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only result was that it became yellow when heated. I next tried it with borax, in which it dissolved very slowly, and did not colour the glass. I was still at a loss to say what it was, but suspecting that it contained some metal, which was indicated by the white sublimate, I tried it with carbonate of soda on charcoal, and it speedily yielded brilliant metallic globules, which tarnished rapidly after cooling; they were malleable, and when flattened out presented the appearance of tin.

No doubt now remained as to its nature; and I have only to add, in corroboration of the assertion of Messrs. Mills, King, and Weaver, that "native oxide of tin" exists in the county Wicklow.

XXIX. *Introduction to an Essay on the Amount and Distribution of the Multiplicity of the Roots of an Algebraic Equation.*
By J. J. SYLVESTER, F.R.S. &c., Professor of Natural Philosophy in University College, London*.

I USE the word *multiplicity* to denote a number, and distinguish between the total and partial multiplicities of the roots of an algebraic equation.

There may be r different roots repeated respectively $h_1 h_2 \dots h_r$ times.

r is the index of distribution.

$h_1 h_2 \dots h_r$ are the partial multiplicities, and if $h = h_1 + h_2 + \dots + h_r$,

h is the *total* multiplicity.

The total multiplicity it is clear may be defined as the difference between the index of the equation and the number of its roots distinguishable from one another.

In this Introduction, I propose merely to consider how existing methods may be applied to determine the amount and distribution of multiplicity in a given equation, and conversely, how equations of condition can be formed which shall imply a *given* distribution and amount.

Let the greatest common factor between $f x$ (the argument of the proposed equation) and $\frac{dfx}{dx}$ be called $f_1 x$.

And in like manner, let the greatest common factor of $f_1 x$ and $\frac{df_1 x}{dx}$ be called $f_2 x$. and so on, till in the end we come to $f_r \cdot x$, which has no common factor with $\frac{df_r x}{dx}$.

Let $k_1 k_2 \dots k_r$ denote the degrees in x of $f x f_1 x \dots f_r \cdot x$ respectively.

* Communicated by the Author.

It is easy to see that

$k_1 - k_2$, partial multiplicities, are less than 2, *i. e.* are each units.

$k_2 - k_3$, partial multiplicities, will be less than 3, and therefore either 1 or 2 in value respectively, and so on till we come to

$k_{r-1} - k_r$ which will severally be between zero and $r-1$, and $k_r - 0$ of values intermediate between zero and r .

Hence there will be

$k_1 - 2 k_2 + k_3$ multiplicities each of the value 1.

$k_2 - 2 k_3 + k_4$ 2.

... ..

$k_{r-1} - 2 k_r$... of the value $r-1$.

and k_r of the value r .

In place of $f x$ with $\frac{d_1 f x}{d x}$ we might employ $\frac{d f x}{d x}$ with $\frac{d^2 f x}{d x^2}$ and so on for the rest; the values of $k_2 k_3 \dots k_r$ will remain unaffected by this change; but the former method would be more expeditious in practice.

The total multiplicity is, of course, $= k_1$.

Suppose now that we propose to ourselves the converse problem to determine the conditions that an algebraic equation may have a given amount of multiplicity distributed in a given manner.

If $h_1 h_2 h_3 \dots h_r$ be used to denote the given number of partial multiplicities which are respectively of the values 1 2 3 ... r , it is easy to see that the quantities derived above by $k_1 k_2 \dots k_r$ are respectively equal to

$$\begin{aligned} h_1 + 2 h_2 + \dots + r h_r \\ h_2 + 2 h_3 + \dots + r h_{r-1} \\ h_3 + 2 h_4 + \dots + r h_{r-2} \\ \dots \dots \dots \dots \dots \\ h_r. \end{aligned}$$

Now from $\frac{d f x}{d x}$ having a factor of the degree k_1 common

with $f x$ we obtain k_1 conditions from $\frac{d f_1 x}{d x}$ having a factor of the degree k_2 common with $f_1 x$ we obtain k_2 more, and so on. So that altogether we obtain in this way

$$k_1 + k_2 + \dots k_r \text{ conditions.}$$

But it may easily be seen that the total multiplicity being k , the number of conditions *need* never to exceed k_1 in number, no matter what its distribution may be. Hence, besides the enormous labour of the process, and the extreme complexity of

the results, we obtain by this method more equations by far than are necessary, and it requires some caution to know which to reject.

In my forthcoming paper (to appear in Phil. Mag. of next month) I shall show, by a most simple means, how without the use of derived or other subsidiary functions, to obtain the simplest equations of condition which correspond to a given distribution of a given amount of multiplicity.

The total multiplicity, say m , being given in as many ways as that number can be broken into parts, so many different systems of m equations can be formed differing each from the other in the dimensions of the terms.

These systems may be arranged in order so that each in the series shall imply all those that follow it, and be implied in all those that go before, without the converse being satisfied.

The subject of the unreciprocal implication of systems of equations is a very curious one, upon which the limits assigned to me prevent me from enlarging at present. It is closely connected with a part of the theory of elimination, which, as far as I am aware, has either been overlooked, or has not met with the attention which it deserves; I mean the theory of *Special Factors*.

An *example* may make what I mean by these clear.

Let C be a function (if my reader please) void of x , which equivalent to zero implies two given equations in x having a common root.

Let C be rid of all irrelevant factors, *i. e.* let C be the simplest form of the determinant, when the coefficients of the two equations are perfectly independent qualities. Now suppose, as is *quite possible in a variety of ways*, that such relations are instituted between the coefficients alluded to as make C split up into factors, so that $C = L \times M \times N = 0$.

Only one of the factors, L , M , N will satisfy the condition of the co-existence of the two given equations: the others are clearly, however, not to be confounded with factors of solution, or irrelevant factors, as they are termed, but are of quite a different nature, and enjoy remarkable properties, which point to an enlarged theory of elimination, and constitute what I call special or singular factors.

I shall feel much obliged to any of the readers of your widely circulated Journal, interested in the subject of this paper, who would do me the honour of communicating with me upon it, and especially if they would (between now and the next coming out of the Magazine) inform me whether any thing, and if so how much, different from what is here stated

has been done in the matter of determining the relations between the coefficients of an equation corresponding to a given amount and distribution of multiplicity in its roots.

I ought to add, that my method enables me not merely to determine the conditions of multiplicity, but also to decompose the equations containing multiple roots into others free of multiplicity, *i. e.* to find, *à priori*, the values of the several quantities

$$\frac{f x \cdot f_2 \cdot x}{(f_1 \cdot x)^2}, \frac{f_1 \cdot x \cdot f_3 \cdot x}{(f_2 \cdot x)^2}, \dots, \frac{(f_{r-1} \cdot x)}{(f_r \cdot x)^2}, f_r \cdot x.$$

Moreover, other decompositions, not necessary to be enlarged upon in this place, may be obtained with equal facility.

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[To be continued.]

XXX. *Proceedings of Learned Societies.*

ROYAL SOCIETY.

[Continued from vol. xvii. p. 387.]

Nov. 19, **A** PAPER was read, entitled, Supplement to a paper 1840. "On the Theoretical Explanation of an apparent new Polarity in Light;" by George B. Airy, Esq., M.A., F.R.S., Astronomer Royal.

In a paper published in the second part of the Philosophical Transactions for 1840*, the author explained, on the undulatory theory of light, the phenomena observed by Sir David Brewster, and apparently indicating a new polarity in light. That explanation was founded on the assumption that the spectrum was viewed out of focus; an assumption which corresponded with the observation of the author and of other persons. But the author having, since the publication of that memoir, been assured by Sir David Brewster that the phenomenon was most certainly observed with great distinctness when the spectrum was viewed so accurately in focus that many of Fraunhofer's finer lines could be seen, he has continued the theoretical investigation for that case, which had been omitted in the former memoir, namely, when the spectrum is viewed in focus; and he has arrived at a result, which appears completely to reconcile the seemingly conflicting statements, and to dispel the obscurity in which the subject had hitherto been enveloped.

Nov. 26, 1840.—Description of the Electro-magnetic Clock. By C. Wheatstone, Esq., F.R.S.

The object of the apparatus forming the subject of this communication, is stated by the author to be that of enabling a single clock to indicate exactly the same time in as many different places, distant from each other, as may be required. Thus, in an astronomical

* An abstract of which appeared in L. E. & D. Phil. Mag., vol. xvii. p. 381.—EDIT.