Chap. v., p. 103), and probably by Hamilton, for matrices of the third order.

It is, of course, only necessary to show that the sum of the k^{th} powers of the latent roots of mn [*i.e.*, the sum of the latent roots of $(mn)^k$] is equal to the sum of the k^{th} powers of the latent roots of nm [*i.e.*, the sum of the latent roots of $(nm)^k$]. But

$$S\,(mn)^k = S\,.\,m\,(nm)^{k-1}\,n = S\,.\,nm\,(nm)^{k-1} = S\,(nm)^k.$$

Note.—By an oversight the word "involutant" has been used above in § 4 for "square of the involutant." Thus, the above resultants I and J are the *squares* of the corresponding involutants.

On the Reversion of Partial Differential Expressions with two Independent and two Dependent Variables. By E. B. Elliott.

[*Read Dec.* 11*th*, 1890.]

1. The theory of the interchange of the dependent and independent variables, in functions of the derivatives of one variable with regard to another of which it is by supposition a function, has of late years been studied with great elaboration on the basis of Prof. Sylvester's new discovery of reciprocants. The last five volumes of our *Proceedings* abound in valuable contributions to the study.

Mr. A. Berry, in his paper on "Simultaneous Reciprocants" (*Quarterly Journal*, 1888, p. 260), has initiated an analogous theory as to the reversion, by interchange of x and y and of x' and y', of functions of the two sets of ordinary derivatives

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \dots\quad \frac{dy'}{dx'},\ \frac{d^2y'}{dx'^2},\ \frac{d^3y'}{dx'^3},\ \dots\,;$$

his supposition being that x, y are connected by one relation, and x', y' by another.

It would appear that the theory of the interchange of the dependent and independent pairs in functions of the partial derivatives of two variables with regard to two others, the four being connected by two relations of perfectly arbitrary form, has not hitherto been dealt with in the light of recent researches. It is on the development of this theory that I now propose to embark.

2. Let x, y be two variables connected with two others x', y' by two relations

$$f_1(x, y, x', y') = 0, \quad f_2(x, y, x', y') = 0,$$

whose form is immaterial and need not be known. We may regard either x, y as independent and x', y' as functions of them, or x', y' as independent and x, y as dependent. It is required to investigate connexions between the various partial derivatives in the one case, and in the other. Two important branches of the investigation will be the search for the invariants or reciprocants of the reversion,—*i.e.*, functions of the derivatives which have the same form, but for a simple factor in the two cases,—and the consideration of the reversion of operators linear in symbols of partial differentiation with regard to the partial derivatives.

Let us once for all denote, for zero and positive integral values of r and s,

$$\frac{1}{r\,!\,s\,!}\,\frac{d^{r+s}x}{dx'^r dy'^s}, \quad \frac{1}{r\,!\,s\,!}\,\frac{d^{r+s}y}{dx'^r dy'^s},$$

$$\frac{1}{r\,!\,s\,!}\,\frac{d^{r+s}x'}{dx^r dy^s}, \quad \frac{1}{r\,!\,s\,!}\,\frac{d^{r+s}y'}{dx^r dy^s},$$

by x_{rs}, y_{rs}, x'_{rs}, y'_{rs}, respectively.

The values of either pair of first derivatives in terms of the other pair follow at once by a well-known method from the identity of the pair of equalities

$$dx = x_{10}dx' + x_{01}dy',$$

$$dy = y_{10}dx' + y_{01}dy',$$

with the pair

$$dx' = x'_{10}dx + x'_{01}dy,$$

$$dy' = y'_{10}dx + y'_{01}dy.$$

We are thus given that

$$\left.\begin{aligned} x_{10}x'_{10} + x_{01}y'_{10} &= 1 \\ x_{10}x'_{01} + x_{01}y'_{01} &= 0 \\ y_{10}x'_{10} + y_{01}y'_{10} &= 0 \\ y_{10}x'_{01} + y_{01}y'_{01} &= 1 \end{aligned}\right\} \quad \text{.........................} \quad (1),$$

so that

$$\frac{x_{10}}{y'_{01}} = -\frac{x_{01}}{x'_{01}} = -\frac{y_{10}}{y'_{10}} = \frac{y_{01}}{x'_{10}} = x_{10}y_{01} - x_{01}y_{10} = \frac{1}{x'_{10}y'_{01} - x'_{01}y'_{10}} \quad \text{......(2).}$$

The successive derivation of expressions for higher derivatives x_{mn}, y_{mn} is, by means of the operator equivalents,

$$\left.\begin{aligned} \frac{d}{dx} &= (x_{10}y_{01} - x_{01}y_{10})^{-1}\left\{ \quad y_{01}\frac{d}{dx'} - y_{10}\frac{d}{dy'}\right\} \\ \frac{d}{dy} &= (x_{10}y_{01} - x_{01}y_{10})^{-1}\left\{ -x_{01}\frac{d}{dx'} + x_{10}\frac{d}{dy'}\right\} \end{aligned}\right\} \quad \text{..............(3),}$$

which follow from the facts,

$$\left.\begin{aligned} \frac{du}{dx} &= \frac{d(u, y)}{d(x, y)} = \left\{\frac{d(x, y)}{d(x', y')}\right\}^{-1} \frac{d(u, y)}{d(x', y')} \\ \frac{du}{dy} &= -\frac{d(u, x)}{d(x, y)} = -\left\{\frac{d(x, y)}{d(x', y')}\right\}^{-1} \frac{d(u, x)}{d(x', y')} \end{aligned}\right\} \quad \text{......... (4).}$$

In this way, for instance, we obtain in the next place that

$$\begin{aligned} x'_{20} = {} & (x_{10}y_{01} - x_{01}y_{10})^{-3} \\ & \times \left\{x_{01}(y_{01}^2 y_{20} - y_{01}y_{10}y_{11} + y_{10}^2 y_{02}) - y_{01}(y_{01}^2 x_{20} - y_{01}y_{10}x_{11} + y_{10}^2 x_{02})\right\} \\ & \text{.....................(5),} \end{aligned}$$

$$\begin{aligned} x'_{11} = {} & -(x_{10}y_{01} - x_{01}y_{10})^{-3}\left\{x_{01}(2y_{01}x_{01}y_{20} - \overline{y_{01}x_{10} + y_{10}x_{01}}\, y_{11} + 2y_{10}x_{10}y_{02})\right. \\ & \left. - y_{01}(2y_{01}x_{01}x_{20} - \overline{y_{01}x_{10} + y_{10}x_{01}}\, x_{11} + 2y_{10}x_{10}x_{02})\right\} \text{.............(6),} \end{aligned}$$

$$\begin{aligned} x'_{02} = {} & (x_{10}y_{01} - x_{01}y_{10})^{-3} \\ & \times \left\{x_{01}(x_{01}^2 y_{20} - x_{01}x_{10}y_{11} + x_{10}^2 y_{02}) - y_{01}(x_{01}^2 x_{20} - x_{01}x_{10}x_{11} + x_{10}^2 x_{02})\right\} \\ & \text{.....................(7),} \end{aligned}$$

$$\begin{aligned} y'_{20} = {} & (x_{10}y_{01} - x_{01}y_{10})^{-3} \\ & \times \left\{-x_{10}(y_{01}^2 y_{20} - y_{01}y_{10}y_{11} + y_{10}^2 y_{02}) + y_{10}(y_{01}^2 x_{20} - y_{01}y_{10}x_{11} + y_{10}^2 x_{02})\right\} \\ & \text{.....................(8),} \end{aligned}$$

$$\begin{aligned} y'_{11} = {} & -(x_{10}y_{01} - x_{01}y_{10})^{-3}\left\{-x_{10}(2y_{01}x_{01}y_{20} - \overline{y_{01}x_{10} + y_{10}x_{01}}\, y_{11} + 2y_{10}x_{10}y_{02})\right. \\ & \left. + y_{10}(2y_{01}x_{01}x_{20} - \overline{y_{01}x_{10} + y_{10}x_{01}}\, x_{11} + 2y_{10}x_{10}x_{02})\right\} \text{...... (9),} \end{aligned}$$

$$\begin{aligned} y'_{02} = {} & (x_{10}y_{01} - x_{01}y_{10})^{-3} \\ & \times \left\{-x_{10}(x_{01}^2 y_{20} - x_{01}x_{10}y_{11} + x_{10}^2 y_{02}) + y_{10}(x_{01}^2 x_{20} - x_{01}x_{10}x_{11} + x_{10}^2 x_{02})\right\} \\ & \text{.....................(10).} \end{aligned}$$

The general formulæ are

$$\left.\begin{aligned}(m+1)\,!\;n\,!\;x'_{m+1\,n} &= \partial^m\partial'^n\left\{\ \ (x_{10}y_{01}-x_{01}y_{10})^{-1}y_{01}\right\}\\ m\,!\;(n+1)\,!\;x'_{m\,n+1} &= \partial^m\partial'^n\left\{-(x_{10}y_{01}-x_{01}y_{10})^{-1}x_{01}\right\}\\ (m+1)\,!\;n\,!\;y'_{m+1\,n} &= \partial^m\partial'^n\left\{-(x_{10}y_{01}-x_{01}y_{10})^{-1}y_{10}\right\}\\ m\,!\;(n+1)\,!\;y'_{m\,n+1} &= \partial^m\partial'^n\left\{\ \ (x_{10}y_{01}-x_{01}y_{10})^{-1}x_{10}\right\}\end{aligned}\right\}\ldots\ldots(11),$$

where ∂, ∂' are the two dexters of the equivalences (3). In these identities (11), m and n may both or either be zero.

3. A function of the unaccented derivatives whose equivalent in terms of the accented is of exactly the same form, but for a sign factor at most, is, in accordance with the nomenclature of analogous theories, called an *absolute reciprocant.* A function whose equivalent is of the same form, but for a power of the Jacobian

$$x'_{10}y'_{01}-x'_{01}y'_{10}$$

as factor, and also it may be by a sign factor, is called a *reciprocant.* If there be a negative sign factor, a reciprocant is said to be of negative character; otherwise, of positive.

From the identities (2) we have at once the means of writing down five fundamental absolute reciprocants of the first order, namely,

$$\alpha\equiv\log\,(x_{10}y_{01}-x_{01}y_{10})\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (12),$$

$$\beta\equiv\quad(x_{10}y_{01}-x_{01}y_{10})^{-\frac{1}{2}}x_{01}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (13),$$

$$\gamma\equiv\quad(x_{10}y_{01}-x_{01}y_{10})^{-\frac{1}{2}}y_{10}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (14),$$

$$\delta\equiv\quad(x_{10}y_{01}-x_{01}y_{10})^{-\frac{1}{2}}(x_{10}-y_{01})\ \ldots\ldots\ldots\ldots\ (15),$$

$$\epsilon\equiv\quad(x_{10}y_{01}-x_{01}y_{10})^{-\frac{1}{2}}(x_{10}+y_{01})\ \ldots\ldots\ldots\ldots\ (16),$$

whose characters, respectively, are

$$-,\quad -,\quad -,\quad -,\quad +.$$

The last four are connected by a relation, namely,

$$\epsilon^2-\delta^2-4\beta\gamma=4\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

Thus, in one sense, the four of negative character form a complete system of the first order. There must, however, be two not explicitly known reciprocants whose Jacobian, namely,

$$(x_{10}y_{01}-x_{01}y_{10})^{\frac{1}{2}}\ldots\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots(18),$$

is of the first order, to which order they must themselves be regarded as prior, depending at any rate partly on the variables themselves, and not on their derivatives only. Their product is

$$\iint (x_{10}y_{01}-x_{01}y_{10})^{\frac{1}{2}}dx'dy' = \iint (dx\,dy\,dx'dy')^{\frac{1}{2}}$$
$$= \iint (x'_{10}y'_{01}-x'_{01}y'_{10})^{\frac{1}{2}}\,dx\,dy \quad \text{............ (19).}$$

Their importance undoubtedly underlies the theory of the reciprocants β, γ, δ, ϵ, and all that can be deduced from them, just as the ordinary reciprocant, $\int \sqrt{t}\,dx$, is of capital importance in Professor Sylvester's theory.

4. The following is the central theorem as to the eduction of reciprocants of higher orders:—"If u, v, w are three absolute reciprocants, then $\frac{d^{m+n}u}{dv^m\,dw^n}$ is another, m and n having any positive integral or zero values."

This is clear, for $\frac{d}{dv}$ and $\frac{d}{dw}$ must be reciprocant operators; or again, because $\frac{1}{m\,!\;n\,!}\,\frac{d^{m+n}u}{dv^m\,dw^n}$ is the general coefficient in the expansion of an increment of u in terms of increments of v and w.

Accordingly, since, as in (4),

$$\frac{d\phi}{dv} = \frac{d(\phi, w)}{d(x', y')}\left\{\frac{d(v, w)}{d(x', y')}\right\}^{-1},$$

and

$$\frac{d\phi}{dw} = -\frac{d(\phi, v)}{d(x', y')}\left\{\frac{d(v, w)}{d(x', y')}\right\}^{-1},$$

it follows that

$$\left\{\left(\frac{dv}{dx'}\frac{dw}{dy'}-\frac{dw}{dx'}\frac{dv}{dy'}\right)^{-1}\left(\frac{dw}{dy'}\frac{d}{dx'}-\frac{dw}{dx'}\frac{d}{dy'}\right)\right\}^m$$
$$\times\left\{\left(\frac{dv}{dx'}\frac{dw}{dy'}-\frac{dw}{dx'}\frac{dv}{dy'}\right)^{-1}\left(\frac{dv}{dy'}\frac{d}{dx'}-\frac{dv}{dx'}\frac{d}{dy'}\right)\right\}^n u \quad \text{............ (20)}$$

is the type of an infinite series of absolute reciprocants educed from u by means of v and w.

If $(-1)^{k_1}$, $(-1)^{k_2}$, $(-1)^{k_3}$ be the characters of u, v, w, the character of $\frac{d^{m+n}u}{dv^m\,dw^n}$ is $(-1)^{mk_2+nk_3-k_1}$.

5. A simpler form may be given to the aggregate (20) by aid of the theorem that the Jacobian of two absolute reciprocants is a reciprocant. This is clear, for

$$\frac{d(u, v)}{d(x, y)} = \frac{d(x', y')}{d(x, y)} \frac{d(u, v)}{d(x', y')};$$

so that, remembering the last equality in (2),

$$(x'_{10}y'_{01} - x'_{01}y'_{10})^{-\frac{1}{2}} \frac{d(u, v)}{d(x, y)} = (x_{10}y_{01} - x_{01}y_{10})^{-\frac{1}{2}} \frac{d(u, v)}{d(x', y')} \quad \text{......... (21).}$$

We may express this by saying that the Jacobian of any two absolute reciprocants is a reciprocant of *index* $\frac{1}{2}$, the index being that power of $x_{10}y_{01} - x_{01}y_{10}$ by which a reciprocant has to be divided to make it absolute.

We have, hence, as the type of a system of absolute reciprocants, equivalent in the aggregate to (20),

$$\left\{ (x_{10}y_{01} - x_{01}y_{10})^{-\frac{1}{2}} \left(\frac{dw}{dy'} \frac{d}{dx'} - \frac{dw}{dx'} \frac{d}{dy'} \right) \right\}^{m}$$

$$\times \left\{ (x_{10}y_{01} - x_{01}y_{10})^{-\frac{1}{2}} \left(\frac{dv}{dy'} \frac{d}{dx'} - \frac{dv}{dx'} \frac{d}{dy'} \right) \right\}^{n} u \quad \text{..............(22).}$$

Again, we can use the absolute reciprocants (13) to (16) to produce further apparent simplification. Thus, for instance, we may replace, in (22),

$$(x_{10}y_{01} - x_{01}y_{10})^{\frac{1}{2}}$$

by $\qquad x_{01}, \quad y_{10}, \quad x_{10} - y_{01} \quad \text{or} \quad x_{10} + y_{01}.$

There will, it is naturally to be expected, be *pure* reciprocants, *i.e.*, reciprocants in which the first derivatives are absent and higher ones alone occur. Theorems as to such will best follow the investigation of the reversion of linear differential operators, to which we next proceed.

6. The object now before us is to express such an operator as

$$P_{10} \frac{d}{dx_{10}} + P_{01} \frac{d}{dx_{01}} + P_{20} \frac{d}{dx_{20}} + \dots + Q_{10} \frac{d}{dy_{10}} + Q_{01} \frac{d}{dy_{01}} + Q_{20} \frac{d}{dy_{20}} + \dots,$$

supposed to act on any function of the unaccented derivatives, as a sum of multiples of the symbols

$$\frac{d}{dx'_{10}}, \; \frac{d}{dx'_{01}}, \; \frac{d}{dx'_{20}}, \dots \frac{d}{dy'_{10}}, \; \frac{d}{dy'_{01}}, \; \frac{d}{dy'_{20}}, \dots,$$

operating on the equivalent function of the accented derivatives.

Let ξ, η, ξ', η' be possible simultaneous finite increments of x, y, x', y'. The pair of relations connecting them may, by Taylor's theorem, be given the forms

$$\xi' = x'_{10}\xi + x'_{01}\eta + x'_{20}\xi^2 + x'_{11}\xi\eta + x'_{02}\eta^2 + \dots \quad \text{(23)},$$

$$\eta' = y'_{10}\xi + y'_{01}\eta + y'_{20}\xi^2 + y'_{11}\xi\eta + y'_{02}\eta^2 + \dots \quad \text{(24)},$$

or, again, the forms

$$\xi = x_{10}\xi' + x_{01}\eta' + x_{20}\xi'^2 + x_{11}\xi'\eta' + x_{02}\eta'^2 + \dots \quad \text{(25)},$$

$$\eta = y_{10}\xi' + y_{01}\eta' + y_{20}\xi'^2 + y_{11}\xi'\eta' + y_{02}\eta'^2 + \dots \quad \text{(26)}.$$

Now, the forms of the relations

$$f_1(x, y, x', y') = 0, \quad f_2(x, y, x', y') = 0,$$

which connect x, y, x', y', being arbitrary, we may, if we please, look upon

$$\xi, \eta, \quad x'_{10}, x'_{01}, x'_{20}, \dots y'_{10}, y'_{01}, y'_{20}, \dots$$

as independent quantities,

and upon $\quad \xi', \eta', \quad x_{10}, x_{01}, x_{20}, \dots y_{10}, y_{01}, y_{20}, \dots$

as quantities dependent on them, and determinate in terms of them; the first two by (23) and (24), and the rest by the equivalence of the relations (25) and (26) with (23) and (24).

Of the quantities

$$x'_{10}, x'_{01}, \dots y'_{10}, y'_{01}, \dots$$

keep, then, all constant except x'_{rs}, to which give an infinitesimal variation $\delta x'_{rs}$. Moreover, keep ξ and η constant. By (23) and (24), we get under these circumstances

$$\delta\xi' = \xi^r\eta^s\,\delta x'_{rs} \quad \text{(27)},$$

$$\delta\eta' = 0 \quad \text{(28)};$$

and by (25) and (26),

$$0 = \left\{ \frac{dx_{10}}{dx'_{rs}}\xi' + \frac{dx_{01}}{dx'_{rs}}\eta' + \frac{dx_{20}}{dx'_{rs}}\xi'^2 + \frac{dx_{11}}{dx'_{rs}}\xi'\eta' + \frac{dx_{02}}{dx'_{rs}}\eta'^2 + \dots \right\} \delta x'_{rs}$$

$$+ \left\{ x_{10} + 2x_{20}\xi' + x_{11}\eta' + 3x_{30}\xi'^2 + 2x_{21}\xi'\eta' + x_{12}\eta'^2 + \dots \right\} \delta\xi'$$

$$+ \left\{ x_{01} + x_{11}\xi' + 2x_{02}\eta' + x_{21}\xi'^2 + 2x_{12}\xi'\eta' + 3x_{03}\eta'^2 + \dots \right\} \delta\eta'$$

$$\text{(29)};$$

$$0=\left\{\frac{dy_{10}}{dx'_{rs}}\xi'+\frac{dy_{01}}{dx'_{rs}}\eta'+\frac{dy_{20}}{dx'_{rs}}\xi'^2+\frac{dy_{11}}{dx'_{rs}}\xi'\eta'+\frac{dy_{02}}{dx'_{rs}}\eta'^2+\dots\right\}\delta x'_{rs}$$
$$+\{y_{10}\quad+2y_{20}\xi'+y_{11}\eta'\quad+3y_{30}\xi'^2+2y_{21}\xi'\eta'+y_{02}\eta'^2\quad+\dots\quad\}\delta\xi'$$
$$+\{\quad y_{01}\quad+y_{11}\xi'+2y_{02}\eta'\quad+y_{21}\xi'^2+2y_{12}\xi'\eta'+3y_{03}\eta'^2\;+\dots\}\delta\eta' \quad\dots\dots(30).$$

The two pairs of conditions must be identical. Expressing this identity, we obtain the equivalences

$$\frac{dx_{10}}{dx'_{rs}}\xi'+\frac{dx_{01}}{dx'_{rs}}\eta'+\frac{dx_{20}}{dx'_{rs}}\xi'^2+\frac{dx_{11}}{dx'_{rs}}\xi'\eta'+\frac{dx_{02}}{dx'_{rs}}\eta'^2+\dots$$
$$=-\{x_{10}+2x_{20}\xi'+x_{11}\eta'+3x_{30}\xi'^2+2x_{21}\xi'\eta'+x_{12}\eta'^2+\dots\}\xi^r\eta^s$$
$$=-\frac{d\xi}{d\xi'}\xi^r\eta^s\quad\dots\dots(31);$$

and

$$\frac{dy_{10}}{dx'_{rs}}\xi'+\frac{dy_{01}}{dx'_{rs}}\eta'+\frac{dy_{20}}{dx'_{rs}}\xi'^2+\frac{dy_{11}}{dx'_{rs}}\xi'\eta'+\frac{dy_{02}}{dx'_{rs}}\eta'^2+\dots$$
$$=-\{y_{10}+2y_{20}\xi'+y_{11}\eta'+3y_{30}\xi'^2+2y_{21}\xi'\eta'+y_{12}\eta'^2+\dots\}\xi^r\eta^s$$
$$=-\frac{d\eta}{d\xi'}\xi^r\eta^s\quad\dots\dots(32),$$

in which $\frac{d}{d\xi'}$ is indicative of strictly partial differentiation of the expressions for ξ, η in (25) and (26).

If to ξ and η, in the right of these equalities (31) and (32), we give their expanded values from (25) and (26), and if we then expand and arrange by powers and products of ξ' and η', the multipliers of products obtained must of course be equal to those of the same products in the left-hand members of the equalities. Thus, we conclude that, for any m and n,

$$\frac{dx_{mn}}{dx'_{rs}}=\text{co. }\xi'^m\eta'^n\text{ in expansion of}$$
$$-\frac{1}{r+1}\{y_{10}\xi'+y_{01}\eta'+2y_{20}\xi'^2+\dots\}^s\frac{d}{d\xi'}\{x_{10}\xi'+x_{01}\eta'+2x_{20}\xi'^2+\dots\}^{r+1}\quad\dots\dots(33),$$

$$\frac{dy_{mn}}{dx'_{rs}}=\text{co. }\xi'^m\eta'^n\text{ in expansion of}$$
$$-\frac{1}{s+1}\{x_{10}\xi'+x_{01}\eta'+2x_{20}\xi'^2+\dots\}^r\frac{d}{d\xi'}\{y_{10}\xi'+y_{01}\eta'+2y_{20}\xi'^2+\dots\}^{s+1}\quad\dots\dots(34).$$

We have thus the means of obtaining the partial differential coefficient with regard to x'_{rs} of any function of the unaccented derivatives. The conclusion is thrown into a very simple form, as follows. Add (31) to λ times (32), where λ is perfectly arbitrary. On the left we thus get an expression in which, on substitution for

$$\xi',\ \lambda\xi',\ \eta',\ \lambda\eta',\ \xi'^2,\ \lambda\xi'^2,\ \xi'\eta',\ \lambda\xi'\eta',\ \dots\ \xi'^m\eta'^n,\ \lambda\xi'^m\eta'^n,\ \dots$$

of $$\frac{d}{dx_{10}},\ \frac{d}{dy_{10}},\ \frac{d}{dx_{01}},\ \frac{d}{dy_{01}}\ \frac{d}{dx_{20}},\ \frac{d}{dy_{20}},\ \frac{d}{dx_{11}},\ \frac{d}{dy_{11}},\ \dots\ \frac{d}{dx_{mn}},\ \frac{d}{dy_{mn}},\ \dots,$$

we obtain

$$\frac{dx_{10}}{dx'_{rs}}\frac{d}{dx_{10}}+\frac{dy_{10}}{dx'_{rs}}\frac{d}{dy_{10}}+\frac{dx_{01}}{dx'_{rs}}\frac{d}{dx_{01}}+\frac{dy_{01}}{dx'_{rs}}\frac{d}{dy_{01}}+\dots,$$

which is the equivalent of the operative symbol $\frac{d}{dx'_{rs}}$. The same, then, must be the case on the right. In other words, we obtain the equivalence, in which on the left the operation is upon a function of the accented, and on the right upon the equal function of the accented variables,

$$\frac{d}{dx'_{rs}}=-\xi'^r\eta'^s\left\{\frac{d\xi}{d\xi'}+\lambda\frac{d\eta}{d\xi'}\right\}\ \dots\dots\dots\dots\dots\dots(35),$$

the notation on the right being symbolical, and the meaning being that by aid of (25) and (26) the expansion as a sum of multiples of products of powers of ξ' and η' is to be effected, and that then, for every product, $\xi'^m\eta'^n$, as it occurs linearly without a λ factor, is to be written $\frac{d}{dx_{mn}}$, while, for every product, $\lambda\xi'^m\eta'^n$ is to be written $\frac{d}{dy_{mn}}$.

In precisely the same way, by giving y'_{rs} instead of x'_{rs} a variation in (23) to (26), we arrive at

$$\frac{dx_{mn}}{dy'_{rs}}=\text{co. }\xi'^m\eta'^n\text{ in expansion of}$$

$$-\frac{1}{r+1}\left\{y_{10}\xi'+y_{01}\eta'+y_{20}\xi'^2+\dots\right\}^s\frac{d}{d\eta'}\left\{x_{10}\xi'+x_{01}\eta'+x_{20}\xi'^2+\dots\right\}^{r+1}$$
$$\dots\dots\dots\dots\dots(36),$$

and $\frac{dy_{mn}}{dy'_{rs}} = \text{co. } \xi'^m \eta'^n$ in expansion of

$$-\frac{1}{s+1}\left\{x_{10}\xi'+x_{01}\eta'+x_{20}\xi'^2+\dots\right\}^r \frac{d}{d\eta'}\left\{y_{10}\xi'+y_{01}\eta'+y_{20}\xi'^2+\dots\right\}^{s+1} \quad \dots\dots\dots(37),$$

and at the general conclusion, symbolically written as in (35), that

$$\frac{d}{dy'_{rs}} = -\xi^r\eta^s\left\{\frac{d\xi}{d\eta'}+\lambda\frac{d\eta}{d\eta'}\right\} \quad \dots\dots\dots(38).$$

To obtain, then, the transformed linear operator equivalent to

$$\Sigma P'_{rs}\frac{d}{dx'_{rs}}+\Sigma Q'_{rs}\frac{d}{dy'_{rs}},$$

it is effectual to put $\xi^r\eta^s$ both for $\frac{d}{dx'_{rs}}$ and $\frac{d}{dy'_{rs}}$ for all values of r and s involved, to then multiply the first sum by $-\frac{d\xi}{d\xi'}-\lambda\frac{d\eta}{d\xi'}$, and the second by $-\frac{d\xi}{d\eta'}-\lambda\frac{d\eta}{d\eta'}$, to expand the result in terms of ξ' and η' by (25) and (26), and finally in the expansion to put, for every $\lambda\xi'^m\eta'^n$, $\frac{d}{dy_{mn}}$, and for every $\xi'^m\eta'^n$ without a λ, $\frac{d}{dx_{mn}}$.

7. With a view to the utilization of these formulæ now arrived at, it is important to notice that, by first notions of partial differentiation,

$$\left.\begin{aligned}\frac{d\xi}{d\xi'}\cdot\frac{d\xi'}{d\xi}+\frac{d\xi}{d\eta'}\cdot\frac{d\eta'}{d\xi}&=1\\ \frac{d\xi}{d\xi'}\cdot\frac{d\xi'}{d\eta}+\frac{d\xi}{d\eta'}\cdot\frac{d\eta'}{d\eta}&=0\\ \frac{d\eta}{d\xi'}\cdot\frac{d\xi'}{d\xi}+\frac{d\eta}{d\eta'}\cdot\frac{d\eta'}{d\xi}&=0\\ \frac{d\eta}{d\xi'}\cdot\frac{d\xi'}{d\eta}+\frac{d\eta}{d\eta'}\cdot\frac{d\eta'}{d\eta}&=1\end{aligned}\right\} \quad \dots\dots\dots(39);$$

for, the derivatives ... $x_{rs}, y_{rs}, x'_{rs}, y'_{rs}$, ... being all regarded as constant, (23) and (24) or (25) and (26) are merely two equations connecting ξ and η with ξ' and η'.

The same relations may also be written

$$\frac{\dfrac{d\xi}{d\xi'}}{\dfrac{d\eta'}{d\eta}} = -\frac{\dfrac{d\xi}{d\eta'}}{\dfrac{d\xi'}{d\eta}} = -\frac{\dfrac{d\eta}{d\xi'}}{\dfrac{d\eta'}{d\xi}} = \frac{\dfrac{d\eta}{d\eta'}}{\dfrac{d\xi'}{d\xi}}$$

$$= \frac{\dfrac{d\xi}{d\xi'}\cdot\dfrac{d\eta}{d\eta'} - \dfrac{d\xi}{d\eta'}\cdot\dfrac{d\eta}{d\xi'}}{1} = \frac{1}{\dfrac{d\xi'}{d\xi}\cdot\dfrac{d\eta'}{d\eta} - \dfrac{d\xi'}{d\eta}\cdot\dfrac{d\eta'}{d\xi}} \quad \ldots\ldots\ldots \ (40).$$

8. Let us now first apply what precedes to the reversion of operators included in the form

$$\Sigma\,(a+br+cs)\,x'_{rs}\frac{d}{dx'_{rs}} + \Sigma\,(a'+b'r+c's)\,y'_{rs}\frac{d}{dy'_{rs}} \quad \ldots\ldots\ldots (41),$$

where the summations extend over all positive integral (and zero) values of r and s for which $r+s>0$.

The symbolical form of the reversed operator is, by § 6,

$$-\left\{a\xi' + b\xi\frac{d\xi'}{d\xi} + c\eta\frac{d\xi'}{d\eta}\right\}\left(\frac{d\xi}{d\xi'} + \lambda\frac{d\eta}{d\xi'}\right)$$

$$-\left\{a'\eta' + b'\xi\frac{d\eta'}{d\xi} + c'\eta\frac{d\eta'}{d\eta}\right\}\left(\frac{d\xi}{d\eta'} + \lambda\frac{d\eta}{d\eta'}\right);$$

which may be written

$$-a\left\{\xi'\frac{d\xi}{d\xi'} + \lambda\xi'\frac{d\eta}{d\xi'}\right\} - a'\left\{\eta'\frac{d\xi}{d\eta'} + \lambda\eta'\frac{d\eta}{d\eta'}\right\}$$

$$-\frac{b+b'}{2}\left\{\frac{d\xi'}{d\xi}\frac{d\xi}{d\xi'} + \frac{d\eta'}{d\xi}\frac{d\xi}{d\eta'} + \lambda\left(\frac{d\xi'}{d\xi}\frac{d\eta}{d\xi'} + \frac{d\eta'}{d\xi}\frac{d\eta}{d\eta'}\right)\right\}\xi$$

$$-\frac{c+c'}{2}\left\{\frac{d\xi'}{d\eta}\frac{d\xi}{d\xi'} + \frac{d\eta'}{d\eta}\frac{d\xi}{d\eta'} + \lambda\left(\frac{d\xi'}{d\eta}\frac{d\eta}{d\xi'} + \frac{d\eta'}{d\eta}\frac{d\eta}{d\eta'}\right)\right\}\eta$$

$$-\frac{b-b'}{2}\left\{\frac{d\xi'}{d\xi}\frac{d\xi}{d\xi'} - \frac{d\eta'}{d\xi}\frac{d\xi}{d\eta'} + \lambda\left(\frac{d\xi'}{d\xi}\frac{d\eta}{d\xi'} - \frac{d\eta'}{d\xi}\frac{d\eta}{d\eta'}\right)\right\}\xi$$

$$-\frac{c-c'}{2}\left\{\frac{d\xi'}{d\eta}\frac{d\xi}{d\xi'} - \frac{d\eta'}{d\eta}\frac{d\xi}{d\eta'} + \lambda\left(\frac{d\xi'}{d\eta}\frac{d\eta}{d\xi'} - \frac{d\eta'}{d\eta}\frac{d\eta}{d\eta'}\right)\right\}\eta,$$

and, by (39) or (40), is consequently equal to

$$-a\left\{\xi'\frac{d\xi}{d\xi'}+\lambda\xi'\frac{d\eta}{d\xi'}\right\}-a'\left\{\eta'\frac{d\xi}{d\eta'}+\lambda\eta'\frac{d\eta}{d\eta'}\right\}-\frac{b+b'}{2}\xi-\frac{c+c'}{2}\lambda\eta$$

$$-\frac{b-b'}{2}\left\{\xi\frac{\frac{d\xi}{d\xi'}\frac{d\eta}{d\eta'}+\frac{d\xi}{d\eta'}\frac{d\eta}{d\xi'}}{\frac{d\xi}{d\xi'}\frac{d\eta}{d\eta'}-\frac{d\xi}{d\eta'}\frac{d\eta}{d\xi'}}+\lambda\xi\frac{2\frac{d\eta}{d\xi}\frac{d\eta}{d\eta'}}{\frac{d\xi}{d\xi'}\frac{d\eta}{d\eta'}-\frac{d\xi}{d\eta'}\frac{d\eta}{d\xi'}}\right\}$$

$$+\frac{c-c'}{2}\left\{\eta\frac{2\frac{d\xi}{d\xi'}\frac{d\xi}{d\eta'}}{\frac{d\xi}{d\xi'}\frac{d\eta}{d\eta'}-\frac{d\xi}{d\eta'}\frac{d\eta}{d\xi'}}+\lambda\eta\frac{\frac{d\xi}{d\xi'}\frac{d\eta}{d\eta'}+\frac{d\xi}{d\eta'}\frac{d\eta}{d\xi'}}{\frac{d\xi}{d\xi'}\frac{d\eta}{d\eta'}-\frac{d\xi}{d\eta'}\frac{d\eta}{d\xi'}}\right\} \quad \ldots (42).$$

The operators which in this multiply $\frac{1}{2}(b-b')$ and $\frac{1}{2}(c-c')$ are o some complexity. The preceding terms, however, represent

$$-a\left\{\Sigma\,.\,rx_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,ry_{rs}\frac{d}{dy_{rs}}\right\}-a'\left\{\Sigma\,.\,sx_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,sy_{rs}\frac{d}{dy_{rs}}\right\}$$

$$-\tfrac{1}{2}(b+b')\,\Sigma\,.\,x_{rs}\frac{d}{dx_{rs}}-\tfrac{1}{2}(c+c')\,\Sigma\,.\,y_{rs}\frac{d}{dy_{rs}}.$$

Accordingly, putting $c=c'$ and $b=b'$, we obtain the formula of reversion

$$a\Sigma\,.\,x'_{rs}\frac{d}{dx'_{rs}}+a'\Sigma\,.\,y'_{rs}\frac{d}{dy'_{rs}}+b\Sigma\left\{rx'_{rs}\frac{d}{dx'_{rs}}+ry'_{rs}\frac{d}{dy'_{rs}}\right\}$$

$$+c\Sigma\left\{sx'_{rs}\frac{d}{dx'_{rs}}+sy'_{rs}\frac{d}{dy'_{rs}}\right\}$$

$$=-a\Sigma\left\{rx_{rs}\frac{d}{dx_{rs}}+ry_{rs}\frac{d}{dy_{rs}}\right\}-a'\Sigma\left\{sx_{rs}\frac{d}{dx_{rs}}+sy_{rs}\frac{d}{dy_{rs}}\right\}$$

$$-b\Sigma\,.\,x_{rs}\frac{d}{dx_{rs}}-c\Sigma\,.\,y_{rs}\frac{d}{dy_{rs}}\ \ldots\ldots (43).$$

The operators, therefore, which multiply a, a', b, c on the two sides of this identity are equivalent in pairs.

Now, a function of either unaccented or accented derivatives can have two kinds of homogeneity—in suffixed x's and in suffixed y's. It can also have four kinds of isobarism—in first and second suffixes of x's, and in first and second suffixes of y's. Denote by $i(x)$, $i(y)$, $w_1(x)$, $w_2(x)$, $w_1(y)$, $w_2(y)$ the several degrees and weights of a function of the unaccented derivatives, and let $i(x')$, $i(y')$, $w_1(x')$, &c. have like meanings as to the expression in terms of accented derivatives. The facts now arrived at may, it will be seen, be stated

$$\left.\begin{aligned} i(x') &= -w_1(x) - w_1(y) \\ i(y') &= -w_2(x) - w_2(y) \\ w_1(x') + w_1(y') &= -i(x) \\ w_2(x') + w_2(y') &= -i(y) \end{aligned}\right\} \qquad (44),$$

the last two being the conjugates, and so affording verifications, of the first two.

For instance, the meaning of the first equality is that a function of the accented derivatives which is homogeneous in those of x has for its reversion a function of the unaccented derivatives, throughout which the sum of first suffixes is constant and equal to the negative of the given degree. Similarly for the others. By addition of the first two, we see that the reversion of a function which is homogeneous on the whole, is on the whole isobaric, and conversely.

9. Closely akin to the operators reversed in the last article are the two

$$\Sigma \,.\, x'_{rs} \frac{d}{dy'_{rs}}, \quad \Sigma \,.\, y'_{rs} \frac{d}{dx'_{rs}}.$$

The transformation of the former is

$$-\Sigma \,.\, x'_{rs} \xi^r \eta^s \left(\frac{d\xi}{d\eta'} + \lambda \frac{d\eta}{d\eta'}\right)$$

$$= -\xi' \left(\frac{d\xi}{d\eta'} + \lambda \frac{d\eta}{d\eta'}\right)$$

$$= -\Sigma \,.\, s x_{rs} \frac{d}{dx_{r+1, s-1}} - \Sigma \,.\, s y_{rs} \frac{d}{dy_{r+1, s-1}} \qquad (45),$$

while that of $\Sigma \,.\, y'_{rs} \frac{d}{dx'_{rs}}$ is

$$-\eta' \left(\frac{d\xi}{d\xi'} + \lambda \frac{d\eta}{d\xi'}\right)$$

$$= -\Sigma \,.\, r x_{rs} \frac{d}{dx_{r-1, s+1}} - \Sigma \,.\, r y_{rs} \frac{d}{dy_{r-1, s+1}} \qquad (46).$$

10. A function of the second and higher without the first derivatives is called a *pure* function. An aim in the sequel will be to discover properties of a pure function P, of the unaccented derivatives, which is such that, for some value of the constant μ, the reversion of

$$(x_{10}y_{01}-x_{01}y_{10})^{-\mu}P$$

is a pure function of the accented derivatives.

Now, by (2),

$$x_{10}y_{01}-x_{01}y_{10}=\frac{1}{x'_{10}y'_{01}-x'_{01}y'_{10}}=\frac{x_{10}}{y'_{01}}=-\frac{x_{01}}{x'_{01}}=-\frac{y_{10}}{y'_{10}}=\frac{y_{01}}{x'_{10}}.$$

Thus,

$$\frac{d}{dx'_{10}}\{(x_{10}y_{01}-x_{01}y_{10})^{-\mu}P\}=0$$

may be written

$$\frac{d}{dx'_{10}}\left\{\left(\frac{x_{10}}{y'_{01}}\right)^{-\mu}P\right\}=0,$$

i.e.,

$$\frac{d}{dx'_{10}}(x_{10}^{-\mu}P)=0,$$

i.e.,

$$\frac{dP}{dx'_{10}}+\mu x_{10}P=0 \qquad (47);$$

since, as will be seen in the next article,

$$\frac{dx_{10}}{dx'_{10}}=-x_{10}^2.$$

Again,

$$\frac{d}{dx'_{01}}\{(x_{10}y_{01}-x_{01}y_{10})^{-\mu}P\}=0$$

means

$$\frac{d}{dx'_{01}}(x_{10}^{-\mu}P)=0,$$

i.e.,

$$\frac{dP}{dx'_{01}}+\mu y_{10}P=0 \qquad (48);$$

since, as will be seen,

$$\frac{dx_{10}}{dx'_{01}}=-x_{10}y_{10}.$$

Once more,

$$\frac{d}{dy'_{10}}\{(x_{10}y_{01}-x_{01}y_{10})^{-\mu}P\}=0$$

is the same as

$$\frac{d}{dy'_{10}}(x_{10}^{-\mu}P)=0,$$

i.e.,

$$\frac{dP}{dy'_{10}}+\mu x_{01}P=0 \qquad (49);$$

for it will be seen that

$$\frac{dx_{10}}{dy'_{10}} = -x_{10}x_{01}.$$

And lastly, $$\frac{d}{dy'_{01}}\left\{(x_{10}y_{01}-x_{01}y_{10})^{-\mu}P\right\} = 0$$

is $$\frac{d}{dy'_{01}}\left\{\left(\frac{x_{01}}{x'_{01}}\right)^{-\mu}P\right\} = 0,$$

i.e., $$\frac{d}{dy'_{01}}(x_{01}^{-\mu}P) = 0,$$

i.e., $$\frac{dP}{dy'_{01}} + \mu y_{01}P = 0 \quad\text{.........................(50)};$$

since we shall see that

$$\frac{dx_{01}}{dy'_{01}} = -x_{01}y_{01}.$$

11. For settlement of the points here assumed, as well as for future use, it is desirable to examine more closely, by separation of the first and higher derivatives, the reversed operators which are found, as in § 6, as the equivalents of $\frac{d}{dx'_{10}}, \frac{d}{dx'_{01}}, \frac{d}{dy'_{10}}, \frac{d}{dy'_{01}}$.

As the reversed equivalent of $\frac{d}{dx'_{10}}$, we have, by (35),

$$-\xi\left\{\frac{d\xi}{d\xi'} + \lambda\frac{d\eta}{d\xi'}\right\},$$

i.e., $$-\left\{x_{10}\xi'+x_{01}\eta'+(\xi-x_{10}\xi'-x_{01}\eta')\right\}$$
$$\times\left\{x_{10}+\lambda y_{10}+\frac{d}{d\xi'}(\xi-x_{10}\xi'-x_{01}\eta')+\lambda\frac{d}{d\xi'}(\eta-y_{10}\xi'-y_{01}\eta')\right\},$$

i.e., $$-x_{10}^2\xi'-x_{10}x_{01}\eta'-x_{10}y_{10}\lambda\xi'-x_{01}y_{10}\lambda\eta'$$
$$-x_{10}(\xi-x_{10}\xi'-x_{01}\eta')-y_{10}\lambda(\xi-x_{10}\xi'-x_{01}\eta')$$
$$-x_{10}\left\{\xi'\frac{d}{d\xi'}(\xi-x_{10}\xi'-x_{01}\eta')+\lambda\xi'\frac{d}{d\xi'}(\eta-y_{10}\xi'-y_{01}\eta')\right\}$$
$$-x_{01}\left\{\eta'\frac{d}{d\xi'}(\xi-x_{10}\xi'-x_{01}\eta')+\lambda\eta'\frac{d}{d\xi'}(\eta-y_{10}\xi'-y_{01}\eta')\right\}$$
$$-\tfrac{1}{2}\frac{d}{d\xi'}(\xi-x_{10}\xi'-x_{01}\eta')^2-\lambda(\xi-x_{10}\xi'-x_{01}\eta')\frac{d}{d\xi'}(\eta-y_{10}\xi'-y_{01}\eta');$$

that is to say, we have

$$\frac{d}{dx'_{10}} = -x_{10}^2 \frac{d}{dx_{10}} - x_{10}x_{01}\frac{d}{dx_{01}} - x_{10}y_{10}\frac{d}{dy_{10}} - x_{01}y_{10}\frac{d}{dy_{01}}$$

$$-x_{10} \sum_{r+s \nless 2} x_{rs}\frac{d}{dx_{rs}} - y_{10} \sum_{r+s \nless 2} x_{rs}\frac{d}{dy_{rs}}$$

$$-x_{10} \sum_{r+s \nless 2} \left\{ rx_{rs}\frac{d}{dx_{rs}} + ry_{rs}\frac{d}{dy_{rs}} \right\}$$

$$-x_{01} \sum_{r+s \nless 2} \left\{ rx_{rs}\frac{d}{dx_{r-1,s+1}} + ry_{rs}\frac{d}{dy_{r-1,s+1}} \right\}$$

$$-\Sigma\Sigma \left\{ rx_{m-r,n-s}\left[x_{rs}\frac{d}{dx_{m-1,n}} + y_{rs}\frac{d}{dy_{m-1,n}} \right] \right\} \text{ ....(51)},$$

the limits of the last summations with regard to m and n and r and s being such as to allow the inclusion of all cases for which

$$r \nless 0 \text{ and } \ngtr m, \quad s \nless 0 \text{ and } \ngtr n,$$

$$r+s \nless 2 \text{ and } \ngtr m+n-2, \; m \nless 1, \; n \nless 0.$$

In like manner, the equivalent of $\frac{d}{dx'_{01}}$ is

$$-\eta \left\{ \frac{d\xi}{d\xi'} + \lambda \frac{d\eta}{d\xi'} \right\},$$

i.e., $$-\{y_{10}\xi' + y_{01}\eta' + (\eta - y_{10}\xi' - y_{01}\eta')\}$$

$$\times \left\{ x_{10} + \lambda y_{10} + \frac{d}{d\xi'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\frac{d}{d\xi'}(\eta - y_{10}\xi' - y_{01}\eta') \right\},$$

i.e., $$-x_{10}y_{10}\xi' - x_{10}y_{01}\eta' - y_{10}^2\lambda\xi' - y_{10}y_{01}\lambda\eta'$$

$$-x_{10}(\eta - y_{10}\xi' - y_{01}\eta') - y_{10}\lambda(\eta - y_{10}\xi' - y_{01}\eta')$$

$$-y_{10}\left\{ \xi'\frac{d}{d\xi'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\xi'\frac{d}{d\xi'}(\eta - y_{10}\xi' - y_{01}\eta') \right\}$$

$$-y_{01}\left\{ \eta'\frac{d}{d\xi'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\eta'\frac{d}{d\xi'}(\eta - y_{10}\xi' - y_{01}\eta') \right\}$$

$$-(\eta - y_{10}\xi' - y_{01}\eta')\frac{d}{d\xi'}(\xi - x_{10}\xi' - x_{01}\eta') - \tfrac{1}{2}\lambda\frac{d}{d\xi'}(\eta - y_{10}\xi' - y_{01}\eta')^2;$$

so that

$$\frac{d}{dx'_{01}} = -x_{10}y_{10}\frac{d}{dx_{10}} - x_{10}y_{01}\frac{d}{dx_{01}} - y_{10}^2\frac{d}{dy_{10}} - y_{10}y_{01}\frac{d}{dy_{01}}$$

$$-x_{10}\sum_{r+s \not< 2} y_{rs}\frac{d}{dx_{rs}} - y_{10}\sum_{r+s \not< 2} y_{rs}\frac{d}{dy_{rs}}$$

$$-y_{10}\sum_{r+s \not< 2}\left\{rx_{rs}\frac{d}{dx_{rs}} + ry_{rs}\frac{d}{dy_{rs}}\right\}$$

$$-y_{01}\sum_{r+s \not< 2}\left\{rx_{rs}\frac{d}{dx_{r-1,s+1}} + ry_{rs}\frac{d}{dy_{r-1,s+1}}\right\}$$

$$-\Sigma\Sigma\left\{ry_{m-r,n-s}\left[x_{rs}\frac{d}{dx_{m-1,n}} + y_{rs}\frac{d}{dy_{m-1,n}}\right]\right\} \quad \ldots\ldots\ldots\ldots (52),$$

the limits of the last summations being as in (51).

Once more, as the equivalent of $\frac{d}{dy'_{10}}$ we have

$$-\xi\left\{\frac{d\xi}{d\eta'} + \lambda\frac{d\eta}{d\eta'}\right\},$$

i.e.,
$$-\left\{x_{10}\xi' + x_{01}\eta' + (\xi - x_{10}\xi' - x_{01}\eta')\right\}$$

$$\times\left\{x_{01} + \lambda y_{01} + \frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\frac{d}{d\eta'}(\eta - y_{10}\xi' - y_{01}\eta')\right\},$$

i.e.,
$$-x_{10}x_{01}\xi' - x_{01}^2\eta' - x_{10}y_{01}\lambda\xi' - x_{01}y_{01}\lambda\eta'$$

$$-x_{01}(\xi - x_{10}\xi' - x_{01}\eta') - y_{01}\lambda(\xi - x_{10}\xi' - x_{01}\eta')$$

$$-x_{10}\left\{\xi'\frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\xi'\frac{d}{d\eta'}(\eta - y_{10}\xi' - y_{01}\eta')\right\}$$

$$-x_{01}\left\{\eta'\frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\eta'\frac{d}{d\eta'}(\eta - y_{10}\xi' - y_{01}\eta')\right\}$$

$$-\frac{1}{2}\frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta')^2 - \lambda(\xi - x_{10}\xi' - x_{01}\eta')(\eta - y_{10}\xi' - y_{01}\eta');$$

so that

$$\frac{d}{dy'_{10}} = -x_{10}x_{01}\frac{d}{dx_{10}} - x_{01}^2\frac{d}{dx_{01}} - x_{10}y_{01}\frac{d}{dy_{10}} - x_{01}y_{01}\frac{d}{dy_{01}}$$

$$-x_{01}\sum_{r+s \not< 2} x_{rs}\frac{d}{dx_{rs}} - y_{01}\sum_{r+s\not<2} x_{rs}\frac{d}{dy_{rs}}$$

$$-x_{10}\sum_{r+s\not<2}\left\{sx_{rs}\frac{d}{dx_{r+1,s-1}} + sy_{rs}\frac{d}{dy_{r+1,s-1}}\right\}$$

$$-x_{01}\sum_{r+s\not<2}\left\{sx_{rs}\frac{d}{dx_{rs}} + sy_{rs}\frac{d}{dy_{rs}}\right\}$$

$$-\Sigma\Sigma\left\{sx_{m-r,n-s}\left[x_{rs}\frac{d}{dx_{m,n-1}} + y_{rs}\frac{d}{dy_{m,n-1}}\right]\right\} \quad \text{.................(53)},$$

the limits of the last summations being as in (51), except for

$$m \not< 0, \quad n \not< 1,$$

instead of $\qquad m \not< 1, \quad n \not< 0.$

And lastly, since $\dfrac{d}{dy'_{01}}$ reverses into

$$-\eta\left\{\frac{d\xi}{d\eta'} + \lambda\frac{d\eta}{d\eta'}\right\},$$

i.e.,
$$-\left\{y_{10}\xi' + y_{01}\eta' + (\eta - y_{10}\xi' - y_{01}\eta')\right\}$$
$$\times\left\{x_{01} + \lambda y_{01} + \frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\frac{d}{d\eta'}(\eta - y_{10}\xi' - y_{01}\eta')\right\},$$

i.e.,
$$-x_{01}y_{10}\xi' - x_{01}y_{01}\eta' - y_{10}y_{01}\lambda\xi' - y_{01}^2\lambda\eta'$$
$$-x_{01}(\eta - y_{10}\xi' - y_{01}\eta') - y_{01}\lambda(\eta - y_{10}\xi' - y_{01}\eta')$$
$$-y_{10}\left\{\xi'\frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\xi'\frac{d}{d\eta'}(\eta - y_{10}\xi' - y_{01}\eta')\right\}$$
$$-y_{01}\left\{\eta'\frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta') + \lambda\eta'\frac{d}{d\eta'}(\eta - y_{10}\xi' - y_{01}\eta')\right\}$$
$$-(\eta - y_{10}\xi' - y_{01}\eta')\frac{d}{d\eta'}(\xi - x_{10}\xi' - x_{01}\eta') - \tfrac{1}{2}\lambda\frac{d}{d\eta'}(\eta - y_{10}\xi' - y_{01}\eta')^2,$$

we have

$$\frac{d}{dy'_{01}} = -x_{01}y_{10}\frac{d}{dx_{10}} - x_{01}y_{01}\frac{d}{dx_{01}} - y_{10}y_{01}\frac{d}{dy_{10}} - y_{01}^2\frac{d}{dy_{01}}$$

$$-x_{01}\sum_{r+s \nless 2} y_{rs}\frac{d}{dx_{rs}} - y_{01}\sum_{r+s \nless 2} y_{rs}\frac{d}{dy_{rs}}$$

$$-y_{10}\sum_{r+s \nless 2}\left\{sx_{rs}\frac{d}{dx_{r+1,s-1}} + sy_{rs}\frac{d}{dy_{r+1,s-1}}\right\}$$

$$-y_{01}\sum_{r+s \nless 2}\left\{sx_{rs}\frac{d}{dx_{rs}} + sy_{rs}\frac{d}{dy_{rs}}\right\}$$

$$-\Sigma\Sigma\left\{sy_{m-r,n-s}\left[x_{rs}\frac{d}{dy_{m,n-1}} + y_{rs}\frac{d}{dy_{m,n-1}}\right]\right\} \quad \ldots\ldots\ldots(54),$$

with limits of the last summations, as in (53).

12. As a first application of this last article, notice that, as partially used in § 10, .

$$\left.\begin{array}{llll}
\frac{d}{dx'_{10}}x_{10} = -x_{10}^2, & \frac{d}{dx'_{10}}x_{01} = -x_{10}x_{01}, & \frac{d}{dx'_{10}}y_{10} = -x_{10}y_{10}, & \frac{d}{dx'_{10}}y_{01} = -x_{01}y_{10} \\
\frac{d}{dx'_{01}}x_{10} = -x_{10}y_{10}, & \frac{d}{dx'_{01}}x_{01} = -x_{10}y_{01}, & \frac{d}{dx'_{01}}y_{10} = -y_{10}^2, & \frac{d}{dx'_{01}}y_{01} = -y_{10}y_{01} \\
\frac{d}{dy'_{10}}x_{10} = -x_{10}y_{01}, & \frac{d}{dy'_{10}}x_{01} = -x_{01}^2, & \frac{d}{dy'_{10}}y_{10} = -x_{10}y_{01}, & \frac{d}{dy'_{10}}y_{01} = -x_{01}y_{01} \\
\frac{d}{dy'_{01}}x_{10} = -x_{01}y_{10}, & \frac{d}{dy'_{01}}x_{01} = -x_{01}y_{01}, & \frac{d}{dy'_{10}}y_{10} = -y_{10}y_{01}, & \frac{d}{dy'_{01}}y_{01} = -y_{01}^2
\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(55).$$

13. We can now exhibit in their full cogency the conditions of § 10, that the reversion of

$$(x_{10}y_{01} - x_{01}y_{10})^{-\mu}P$$

be independent of all the first derivatives

$$x'_{10},\ x'_{01},\ y'_{10},\ y'_{01}.$$

For, upon inserting in (47) to (50) from (51) to 54), we get vanishing expressions in which the terms free from first derivatives and the terms multiplying

$$x_{10},\ x_{01},\ y_{10},\ y_{01}$$

must separately vanish, since no term of any one set can go out against a term of any other. The conclusions are found to be that the pure function P must be annihilated separately by each of the twelve operators

$$\Sigma\,.\,x_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,rx_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,ry_{rs}\frac{d}{dy_{rs}}-\mu \quad\ldots\ldots\ldots\ldots (56),$$

$$\Sigma\,.\,y_{rs}\frac{d}{dy_{rs}}+\Sigma\,.\,rx_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,ry_{rs}\frac{d}{dy_{rs}}-\mu \quad\ldots\ldots\ldots\ldots (57),$$

$$\Sigma\,.\,x_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,sx_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,sy_{rs}\frac{d}{dy_{rs}}-\mu \quad\ldots\ldots\ldots\ldots (58),$$

$$\Sigma\,.\,y_{rs}\frac{d}{dy_{rs}}+\Sigma\,.\,sx_{rs}\frac{d}{dx_{rs}}+\Sigma\,.\,sy_{rs}\frac{d}{dy_{rs}}-\mu \quad\ldots\ldots\ldots\ldots (59),$$

$$\Sigma\,.\,x_{rs}\frac{d}{dy_{rs}} \quad\ldots\ldots\ldots\ldots (60),$$

$$\Sigma\,.\,y_{rs}\frac{d}{dx_{rs}} \quad\ldots\ldots\ldots\ldots (61),$$

$$\Sigma\,.\,rx_{rs}\frac{d}{dx_{r-1,s+1}}+\Sigma\,.\,ry_{rs}\frac{d}{dy_{r-1,s+1}} \quad\ldots\ldots\ldots\ldots (62),$$

$$\Sigma\,.\,sx_{rs}\frac{d}{dx_{r+1,s-1}}+\Sigma\,.\,sy_{rs}\frac{d}{dy_{r+1,s-1}} \quad\ldots\ldots\ldots\ldots (63),$$

$$\Sigma\Sigma\left\{rx_{m-r,n-s}\left[x_{rs}\frac{d}{dx_{m-1,n}}+y_{rs}\frac{d}{dy_{m-1,n}}\right]\right\} \quad\ldots\ldots\ldots\ldots (64),$$

$$\Sigma\Sigma\left\{ry_{m-r,n-s}\left[x_{rs}\frac{d}{dx_{m-1,n}}+y_{rs}\frac{d}{dy_{m-1,n}}\right]\right\} \quad\ldots\ldots\ldots\ldots (65),$$

$$\Sigma\Sigma\left\{sx_{m-r,n-s}\left[x_{rs}\frac{d}{dx_{m,n-1}}+y_{rs}\frac{d}{dy_{m,n-1}}\right]\right\} \quad\ldots\ldots\ldots\ldots (66),$$

$$\Sigma\Sigma\left\{sy_{m-r,n-s}\left[x_{rs}\frac{d}{dx_{m,n-1}}+y_{rs}\frac{d}{dy_{m,n-1}}\right]\right\} \quad\ldots\ldots\ldots\ldots (67).$$

The extent of every summation may be concisely expressed by saying, that both in the coefficients and the symbols of partial differentiation none but second and higher derivatives occur, and that no possible term, with such a limitation, is absent.

Of the twelve annihilators it is not at once clear how many are independent. The first four are equivalent to three only, since

$$(56)-(57)-(58)+(59)$$

is identically zero.

14. Most of these annihilators admit of ready interpretation.

The first four (three independent) express facts with regard to the degrees and weights of P; namely, that throughout P the degree in derivatives of x is equal to that in derivatives of y, that the sum of first suffixes (of x's and y's together) is equal to the sum of second suffixes, and that the sum of either of the equal degree sums and either of the equal weight sums is constant and equal to μ. These may be expressed

$$i\,(x) = i\,(y) = \mu - w_1\,(x) - w_1\,(y) = \mu - w_2\,(x) - w_2\,(y) \;\ldots\; (68).$$

That these equal characteristics are all constant will be established in the following article.

Again, (60) and (61) tell us that P is a function of the determinants included in the system

$$\left\|\begin{matrix} x_{20}, & x_{11}, & x_{02}, & x_{30}, & x_{21}, & x_{12}, & x_{03}, & \ldots \\ y_{20}, & y_{11}, & y_{02}, & y_{30}, & y_{21}, & y_{12}, & y_{03}, & \ldots \end{matrix}\right\| \;\ldots\ldots\ldots\ldots\ldots\; (69);$$

in other words, that P is unaltered, but for a power of the modulus by any linear transformation of the cogredient pairs of quantities x_{20}, y_{20}; x_{11}, y_{11}; x_{02}, y_{02}; ...; that is to say, by such a substitution as

$$\left.\begin{aligned} x+px'+qy'+r &= lX+mY \\ y+p'x'+q'y'+r' &= l'X+m'Y \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (70).$$

Once more, (62) and (63) require that P be a full invariant of the two systems of quantics

$$\left.\begin{aligned} &(x_{20}, x_{11}, x_{02}\between \xi', \eta')^2 \\ &(x_{30}, x_{21}, x_{12}, x_{03}\between \xi', \eta')^3 \\ &\&c. \qquad \&c. \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (71),$$

$$\left.\begin{aligned} &(y_{20}, y_{11}, y_{02}\between \xi', \eta')^2 \\ &(y_{30}, y_{21}, y_{12}, y_{03}\between \xi', \eta')^3 \\ &\&c. \qquad \&c. \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (72).$$

Lastly, (64) to (67) are annihilators closely related to the annihilators

$$V_1 \equiv \Sigma\Sigma\, r x_{rs} x_{m-r,\,n-s} \frac{d}{dx_{m-1,\,n}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (73),$$

$$V_2 \equiv \Sigma\Sigma\, s x_{rs} x_{m-r,\,n-s} \frac{d}{dx_{m,\,n-1}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (74),$$

of pure cyclicants or ternary reciprocants. A detailed study of this connexion would undoubtedly be productive of valuable results.

15. It will now be established that the pure functions P and P' which satisfy

$$(x_{10}y_{01}-x_{01}y_{10})^{-\mu}P = P' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (75),$$

which may also be written

$$(x_{10}y_{01}-x_{01}y_{10})^{-\frac{1}{2}\mu}P = (x'_{10}y'_{01}-x'_{01}y'_{10})^{-\frac{1}{2}\mu}P' \ldots\ldots\ldots\ldots (76),$$

or, again,

$$P = (x'_{10}y'_{01}-x'_{01}y'_{10})^{-\mu}P' \ldots\ldots\ldots\ldots (77),$$

must be necessarily the same functions of the unaccented and accented derivatives respectively; in other words, that such a P is a pure reciprocant in two independent and two dependent variables.

P must be homogeneous, and isobaric both in first and in second suffixes, by the fact that it is an invariant of the system of quantics (71) and (72). Moreover, by a particular case of the invariant property for transformations (70), P is unaffected, except perhaps as to sign, by the interchange of x and y. Thus the relations (68) may be replaced by the far more stringent system

$$2w_1(x) = 2w_1(y) = 2w_2(x) = 2w_2(y)$$
$$= \mu - i(x) = \mu - i(y) = \text{constant} \ldots\ldots\ldots (78).$$

For exactly the same reasons, regarding (77) instead of its equivalent (75), the like facts hold as to the characteristics of P'; viz.,

$$2w_1(x') = 2w_1(y') = 2w_2(x') = 2w_2(y')$$
$$= \mu - i(x') = \mu - i(y') = \text{constant} \ldots\ldots\ldots (79).$$

The constant degrees and weights of P' must, moreover, be the same as those of P. For, with regard to P,

$$i(x) = \mu - w_1(x) - w_1(y)$$

$$= \mu + \text{degree of } (x'_{10}y'_{01} - x'_{01}y'_{10})^{-\mu} P' \text{ in suffixed } x\text{'s},$$

by (44) and (77),

$$= \mu + i(x') - \mu,$$

where $i(x')$ refers to P',

$$= i(x') \quad \text{...(80).}$$

Thus, the equal constant characteristics in (78) and those in (79) are equal to the same constant.

That P and P' are of the same form may now be seen, as follows. We observe that the equalities (5) to (10) may be expressed by saying that, for the values 2, 0; 1, 1; 0, 2 of r and s,

$$r!\,s!\,x'_{rs} = (x_{10}y_{01} - x_{01}y_{10})^{-r-s-1}\left(m'\frac{d}{dx'} - m\frac{d}{dy'}\right)^r \left(-l'\frac{d}{dx'} + l\frac{d}{dy'}\right)^s$$

$$\times (l'y - m'x) \quad \text{.............................(81),}$$

$$r!\,s!\,y'_{rs} = (x_{10}y_{01} - x_{01}y_{10})^{-r-s-1}\left(m'\frac{d}{dx'} - m\frac{d}{dy'}\right)^r \left(-l'\frac{d}{dx'} + l\frac{d}{dy'}\right)^s$$

$$\times (-ly + mx) \quad \text{..........................(82);}$$

in which, after the operations are performed, l, l', m, m' are given the values x_{10}, x_{01}, y_{10}, y_{01}.

It is to be observed that the second—*i.e.*, $(r+s)^{\text{th}}$—derivatives of x and y are involved linearly in these expressions, and that there is no term free from them.

Now, referring to (3), $x'_{r+1,s}$, $y'_{r+1,s}$ are derived from x'_{rs}, y'_{rs} by the operator

$$\frac{1}{r+1}(x_{10}, y_{01} - x_{01}y_{10})^{-1}\left(y_{01}\frac{d}{dx'} - y_{10}\frac{d}{dy'}\right),$$

and $x'_{r,s+1}$, $y'_{r,s+1}$ by the operator

$$\frac{1}{s+1}(x_{10}y_{01} - x_{01}y_{10})^{-1}\left(-x_{01}\frac{d}{dx'} + x_{10}\frac{d}{dy'}\right).$$

By repetition of these processes of derivation we obtain, in the notation of (81) and (82), that for higher values than 2 of the sum $r+s$ the values of $r!\,s!\,x'_{rs}$ and $r!\,s!\,y'_{rs}$ consist of terms given by the expressions on the dexter of (81) and (82), which are accordingly linear in the $(r+s)^{\text{th}}$ derivatives and involve besides only first derivatives, with the addition of terms which are of higher degree than the first in second and higher derivatives.

Now, $(x_{10}y_{01}-x_{01}y_{10})^{-\mu}P$ being a function equal to P' and so homogeneous, and of degree i say, in the values of $x'_{20}, y'_{20}, \ldots x'_{rs}, y'_{rs}, \ldots$, which is also homogeneous and of the same degree i in $x_{20}, y_{20}, \ldots x_{rs}, y_{rs}, \ldots$, it follows that the additional terms not written in (81) and (82), as these could only give rise to terms of degree higher than i, must cancel. P will therefore be, but for a power of $x_{10}y_{01}-x_{01}y_{10}$ as factor, the same function of

$$\frac{1}{r!\,s!}\left(m'\frac{d}{dx'}-m\frac{d}{dy'}\right)^r\left(-l'\frac{d}{dx'}+l\frac{d}{dy'}\right)^s(l'y-m'x) \quad\ldots\ldots (83),$$

$$\frac{1}{r!\,s!}\left(m'\frac{d}{dx'}-m\frac{d}{dy'}\right)^r\left(-l'\frac{d}{dx'}+l\frac{d}{dy'}\right)^s(-ly+mx) \ldots\ldots (84),$$

for different values of r and s, as P' is of x'_{rs}, y'_{rs}, and the other corresponding accented derivatives.

But, whatever z be,

$$\frac{1}{r!\,s!}\left(m'\frac{d}{dx'}-m\frac{d}{dy'}\right)^r\left(-l'\frac{d}{dx'}+l\frac{d}{dy'}\right)^s z$$

$$=\frac{1}{r!\,s!}\Big\{(-1)^s(r+s)!\,m'^r l'^s z_{r+s,0}$$

$$+(-1)^{s+1}(r+s-1)!\,1!\left(l\frac{d}{dl'}+m\frac{d}{dm'}\right)(m'^r l'^s)\,z_{r+s-1,1}$$

$$+(-1)^{s+2}(r+s-2)!\,2!\,\frac{1}{2!}\left(l\frac{d}{dl'}+m\frac{d}{dm'}\right)^2(m'^r l'^s)\,z_{r+s-2,2}+\ldots\Big\}$$

$=$ co. $\xi^r\eta^s$ in

$$z_{r+s,0}(m'\xi-l'\eta)^{r+s}+z_{r+s-1,1}(m'\xi-l'\eta)^{r+s-1}(-m\xi+l\eta)$$
$$+z_{r+s-2,2}(m'\xi-l'\eta)^{r+s-2}(-m\xi+l\eta)^2+\ldots \quad\ldots (85).$$

Accordingly, restoring to l, l', m, m' their values, we see that P is, but for a power of $x_{10}y_{01}-x_{01}y_{10}$ as factor, the same function of the various coefficients in the set of pairs of quantics

$$\left.\begin{array}{c}
(x_{01}y_{20}-y_{01}x_{20},\quad x_{01}y_{11}-y_{01}x_{11},\quad x_{01}y_{02}-y_{01}x_{02}\between \\
\times(y_{01}\xi'-x_{01}\eta',\ -y_{10}\xi'+x_{10}\eta')^2 \\
(-x_{10}y_{20}+y_{10}x_{20},\ -x_{10}y_{11}+y_{10}x_{11},\ -x_{10}y_{02}+y_{10}x_{02}\between \\
\times(y_{01}\xi'-x_{01}\eta',\ -y_{10}\xi'+x_{10}\eta')^2 \\
(x_{01}y_{30}-y_{01}x_{30},\quad x_{01}y_{21}-y_{01}x_{21},\quad x_{01}y_{12}-y_{01}x_{12},\quad x_{01}y_{03}-y_{01}x_{03}\between \\
\times(y_{01}\xi'-x_{01}\eta',\ -y_{10}\xi'+x_{10}\eta')^3 \\
(-x_{10}y_{30}+y_{10}x_{30},\ -x_{10}y_{21}+y_{10}x_{21},\ -x_{10}y_{12}+y_{10}x_{12},\ -x_{10}y_{03}+y_{10}x_{03}\between \\
\times(y_{01}\xi'-x_{01}\eta',\ -y_{10}\xi'+x_{10}\eta')^3 \\
\&c.,\qquad \&c.
\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots(86),$$

as P' is of those of the set of pairs of quantics

$$\left.\begin{array}{r}
(x'_{20},\ x'_{11},\ x'_{02}\between(u,\ v)^2 \\
(y'_{20},\ y'_{11},\ y'_{02}\between(u,\ v)^2 \\
(x'_{30},\ x'_{21},\ x'_{12},\ x'_{03}\between(u,\ v)^3 \\
(y'_{30},\ y'_{21},\ y'_{12},\ y'_{03}\between(u,\ v)^3 \\
\&c.,\qquad \&c.
\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (87).$$

We know, however, by the invariant property expressed by (62) and (63), and by that expressed by (60) and (61) (*cf.* § 14), that P is the same function, but for a power of $x_{10}y_{01}-x_{01}y_{10}$ as factor, of the coefficients of the pairs of quantics

$$\left.\begin{array}{r}
(x_{20},\ x_{11},\ x_{02}\between(\xi',\ \eta')^2 \\
(y_{20},\ y_{11},\ y_{02}\between(\xi',\ \eta')^2 \\
(x_{30},\ x_{21},\ x_{12},\ x_{03}\between(\xi',\ \eta')^3 \\
(y_{30},\ y_{21},\ y_{12},\ y_{03}\between(\xi',\ \eta')^3 \\
\&c.,\qquad \&c.
\end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (88),$$

as it is of the pairs of quantics (86).

Hence, finally, in (75), (76), or (77), P and P' are the same functions of the unaccented and accented second and higher derivatives respectively.

16. Any discussion of the multiplicity of pure reciprocants P, whose existence is here conditioned, must find place in a possible future communication. The investigation will naturally be on the lines, and by aid of the results, of my papers on pure cyclicants. I here note only the first specimen,

$$(x_{20}y_{11}-x_{11}y_{20})(x_{11}y_{02}-x_{02}y_{11})-(x_{20}y_{02}-x_{02}y_{20})^2 \quad \text{.........(90)},$$

the eliminant of the first pair of quantics in (88). It may also be written

$$4\,(x_{20}x_{02}-\tfrac{1}{4}x_{11}^2)(y_{20}y_{02}-\tfrac{1}{4}y_{11}^2)-(x_{20}y_{02}+x_{02}y_{20}-\tfrac{1}{2}x_{11}y_{11})^2 \quad \text{......(91)},$$

in which form its connexion with the first pure cyclicant $x_{20}x_{02}-\frac{1}{4}x_{11}^2$ is exhibited.

On Newton's Classification of Cubic Curves.

By Mr. W. W. Rouse Ball.

[*Read December* 11*th*, 1890.]

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