\nu-1}-\phi_2| \leqslant \sum_1^{\nu-1}|a_n|\left|1-\frac{1-x^{n+1}}{(n+1)(1-x)}\right| < \tfrac{1}{2}|1-x|\sum_1^{\nu-1}|na_n|$$
$$< \nu|1-x| \leqslant 1.$$

From (2.21), (2.22), and (2.23), we obtain
$$|s_{\nu-1}-\Phi(x)| < 5;$$
which proves both parts of the lemma, since ν passes through an unbroken sequence of integral values when $x \to 1$ along C.

2.3. The remainder of the proof of Theorem **Q** depends upon certain lemmas in the theory of analytic functions, of the general type associated particularly with the names of Phragmén and Lindelöf.†

* It is easily verified that $y[1/y] > \frac{1}{2}$ if $0 < y < 1$.

† See in particular the memoir "Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogènes dans le voisinage d'un point singulier", *Acta Mathematica*, Vol. 31, 1908, pp. 381–406. The chief results of this memoir may now be regarded as classical.

LEMMA β.*—*Suppose that T is the semi-infinite strip, in the plane of the complex variable $w = u+iv$, defined by the inequalities $0 \leqslant u \leqslant \pi$, $v \geqslant v_0$; that* **A** *and* **B** *are the left and right-hand edges of the strip, and that* **C** *is a simple Jordan curve which lies entirely inside T, extends to infinity, and divides T into two regions L and R. Suppose further that $f(w)$ is regular inside T and continuous and bounded throughout T.*

Finally, suppose that

$$(2.31) \qquad \overline{\lim}\ |f(w)| \leqslant h,$$

where $h > 0$, when w tends to infinity along **A** *(or* **B***) and along* **C**. *Then (2.31) holds when w tends to infinity in any manner inside L (or R).*

Let us suppose that the data refer to **A** and **C**. We can choose $v_1(\epsilon)$ so that

$$(2.33) \qquad |f| < h+\epsilon,$$

at all points of **A** and **C** for which $v > v_1$. We can then draw a cross-cut (*Querschnitt*) Q, cutting off from L an infinite region L_Q; and (2.33) will be satisfied at all points of $\mathbf{A}_Q$ and $\mathbf{C}_Q$, the parts of **A** and **C** which belong to the boundary of L_Q.

Let K be the upper bound of $|f|$ on Q. We can now choose

$$p(\epsilon, v_1, K) = p(\epsilon),$$

so that
$$\left|\frac{w}{w+p}\right| < 1,$$

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* We state this and the following lemmas in the special forms in which they are required for our immediate purpose. All of them, naturally, are capable of wide generalisation. The most interesting of these generalisations is the following:

If (1) *T is the strip $\alpha \leqslant u \leqslant \beta$, $v \geqslant v_0$;* **A** *and* **B** *its edges;* $\mathbf{C}_\alpha$ *and* $\mathbf{C}_\beta$ *simple non-intersecting Jordan curves interior to T and asymptotic to* **A** *and* **B**; *and* **C** *a similar curve asymptotic to $u = \nu$, where $\alpha < \nu < \beta$:*

(2) *$f(w)$ is regular inside and continuous throughout the region T' formed by those points of T which lie to the right of* $\mathbf{C}_\alpha$ *and to the left of* $\mathbf{C}_\beta$:

(3) $\overline{\lim}\ |f(w)| \leqslant \mathbf{a}$ *when w tends to infinity along* $\mathbf{C}_\alpha$, *and* $\overline{\lim}\ |f(w)| \leqslant \mathbf{b}$ *when w tends to infinity along* $\mathbf{C}_\beta$:

$$(4) \qquad f(w) = O(e^{e^{cv}}),$$

where $c < \pi/(\beta-\alpha)$, uniformly throughout T': then

$$\overline{\lim}\ |f(w)| \leqslant \mathbf{a}^{(\beta-\nu)/(\beta-\alpha)}\mathbf{b}^{(\nu-\alpha)/(\beta-\alpha)},$$

when w tends to infinity along **C**. *This result still holds when* **a** *or* **b** *or both are zero.*

on $\mathbf{A}_Q$ and $\mathbf{B}_Q$, and
$$\left|\frac{w}{w+p}\right| < \frac{h+\epsilon}{K}$$
on Q. And if we write
$$F = \frac{wf}{w+p},$$
we have
$$|F| < h+\epsilon$$
at all points on the boundary of L_Q; and therefore* at all points of L_Q.

Hence
$$\overline{\lim}\, |F| \leqslant h,$$
when w tends to infinity in any manner inside L; and therefore
$$\overline{\lim}\, |f| \leqslant h,$$
which proves the lemma.

Lemma γ.—*Suppose that* (2.31) *is satisfied on both* **A** *and* **B**, *and that*

(2.34)
$$\overline{\lim}\, |f(w)| \leqslant \delta,$$
where $0 < \delta < h$, *on* **C**. *Then*

(2.35)
$$\overline{\lim}\, |f(w)| \leqslant \sqrt{(\delta h)},$$
when w *tends to infinity along* **M**, *the straight line equidistant between* **A** *and* **B**.

We denote by M_L and M_R those parts of M which lie in L and R respectively.

Write
$$g = e^{kw} f,$$
where
$$e^{k\pi} = h/\delta.$$
Then
$$\overline{\lim}\, |g| \leqslant h,$$
on A, and
$$\overline{\lim}\, |g| \leqslant e^{k\pi}\delta = h,$$
on **C**. Hence
$$\overline{\lim}\, |g| \leqslant h,$$
on M_L; and so
$$\overline{\lim}\, |f| \leqslant he^{-\frac{1}{2}k\pi} = \sqrt{(\delta h)},$$
on M_L. Similarly, using an auxiliary function
$$g = e^{k(\pi - w)} f,$$
we can show that (2.35) holds on M_R, and so on the whole of M.

Lemma δ.—*If* $f(w) = O(1)$ *throughout* T, *and* $f(w) = o(1)$ *on* **C**, *then* $f(w) = o(1)$ *on* M.

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* Phragmén and Lindelöf, *l.c.*, p. 388.

Write $$g = f + \delta,$$

where $0 < \delta < 1$. Then we can choose an h independent of δ and such that the conditions of Lemma γ are satisfied; and

$$\overline{\lim} \mid g \mid \leqslant \sqrt{(\delta h)}$$

on M. Hence $$\overline{\lim} \mid f \mid \leqslant \sqrt{(\delta h)} + \delta$$

on M; which proves the theorem, since δ is arbitrarily small.

2.4. We now transform Lemma δ, by means of the theory of conformal representation, into a proposition suitable for direct application to the theory of power-series.

The equation $$\frac{x-1}{x+1} = ie^{iw}$$

transforms the strip $0 \leqslant v \leqslant \pi$ into the unit circle in the plane of x. The point $w = \frac{1}{2}\pi$ corresponds to $x = 0$, the lines **A** and **B** to the upper and lower halves of the circle, and the line M to the real diameter. The upper and lower ends of the strip correspond to $x = 1$ and $x = -1$ respectively. If, finally, we observe that

$$\left| \frac{1+x}{1-x} \right| = e^{v},$$

we obtain

LEMMA ϵ.—*Suppose that $\Phi(x)$ is regular for $|x| < 1$, and continuous and bounded for $|x| \leqslant 1$, $x \neq 1$. Suppose further that $\Phi(x) = o(1)$ when x tends to 1 along a certain internal path C. Then $\Phi(x) = o(1)$ when x tends to 1 by real values.*

2.5. We can now prove our main theorem. In the first place $\Phi(x)$ is continuous at all points in or on the circle, except perhaps the point $x = 1$.* We may therefore suppose, without loss of generality, that C is an internal path.†

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* The series $$\Sigma \frac{a_n}{n+1}(1-x^{n+1}) = \Sigma\, O\left(\frac{1}{n^2}\right)$$

is plainly uniformly convergent.

† We can replace C by an internal path C', which differs so little from C that $\Phi(x)$ tends to zero along C'.

Since Φ tends to a limit along C, s_n is bounded, by Lemma α; and therefore, by the same lemma, Φ is bounded for $|x| \leqslant 1$, $x \neq 1$. Hence Φ satisfies all the conditions of Lemma ϵ; and so

$$\Phi(x) = o(1) \tag{2.51}$$

when x tends to 1 along the real axis.

Suppose then that x is real. We have

$$\Phi'(x) = \frac{1}{1-x}\Phi(x) - \frac{1}{1-x}f(x),$$

$$\Phi''(x) = \frac{2}{(1-x)^2}\Phi(x) - \frac{2}{(1-x)^2}f(x) - \frac{1}{1-x}f'(x).$$

Now $\Phi(x)$ is bounded; $f(x)$ is bounded, since s_n is bounded; and

$$f'(x) = \Sigma n a_n x^{n-1} = \Sigma O(1)\, x^n = O\left(\frac{1}{1-x}\right).$$

Hence

$$\Phi''(x) = O\left\{\frac{1}{(1-x)^2}\right\}. \tag{2.52}$$

From (2.51) and (2.52), we deduce, by Theorem 8 of our paper **1**,

$$\Phi'(x) = o\left(\frac{1}{1-x}\right);$$

and so
$$f(x) = \Phi(x) - (1-x)\Phi'(x) = o(1).$$

That is to say, *Abel's limit for $f(x)$ exists when $x \to 1$ by real values;* and therefore, by Theorem **F**, the series Σa_n is convergent.

Theorems **R** *and* **S**.

2.6. When C is *regular*,

$$f(x) \to A \tag{L}$$

implies

$$\Phi(x) \to A. \tag{Λ}$$

We have therefore

THEOREM **R**.—*If* (O) *is satisfied, and* $f(x) \to A$ *when* $x \to 1$ *along some regular path* C, *then* Σa_n *converges to the sum* A.

This theorem of course includes **P** as a special case. It is in the most essential respects as complete a generalisation of **F** as could be desired, for, although it involves a considerable limitation as to the nature of C, there is no limitation at all on the contact of C with the circle. The contact may be as close as we please, or C may be the circle itself: and it is questions of contact rather than of regularity which are of the first interest in these investigations.

It is, however, natural to suppose that *no* limitation on C is necessary; and, if this be so, it is very desirable that it should be proved. It would seem, however, that some important change in our argument would be needed for such a proof; for, unless C is subject to considerable restrictions, it is not possible to deduce the behaviour of Φ from that of f by means of its representation as an integral taken along C.

It is interesting to observe that, if we limit C to be a Stolz-path, we can get rid of the restriction that it is to be regular. We shall only sketch the proof, the general lines of which are as follows. We draw two chords of the circle through $x = 1$, including C between them, and we consider the region T formed by points which lie between these chords and within a certain distance of $x = 1$.

By a slight modification of the proof of Theorem **E** given by Landau, we can prove that s_n is bounded, and so that $f(x)$ is bounded in T.* And by an adaptation of the arguments used in §§ 2 . 3–4, we can show that f tends to a limit when $x \to 1$ along any regular Stolz-path inside T. If the real axis satisfies this condition, we can appeal to Theorem **F**; if not, to Theorem **P** or **R**. In any case, we obtain

THEOREM **S.**—*If* (O) *is satisfied and* $f(x) \to A$ *when* $x \to 1$ *along some Stolz-path* C, *then* Σa_n *converges to the sum* A.

3.

THE ABELIAN THEOREMS.

Proof of Theorem **T**.

3 . 1. THEOREM **T.**—*If*

(O) $$a_n = O\left(\frac{1}{n}\right)$$

and

(K) $$\Sigma a_n = A,$$

* The argument fails at this stage if C is not a Stolz-path. We could not prove that f is bounded in T if the boundaries of T touched the circle.

then

$$\text{(A)} \qquad \Phi(x) = \frac{1}{1-x} \Sigma \frac{a_n}{n+1} (1-x^{n+1}) \to A,$$

when $x \to 1$ *in any manner.*

One observation should be made before we proceed to the proof. The enunciation contains no reference to a *path*: we may say, of course, "when $x \to 1$ along any path C"; but the idea of approach along a continuous path is not really relevant. In this respect there is an essential difference between this theorem and the Tauberian theorems.

We may suppose, as before, that $A = 0$ and $|na_n| < 1$. That $\Phi(x)$ is *bounded* follows from Lemma α; but to prove convergence to a limit requires an argument of somewhat greater subtlety.

What we have to prove is that

$$(3.11) \qquad S(x) = \Sigma \frac{a_n}{n+1} (1-x^{n+1}) = o(1-x).$$

We write

$$(3.12) \qquad S = \sum_0^{m-1} + \sum_m^{\infty} = S_1 + S_2,$$

where

$$(3.13) \qquad m = K\nu = K\left[\frac{1}{|1-x|}\right],$$

K being a (large) parameter. It is plain that we may suppose $|1-x|$ small enough to ensure that

$$(3.14) \qquad \frac{K}{2|1-x|} < m \leqslant \frac{K}{|1-x|}.$$

As regards S_2, we have

$$(3.15) \qquad |S_2| < 2 \sum_m^{\infty} \frac{1}{n(n+1)} = \frac{2}{m} < \frac{4}{K} |1-x|.$$

In order to deal with S_1, we require a lemma.

3.2. Lemma ζ.—*If* Σa_n *is convergent, and*

$$t_n = \sum_n^{\infty} \frac{a_\mu}{\mu+1},$$

then (i) $$t_n = o\left(\frac{1}{n}\right),$$

and (ii) Σt_n *converges to the same sum as* Σa_n.

The proof of this lemma is very simple. We suppose, as before, that $A = 0$. We can choose n_0 so that

$$|a_n + a_{n+1} + \ldots + a_\mu| < \delta,$$

for $n_0 \leqslant n < \mu$, and then

(3.21) $$|t_n| = \left| \sum_n^\infty \frac{a_n + a_{n+1} + \ldots + a_\mu}{(\mu+1)(\mu+2)} \right| < \frac{\delta}{n+1}.$$

The same argument shows that

(3.22) $$|t_n| < \frac{\mathbf{m}}{n+1},$$

for all values of n, $\mathbf{m}$ being the maximum of $|a_n + a_{n+1} + \ldots + a_\mu|$ for all values of n and μ.

Finally, we have

$$\sum_0^n t_\mu = \sum_0^n a_\mu + (n+1) \sum_{n+1}^\infty \frac{a_\mu}{\mu+1},$$

and the last term tends to zero. We have thus

(3.23) $$\left| \sum_0^n t_\mu \right| < \delta,$$

for $n \geqslant n_1$.

3.3. Now

(3.31) $$S_1 = \sum_0^{m-1} \frac{a_n}{n+1}(1-x^{n+1}) = \sum_0^{m-1} (t_n - t_{n+1})(1-x^{n+1})$$

$$= (1-x) \sum_0^{m-1} t_n - (1-x) \sum_0^{m-1} t_n (1-x^n) - t_m(1-x^m)$$

$$= S_1' + S_1'' + S_1''',$$

say. Plainly

(3.32) $$|S_1'''| < \frac{2\delta}{m} < \frac{4\delta}{K} |1-x|,$$

if $m \geqslant n_0$, which is certainly so when $|1-x|$ is sufficiently small. Secondly

(3.33) $$|S_1'| < \delta |1-x|,$$

by (3.23), if $m \geqslant n_1$, which is certainly so when $|1-x|$ is sufficiently small.

Finally,

$$(3.34)\quad |S_1''| < |1-x|^2 \sum_0^{m-1} n|t_n| = |1-x|^2 \sum_0^{n_0-1} n|t_n| + |1-x|^2 \sum_{n_0}^{m-1} n|t_n|$$

$$< \mathbf{m} n_0 |1-x|^2 + \delta m |1-x|^2$$

$$< \mathbf{m} n_0 |1-x|^2 + 2K\delta |1-x|.$$

From (3.12), (3.15), and (3.31)–(3.34), it follows that

$$(3.35)\qquad |S| < |1-x| \left(\frac{4}{K} + 2K\delta + \delta + \frac{4\delta}{K} + \mathbf{m} n_0 |1-x|\right).$$

We can choose $K = K(\epsilon)$ so that $4/K < \frac{1}{3}\epsilon$; then $\delta = \delta(\epsilon, K) = \delta(\epsilon)$ so that

$$2K\delta + \delta + \frac{4\delta}{K} < \tfrac{1}{3}\epsilon;$$

and then $\eta = \eta(\epsilon, K, \delta) = \eta(\epsilon)$ so that (3.35) is satisfied, and

$$\mathbf{m} n_0 |1-x| < \tfrac{1}{3}\epsilon$$

if $|1-x| < \eta$. Thus $$|S| < \epsilon |1-x|$$

if $|1-x| < \eta$; which proves the theorem.

Combining Theorems **Q** and **T**, we obtain

THEOREM **U**.—*If the coefficients of the series Σa_n satisfy (O), then the necessary and sufficient condition for its convergence is that $\Phi(x)$ should tend to a limit, either along any particular path C, or along all.*

Proof of Theorems **V** *and* **W**.

3.4. Theorem **T** leaves an important question unanswered. Is it certain that the hypotheses do not imply, what is more* than the theorem asserts, that

$$(L)\qquad f(x) \to A$$

along C?

Theorem **B** shows that this is so when C is a Stolz-path: in this case

* (L) asserts more than (Λ) at any rate when C is regular, that is to say in all ordinary cases (cf. pp. 219–220).

indeed the convergence of the series ensures the existence of the limit without any hypothesis as to the order of a_n. Theorem **C** shows that this at any rate ceases to be true when C is allowed to touch the circle. In the example which we attached to our statement of Theorem **C**, we have to take

$$1-a < b \leqslant 1-\tfrac{1}{2}a;$$

and these inequalities allow b to exceed any number less than 1. It follows* that no condition of the type $a_n = O(n^{-b})$, where $b<1$, is enough, in conjunction with the convergence of the series, to ensure the existence of Abel's limit along all tangential paths.

We shall now show that even the condition

$$(o) \qquad a_n = o\left(\frac{1}{n}\right)$$

is not enough for this purpose. It would be easy to modify our argument in such a manner as to show that *no* condition of the form

$$a_n = O(\chi_n),$$

where χ_n is a steadily decreasing function such that $\Sigma\chi_n$ is divergent, is enough: for simplicity, however, we take

$$\chi_n = \frac{1}{n\log n}.$$

THEOREM **V**.—*It is possible to find a convergent series Σa_n for which*

$$a_n = O\left(\frac{1}{n\log n}\right),$$

and a regular path C, such that $f(x)$ does not tend to a limit when x tends to 1 along C.

We take $$a_n = \frac{1}{n\log n}\sin\frac{n\pi}{j},$$

if $$n_j = e^{e^{e^j}} < n < n_{j+1} = e^{e^{e^{j+1}}}$$

To prove that Σa_n is convergent, we write

$$w_{j,k} = \sum_{n_j < n \leqslant k < n_{j+1}} a_n, \qquad w_j = \sum_{n_j<n<n_{j+1}} a_n,$$

* Compare p. 208, where we used the same series in a different manner to establish the same point about the Tauberian Theorem **F**

so that $$w_j = w_{j,\,[n_{j+1}]}.$$

Since $n \log n$ increases steadily with n, and $\Sigma \sin (n\pi/j)$, taken between any limits, is not greater in absolute value than $2 \operatorname{cosec} (\pi/j)$, we have

$$w_{j,k} = O\left(\frac{j}{n_j \log n_j}\right), \quad w_j = O\left(\frac{j}{n_j \log n_j}\right).$$

Since $$\Sigma \frac{j}{n_j \log n_j}$$

is convergent, Σw_j is absolutely convergent; and since $w_{j,k} = o(1)$, Σa_n is convergent.

Now let $$f(x) = \Sigma a_n x^n,$$

$$\phi(r, \theta) = \mathfrak{I}\left[f(re^{i\theta})\right] = \Sigma a_n r^n \sin n\theta.$$

We shall show that $\phi(r, \theta)$ does not tend to a limit when $x \to 1$ along a regular path which has sufficiently close contact with the upper half of the unit circle.

To prove this we observe first that the series

$$(3\,.\,41) \qquad \Sigma a_n \sin n\theta$$

is convergent when θ is positive. In fact, if we write

$$u_{j,k} = \sum_{n_j < n \leqslant k < n_{j+1}} a_n \sin n\theta, \quad u_j = \sum_{n_j < n < n_{j+1}} a_n \sin n\theta,$$

and j is so large that $\pi/j < \frac{1}{2}\theta$, we find, by the same argument that was used above, that

$$u_{j,k} = \tfrac{1}{2} \Sigma \frac{1}{n \log n} \left\{ \cos n \left(\theta - \frac{\pi}{j}\right) - \cos n \left(\theta + \frac{\pi}{j}\right)\right\}$$

$$= O\left(\frac{1}{\theta n_j \log n_j}\right),$$

and in particular that $$u_j = O\left(\frac{1}{\theta n_j \log n_j}\right).$$

From these relations the convergence of (3.41) follows immediately.

If now we denote the sum of (3 . 41) by $\psi(\theta)$, we have

$$\psi(\theta) = \lim_{r\to 1} \phi(r, \theta),$$

for every positive θ. It is thus sufficient to show that $\psi(\theta)$ does not tend to a limit when $\theta \to 0$. Now

$$(3.42) \quad \psi\left(\frac{\pi}{j}\right) = \sum_{n_j}^{n_{j+1}} \frac{1}{n \log n} \sin^2 \frac{n\pi}{j} + \sum_{k\neq j} \sum_{n_k}^{n_{k+1}} \frac{1}{n \log n} \sin \frac{n\pi}{j} \sin \frac{n\pi}{k}$$

$$= \psi_1 + \psi_2,$$

say. Suppose j even. Then, if j is large, there are, between n_j and n_{j+1}, more than $n_{j+1}/2j$ numbers n of the form $(m+\frac{1}{2})j$, where m is an integer; and the sum

$$\Sigma \frac{1}{n \log n},$$

extended to these values of n, is greater than

$$\frac{1}{4j} \log \left(\frac{\log n_{j+1}}{\log n_j}\right) = \frac{e-1}{4j} e^j.$$

Thus

$$(3.43) \qquad | \psi_1 | > \frac{e-1}{4j} e^j.$$

On the other hand, the inner sum in ψ_2 is less than a constant multiple of

$$\frac{jk}{|j-k|} \frac{1}{n_k \log n_k} \leqslant \frac{jk}{n_k \log n_k},$$

and so

$$(3.44) \qquad | \psi_2 | = O\left(j \Sigma \frac{k}{n_k \log n_k}\right) = O(j).$$

Finally, from (3 . 42)–(3 . 44), it follows that

$$\psi\left(\frac{\pi}{j}\right) = \psi_1 + \psi_2 \to \infty,$$

which proves the theorem.

It is hardly necessary to point out that Theorem **V** greatly increases the interest of Theorem **T**, and indeed of Theorem 50 of our paper **1**, of which Theorem **T** is a generalisation.

3.5. We can easily adapt the example by which we proved Theorem **V**, so as to prove

THEOREM **W**.—*It is possible to find a convergent series* Σa_n *and a regular path* C, *so that* $\Phi(x)$ *does not tend to a limit when* x *tends to* 1 *along* C.

We take
$$a_n = (n+1)\, b_n,$$
where b_n is the a_n of the last paragraph. The argument by which we proved Σb_n convergent is sufficient to establish the convergence of Σa_n.*

Also
$$\Phi(x) = \frac{1}{1-x}\,\Sigma b_n(1-x^{n+1}) = \frac{B-xg(x)}{1-x},$$
where $B = \Sigma b_n$ and $g(x) = \Sigma b_n x^n$. And as $g(x)$ does not tend to a limit, $\Phi(x)$ does not do so.

This proves the theorem. It will be observed that there is a great deal to spare in the conclusion: we have proved, in fact, that $\Phi(x)$ assumes values of order greater than that of $1/|1-x|$. The fact is that it ought to be possible to prove that series exist which satisfy the conditions of Theorem **W** and whose coefficients are of order $\phi(n)/n$, where $\phi(n)$ is any function which tends to infinity with n. Such an example would (in conjunction with Theorem **V**) prove that Theorem **T** is a best possible theorem in the same sense as (*e.g.*) Theorem **F**, and that neither hypotheses nor conclusion can be improved upon. We have no doubt that this is true, but we have not succeeded in finding an example to prove our point. In our last example the order of a_n is $1/\log n$, which is far from the limit desired. It is therefore not surprising that we should find that our argument carries us some distance over the mark.

At any rate, however, Theorem **W** is enough to show that, in Theorem **T**, some condition beyond that of mere convergence is essential.

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* The function
$$\phi(n) = \frac{n+1}{n\log n}$$
has the properties (i) that it decreases steadily to the limit zero, and (ii) that
$$\Sigma j\phi(n_j)$$
is convergent. These were the only properties of $1/(n\log n)$ used in the proof of the convergence of Σb_n.

4.

FOURIER SERIES.

Proof of Theorem X.

4 . 1. It has been proved by Fatou* that, *if*

$$a_n = o\left(\frac{1}{n}\right), \quad b_n = o\left(\frac{1}{n}\right), \tag{4 . 11}$$

then the necessary and sufficient condition that the series

$$\tfrac{1}{2}A_0 + \Sigma A_n = \tfrac{1}{2}a_0 + \Sigma (a_n \cos n\theta + b_n \sin n\theta), \tag{4 . 12}$$

which is certainly the Fourier series of a summable function $f(\theta)$ †, *should converge to the sum* A, *is that*

$$\frac{1}{2\alpha}\int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt \to A \tag{4 . 13}$$

when $\alpha \to 0$, *or* (what is the same thing) *that*

$$\Sigma A_n \frac{\sin n\alpha}{n\alpha} \to A \tag{4 . 14}$$

when $\alpha \to 0$.

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* Fatou, *l.c.* Fatou does not state the whole of this result explicitly as one theorem, but it is contained in pp. 345–7, 385–7 of his memoir. It is important to observe that, if the conditions (4 . 11) are satisfied and $F(\theta)$ is the integral of $f(\theta)$—or, what is the same thing, the sum of the series obtained by integrating (4 . 12) term-by-term—then

$$F(\theta+\alpha) + F(\theta-\alpha) - 2F(\theta) = o(\alpha)$$

for every value of θ, in virtue of a well known theorem of Riemann (quoted by Fatou, *l.c.*, p. 385). It follows that the three formulæ

$$\frac{1}{\alpha}\int_{\theta}^{\theta+\alpha} f(t)\,dt \to A, \qquad \frac{1}{\alpha}\int_{\theta-\alpha}^{\theta} f(t)\,dt \to A, \qquad \frac{1}{2\alpha}\int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt \to A$$

are equivalent. This ceases to be true when the conditions (4 . 11) are replaced by the more general conditions (4 . 21), as appears at once from the simple example of the series

$$\Sigma \frac{\sin n\theta}{n}.$$

† By the "Riesz-Fischer Theorem", $\Sigma(a_n^2 + b_n^2)$ being convergent. In fact

$$\Sigma(|a_n|^{1+\delta} + |b_n|^{1+\delta})$$

is convergent for every positive δ, from which it follows that $|f(\theta)|^k$ is summable for all positive values of k. See W. H. Young, "On the determination of the summability of a function by means of its Fourier constants", *Proc. London Math. Soc.*, Ser. 2, Vol. 12, 1912, pp. 71–88.

The investigations of Sections 2 and 3 suggest very forcibly that it should be possible to replace the conditions (4 . 11) of Fatou's theorem by the corresponding conditions of the "O" type. We proceed to prove that this is so.

4 . 2. THEOREM **X**.*—*If* a_n *and* b_n *are the Fourier constants of a summable function* $f(\theta)$, *and*

$$a_n = O\left(\frac{1}{n}\right), \quad b_n = O\left(\frac{1}{n}\right), \tag{4.21}$$

then the necessary and sufficient condition that the Fourier series of $f(\theta)$ *should converge to the sum* A *is that*

$$\frac{1}{2a}\int_{\theta-a}^{\theta+a} f(t)\,dt \to A \tag{4.22}$$

when $a \to 0$.

We may plainly take $A = 0$ and suppose that $|nA_n| < 1$.

In the first place, the condition is sufficient. This may be proved in a variety of manners by a mere combination of known theorems.

(*a*) We may prove, by using Poisson's integral in precisely the same way as Fatou, that

$$\Sigma A_n r^n \to 0,$$

when $r \to 1$. The convergence of ΣA_n then follows from Theorem **F**.†

(*b*) It was proved by Lebesgue‡ that the Fourier series of $f(\theta)$ is summable $(C, 2)$, to sum A, for any value of θ for which (4 . 22) is satisfied. The result then follows from the theorem that a series whose general term is of order $1/n$ cannot be summable by Cesàro's means unless it is convergent.§

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* We have already published a proof of this theorem, by a different method, in the *Comptes rendus* of December 24th, 1917 (our paper **5**).

† Fatou, of course, uses Theorem **D**.

‡ H. Lebesgue, "Recherches sur la convergence des séries de Fourier", *Math. Annalen*, Vol. 61, 1905, pp. 251–280 (p. 278).

§ G. H. Hardy, "Theorems relating to the summability and convergence of slowly oscillating series", *Proc. London Math. Soc.*, Ser. 2, Vol. 8, pp. 301–320.

(c) It has been shown by W. H. Young* that if the Fourier series of

$$g(a) = \frac{1}{\sin \frac{1}{2}a} \int_{\theta-a}^{\theta+a} f(t)\,dt$$

is summable (C, r) for $a = 0$, then the Fourier series of $f(\theta)$ is summable $(C, r+1)$. If (4 . 23) is satisfied, $g(a)$ is continuous for $a = 0$, and we can take $r = 1$. The proof may then be completed as under (b).

4 . 3. We have now to prove that the condition is also necessary. We write†

$$\text{(4 . 31)} \qquad \Phi = \sum_1^\infty A_n \frac{\sin na}{na} = \sum_1^m + \sum_{m+1}^\infty = \Phi_1 + \Phi_2,$$

say, where $$m = [K/a],$$

so that $K/2a < m \leqslant K/a$ for all sufficiently small values of a. Then

$$\text{(4 . 32)} \qquad |\Phi_2| < \frac{1}{a} \sum_{m+1}^\infty \frac{1}{n^2} < \frac{1}{ma} < \frac{2}{K}.$$

Again, if we write $$t_n = \sum_n^\infty \frac{A_\mu}{\mu},$$

we have (as in § 3 . 2)

$$\text{(4 . 33)} \qquad |t_n| < \frac{\mathrm{m}}{n}, \qquad |t_n| < \frac{\delta}{n}\ (n \geqslant n_0), \qquad \left|\sum_1^n t_\mu\right| < \delta\ (n \geqslant n_1).$$

Now

$$\text{(4 . 34)} \qquad \Phi_1 = \sum_1^m A_n \frac{\sin na}{na} = \frac{1}{a} \sum_1^m (t_n - t_{n+1}) \sin na$$

$$= \frac{\sin a}{a} \sum_1^m t_n - \frac{\sin a}{a} \sum_1^m t_n \{1 - \cos(n-1)a\}$$

$$- \frac{1-\cos a}{a} \sum_1^m t_n \sin(n-1)a - \frac{1}{a} t_{m+1} \sin ma$$

$$= \Phi_1' + \Phi_1'' + \Phi_1''' + \Phi_1'''',$$

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* W. H. Young, "On the Convergence of a Fourier series and its allied series", *Proc. London Math. Soc.*, Ser. 2, Vol. 10, 1911, pp. 254–272 (pp. 262–266). Young's argument depends only on a series of elementary identities, and includes a new and greatly simplified proof of Lebesgue's theorem quoted above.

† The proof which we give here is (if hardly shorter) considerably simpler in principle than that which we gave in **5**. Naturally Theorem **T** can also be proved by an adaptation of our former method.

say. In the first place

$$(4.35) \qquad |\Phi_1''''| < \frac{\delta}{ma} < \frac{2\delta}{K},$$

if $m \geqslant n_0$, and

$$(4.36) \qquad |\Phi_1'| < \delta,$$

if $m > n_1$; and each of these conditions is satisfied when a is sufficiently small. Secondly,

$$(4.37) \qquad |\Phi_1'''| < \tfrac{1}{2}a \sum_1^m |t_n| < a\mathbf{m} \log m < a\mathbf{m} \log \frac{K}{a}.$$

Finally,

$$(4.38) \qquad |\Phi_1''| < \tfrac{1}{2}a^2 \sum_1^m n^2 |t_n| = \tfrac{1}{2}a^2 \left(\sum_1^{n_0-1} + \sum_{n_0}^m\right) n^2 |t_n|$$

$$< \tfrac{1}{2}a^2\mathbf{m}n_0^2 + \tfrac{1}{2}a^2\delta m^2 < \tfrac{1}{2}a^2\mathbf{m}n_0^2 + \tfrac{1}{2}K^2\delta.$$

From (4.31), (4.32), and (4.34)–(4.38), we deduce

$$(4.39) \qquad |\Phi| < \frac{2}{K} + \tfrac{1}{2}K^2\delta + \delta + \frac{2\delta}{K} + a\mathbf{m}\log\frac{K}{a} + \tfrac{1}{2}a^2\mathbf{m}n_0^2.$$

Given ϵ, we choose $K(\epsilon)$ so that $2/K < \frac{1}{3}\epsilon$; then $\delta(\epsilon, K) = \delta(\epsilon)$ so that

$$\tfrac{1}{2}K^2\delta + \delta + \frac{2\delta}{K} < \tfrac{1}{3}\epsilon;$$

and then $\eta = \eta(\epsilon, K, \delta) = \eta(\epsilon)$ so that (4.39) is satisfied, and

$$a\mathbf{m}\log\frac{K}{a} + \tfrac{1}{2}a^2\mathbf{m}n_0^2 < \tfrac{1}{3}\epsilon,$$

if $0 < a < \eta$. We have then $|\Phi| < \epsilon$,

if $0 < a < \eta$, and the theorem is proved.

4.4. The condition (4.22) is certainly satisfied if $F(\theta)$ has a differential coefficient equal to A. But the convergence of ΣA_n does *not* involve the existence of such a differential coefficient. Thus, if

$$f(\theta) = \Sigma \frac{\sin n\theta}{n},$$

and we consider the particular value $\theta = 0$, the sum of the series is zero, and

$$\frac{1}{2a}\int_{-a}^{a} f(t)\,dt \to 0.$$

But, if a is positive,

$$\frac{1}{a}\int_0^a f(t)\,dt \to \tfrac{1}{2}\pi, \qquad \frac{1}{a}\int_{-a}^0 f(t)\,dt \to -\tfrac{1}{2}\pi.$$

In this respect Theorem **X** differs essentially from Fatou's theorem of which it is the generalisation.

Supplementary Remarks.

4 . 5. We shall conclude the paper with a few theorems of a slightly different character.

The characteristic properties of series which satisfy the condition (o) or (O) are shared to a great extent by series Σa_n such that

(4 . 51) $$\Sigma n\,|a_n|^2,$$

or, more generally,

(4 . 52) $$\Sigma n^p\,|a_n|^{p+1} \quad (p \geqslant 1),$$

are convergent.* It is easy to see that this is true of the properties which have been discussed in this paper.

Let us suppose, for example, that the series (4 . 51) is convergent; and let us return to the proof of Lemma a.

We can choose m so that

(4 . 53) $$\sum_m^\infty n\,|a_n|^2 < \epsilon.$$

This being so, we have

$$|\phi_1| \leqslant \frac{2}{|1-x|}\sum_\nu^\infty \frac{|a_n|}{n+1} \leqslant \frac{2}{|1-x|}\sqrt{\left(\sum_\nu^\infty n\,|a_n|^2 \sum_\nu^\infty \frac{1}{n\,(n+1)^2}\right)}$$

$$\leqslant \frac{2\sqrt{\epsilon}}{\nu\,|1-x|} < 4\sqrt{\epsilon}.$$

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* See L. Fejér, "Über die Konvergenz der Potenzreihe an der Konvergenzgrenze in Fällen der konformen Abbildung auf der schlichte Ebene", *H. A. Schwarz Festschrift*, 1914, pp. 42–53; and our paper **3**. It is to be observed that the theorems which depend upon a condition of this type have all the simpler "o" character, and their proofs do not involve the peculiar difficulties of those of the "O" theorems.

Also

$$(4.54)\qquad |s_{\nu-1}-\phi_1| \leqslant \tfrac{1}{2}|1-x| \sum_1^m n|a_n| + \tfrac{1}{2}|1-x| \sum_{m+1}^{\nu-1} n|a_n|.$$

The first term on the right hand side of (4.54) is less than $\sqrt{\epsilon}$ if x is near enough to 1. The second does not exceed

$$\tfrac{1}{2}|1-x| \sqrt{\left(\sum_m^{\nu-1} n|a_n|^2 \sum_m^{\nu-1} n\right)} \leqslant \tfrac{1}{2}\nu|1-x|\sqrt{\epsilon} < \sqrt{\epsilon}.$$

Hence $$|s_{\nu-1}-\Phi(x)| < 6\sqrt{\epsilon},$$

if x is near enough to 1. As C is continuous, ν passes through an unbroken sequence of integral values as $x \to 1$. We thus obtain

THEOREM **U1**.—*If $\Sigma n|a_n|^2$ is convergent, then the necessary and sufficient condition that Σa_n should be convergent is that $\Phi(x)$ should tend to a limit when x tends to 1, either along any particular path C, or along all.*

There is no difficulty in proving the more general result which holds when the series (4.52) is convergent. It is only necessary to use the generalised form of the Cauchy-Schwarz inequality.*

Similarly we have

THEOREM **X1**.—*If $\Sigma n a_n^2 + b_n^2$ is convergent, then the necessary and sufficient condition for the convergence of the series*

$$\tfrac{1}{2}a_0 + \Sigma(a_n \cos n\theta + b_n \sin n\theta) \sim f(\theta)$$

is that $$\frac{1}{2\alpha}\int_{\theta-\alpha}^{\theta+\alpha} f(t)\,dt$$

should tend to a limit when $\alpha \to 0$.

4.6. We have found that the condition (4.13) is both necessary and sufficient for the convergence of two important classes of Fourier series. There is a third class for whose convergence it is a *necessary*, though not a sufficient condition. This is the class of *Fourier series of bounded functions*.

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* Cf. **3**, p. 136.

We have, in fact,

THEOREM Y.—*If $f(\theta)$ is a summable function, bounded in the neighbourhood of a particular value of θ, and if the Fourier series of $f(\theta)$ is convergent for that value of θ, then*

$$\frac{1}{2a}\int_{\theta-a}^{\theta+a} f(t)\,dt$$

tends to a limit when $a \to 0$.

Suppose for simplicity that $a_0 = 0$, and let

$$F_1(\theta) = -\Sigma \frac{a_n \cos n\theta + b_n \sin n\theta}{n^2}$$

be the second integral of $f(\theta)$. Then, if the series converges to the sum zero,

(4.61) $$\phi(a) = F_1(\theta+a)+F_1(\theta-a)-2F_1(\theta) = o(a^2)$$

when $a \to 0$, in virtue of another well known theorem of Riemann.*

Now $$\phi'(a) = F(\theta+a)-F(\theta-a) = \int_{\theta-a}^{\theta+a} f(t)\,dt.$$

If $f(\theta)$ were everywhere the differential coefficient of its integral, we should have

$$\phi''(a) = f(\theta+a)+f(\theta-a) = O(1),$$

since $f(t)$ is bounded in the neighbourhood of θ; and from (4.61) and the last equation it would follow at once that

$$\phi'(a) = o(a),$$

proving our point. This is not now a valid proof. But it is easy to see that, in such a theorem as

"$\phi(a) = o(a^2)$ *and* $\phi''(a) = O(1)$ *imply* $\phi'(a) = o(a)$",

the second condition may be replaced by the more general condition expressed by the inequality

$$|\phi'(\beta)-\phi'(\gamma)| < K|\beta-\gamma| \dagger:$$

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* See, for example, de la Vallée Poussin, *Cours d'Analyse*, ed. 2, Vol. 2, p. 172.

† See Landau, "Einige Ungleichungen für zweimal Differentiierbare Functionen", *Proc. London Math. Soc.*, Ser. 2, Vol. 13, 1914, pp. 43-49.

a condition obviously fulfilled in this case, since ϕ' is the integral of a bounded function.

Thus Theorem **Y** is proved. It is, in fact, but a special case of

THEOREM **Z**.—*The Fourier series of a function* $f(\theta)$, *bounded in the neighbourhood of the particular value of* θ *under consideration, is either summable by Cesàro means of arbitrarily small positive order, or summable by no Cesàro mean of any order. The necessary and sufficient condition that it should be summable is expressed by the condition* (4 . 13).

The proof of this theorem would, however, carry us too far from the proper subject of the paper.*

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* We have published a sketch of the proof, under the title "On the Fourier series of a bounded function", in the *Records of Proceedings at Meetings* for December 6th, 1917 (*Proc. London Math. Soc.*, Vol. 17, p. xiii).