LETTERS TO THE EDITOR.

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Clausius' Virial Theorem.

THE question raised by Colonel Basevi, in NATURE for August 29, illustrates the importance of keeping in view a clear statement of what a general theorem such as that of Clausius with respect to the *virial* asserts, and the essential relativity of the forces which are regarded as acting on the particles, and of the kinetic energy of the system. The theorem asserts, I think, that if the motion of the system of particles be continued over any interval of time, t_1, the excess of the mean value of the kinetic energy of the system for that interval of time over the virial for the same interval is equal to the excess of the value of $\frac{1}{4t_1}\Sigma m\frac{d(\rho^2)}{dt}$ at the end of the interval over its value at the beginning, ρ being the distance of a specimen particle from the origin and m its mass, and the summation being extended over all the particles of the system.

It may be noticed here that the mean value of the kinetic energy of a system for an interval of time t_1 is equal to the *action* of the system for that interval taken per unit of the time in the interval.

There can be no doubt that the theorem is true, and will be verified by any test case to which it can be applied. The proof given by Clausius himself is perhaps the simplest, but the following mode of arriving at the theorem is instructive in some ways. Refer the particles to a system of rectangular axes in the ordinary way, and adopt the fluxional notation for velocities and accelerations. Thus taking a specimen particle, which is at the point x, y, z, at time t, regarding, as we are at liberty to do, the velocities $\dot{x}, \dot{y}, \dot{z}$, as functions of the position of the particle in the motion, we have

$$m\left(\dot{x}\frac{\partial\dot{x}}{\partial x} + \dot{y}\frac{\partial\dot{x}}{\partial y} + \dot{z}\frac{\partial\dot{x}}{\partial z}\right) = m\ddot{x} = \text{X}$$

and two other equations for Y, Z, which can be written down from this by symmetry. Multiplying these equations by x, y, z respectively, adding, and rearranging, we easily find

$$\frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)dt = -\tfrac{1}{2}\left(\text{X}x + \text{Y}y + \text{Z}z\right)dt + \frac{m}{2}d\left(x\dot{x} + y\dot{y} + z\dot{z}\right).$$

Integrated from $t = 0$ to $t = t_1$, and extended to all the particles, this gives

$$\tfrac{1}{2}\Sigma m\int_0^{t_1}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)dt = -\tfrac{1}{2}\Sigma\int_0^{t_1}(\text{X}x + \text{Y}y + \text{Z}z)dt + \tfrac{1}{2}\Big[\Sigma m(x\dot{x} + y\dot{y} + z\dot{z})\Big]_0^{t_1}.$$

The expression on the left [which may be written

$$\Sigma m\int(\dot{x}dx + \dot{y}dy + \dot{z}dz)]$$

is nowhere asserted, so far as I know, to be kinetic energy, but is the time-integral of the kinetic energy (that is the *action* of the system) for the time-interval t_1. Dividing both sides by t_1 we get the theorem as stated above, namely

$$\frac{1}{t_1}\int_0^{t_1}\text{T}dt = -\frac{1}{2t_1}\Sigma\int_0^{t_1}(\text{X}x + \text{Y}y + \text{Z}z)dt + \frac{1}{4t_1}\Big[\Sigma m\frac{d}{dt}(x^2 + y^2 + z^2)\Big]_0^{t_1},$$

where T denotes the kinetic energy of the system at the instant t.

It is clear that if t_1 be taken very great, and the velocity and the distance of each particle from the origin be always finite, the term on the left is neither infinite nor zero, while the last term on the right becomes vanishingly small. This is Clausius' case of "stationary motion," in which it is justifiable to write

$$\frac{1}{t_1}\int_0^{t_1}\text{T}dt = -\frac{1}{2t_1}\Sigma\int_0^{t_1}(\text{X}x + \text{Y}y + \text{Z}z)dt.$$

The expression on the right is the *virial*, and is in the circumstances stated undoubtedly equal to the time average or mean value of the kinetic energy, as the equation asserts.

If R be the force acting on a particle in the direction *towards* the origin along the line joining the origin with the particle, and ρ the distance of the particle from the origin, we have

$$\text{X}x + \text{Y}y + \text{Z}z = -\text{R}\rho,$$

and the theorem for stationary motion may be stated thus,

$$\text{Mean value of T} = \text{mean value of } \tfrac{1}{2}\Sigma\text{R}\rho,$$

where the summation takes in each particle once, and once only.

Let us apply this to the case taken by Lord Rayleigh, and alleged by Colonel Basevi to contradict the theory, of two particles each of mass m, at a distance apart $r (= 2\rho)$, revolving round their common centre of gravity. Here, taking the origin at the common centre of gravity, we have constant values of the virial and of T, namely $\frac{1}{2}\Sigma\text{R}\rho = \text{R}\rho$ and $\text{T} = m\text{V}^2$. Thus, $m\text{V}^2/\rho = \text{R}$, which, as Lord Rayleigh remarks, agrees with the law of centrifugal force.

If we take the motion relatively to one of the two particles regarded as at rest, we get the same result. The relative velocity of the other particle becomes 2V, and the corresponding kinetic energy $2m\text{V}^2$, the distance of the origin from the other particle 2ρ, and from itself zero. Since the acceleration of the moving particle relatively to the particle now supposed reduced to rest, is double its acceleration relatively to the common centre of gravity, the force now considered as acting on the moving particle must be taken as 2R. Thus we have $2m\text{V}^2 = \frac{1}{2}2\text{R} \times 2\rho$, or as before, $m\text{V}^2/\rho = \text{R}$.

If we do not suppose the origin to coincide with one of the particles reduced to rest in this manner, but to coincide for the moment with the *position* of one of the particles, the velocity of each particle is V, the force towards the origin on that distant from it r is R, and we have $\text{T} = m\text{V}^2$, $\frac{1}{2}\Sigma\text{R}\rho = \frac{1}{2}\text{R}r$, since now $\rho = r$. Hence once more $m\text{V}^2/\rho = \text{R}$.

Similarly, any other origin and axes of reference would give the same result. Colonel Basevi has, it seems to me, overlooked the fact that in the theorem it is the forces acting on each particle relatively to the assumed axes, and the corresponding motions that must be taken into account, and that in the case of a system of particles between which exist forces of mutual attraction, the stress between a given pair can only enter once into the value of $\frac{1}{2}\Sigma\text{R}r$.

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Bangor, September 1.

I THINK the fort will not surrender at Colonel Basevi's summons. We have

$$m\frac{d}{dt}\left(x\frac{dx}{dt}\right) = mx\frac{d^2x}{dt^2} + m\left(\frac{dx}{dt}\right)^2;$$

and if we put $x = u$ and $\frac{dx}{dt} = v$, this may be written

$$m\frac{d}{dt}(uv) = mu\frac{dv}{dt} + mv\frac{du}{dt}$$

and

$$(uv)_t - (uv)_0 = \int_0^t u\frac{dv}{dt}dt + \int_0^t v\frac{du}{dt}dt = \int_0^t udv + \int_0^t vdu,$$

if you please so to write it. This corresponds to Colonel Basevi's equation, except that I have written v for his $\dot{x}$.

But now $m\int vdu$, or $m\int v\frac{du}{dt}dt$, *does* represent kinetic energy. And $-m\int_0^t udv$ or $-m\int_0 x\frac{d^2x}{dt^2}dt$ is the virial. The equation shows that if for a certain time t, the right-hand member, vanishes, then on the average of that time t, the two terms on the right are equal and opposite.

The form $\Sigma\text{R}r$ is a rather slippery one. If in the example which Colonel Basevi quotes from Lord Rayleigh, you put $\text{X}x + \text{Y}y$ for $\text{R}r$, it comes out easily. For we may take for origin the centre of the circle of radius ρ. Then

$$\text{X} = \frac{x}{\rho}f \quad \text{Y} = \frac{y}{\rho}f \text{ and } \text{X}x + \text{Y}y = f\rho.$$

And therefore

$$\Sigma\tfrac{1}{2}mv^2 = \Sigma\tfrac{1}{2}mf\rho,$$

or

$$f = \frac{v^2}{\rho}.$$

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