ON THE INVERSION OF A DEFINITE INTEGRAL*

By H. BATEMAN.

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CONTENTS.

Introduction.

The present memoir is devoted to the discussion of a problem which was considered at some length in a paper which appeared in these *Proceedings* some months ago.

The problem in question is to determine a function $\phi(t)$ so that, for a given range of values of s, we may have

$$f(s) = \int_L \kappa(s, t)\, \phi(t)\, dt$$

where it is supposed that the path of integration and the functions $f(s)$ and $\kappa(s, t)$ are known.

Equations of this type occur in potential problems in which the value of the potential function is given at points on a curve or surface. On this account alone they are worthy of close attention; but there is another object which a systematic theory of these equations would accomplish—it would group together the innumerable isolated results in the subject of definite integrals, thus giving us a means of classifying them, besides indicating the fundamental principles upon which the formulæ depend.

* This paper is an elaboration of one which was presented to the Society on May 9th, and afterwards withdrawn, as the subsequent researches of the author required that it should be remodelled.

With this object in view, I have attempted in § 2 to classify the integral equations themselves according to what may be called the fundamental formulæ on which the solution depends. A practical method of finding the inversion formula depending on the use of a differential equation is then suggested, and is employed to obtain the solutions of a number of particular equations. I am unable, however, to give a rigorous investigation of the theory of this method.

A second method, which also depends upon the use of linear differential equations, is indicated in § 8; it seems to be full of possibilities, and throws some light upon the theory of a certain type of partial differential equation.

§ 10 consists of an extension of the general method which was given by the author,* and an attempt is made to obtain an existence theorem.

1. *The Integral Equation considered as the Limit of a System of Linear Equations.*

The integral equation

$$f(s) = \int_a^b \kappa(s, t)\, \phi(t)\, dt \tag{1}$$

is in many respects analogous to the system of linear equations

$$f_s = \sum_{t=1}^{n} \kappa_{s,t}\, \phi_t \quad (s = 1, 2, \ldots, m), \tag{2}$$

and can, in fact, be obtained from it by a limiting process in which m and n are finally made infinite. The important point is that in this process of passing to the limit many of the properties of the system of linear equations are preserved.

Now the properties of a system of linear equations depend upon the relation between the number of equations and the number of unknown quantities.

(1) If $m > n$, we shall be able to construct a number of relations of the form

$$\sum_{s=1}^{m} a_s \kappa_{st} = 0,$$

which are satisfied for all values of t, and then the equations (2) will be inconsistent unless the quantities f satisfy linear relations of the same type

$$\sum_{s=1}^{m} a_s f_s = 0.$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 4, Part 2 (1906).

In passing to the limit, a relation of this type may take several forms, such as

$$\int_c^d a(s) f(s)\, ds = 0,$$

$$\left[\Sigma A_r \frac{d^r f}{ds_r}\right]_{s=s_0} = 0,$$

$$\Sigma B_r f(s_r) = 0;$$

or it may be expressible as a linear combination of such forms.

In any case the corresponding property possessed by the integral equation is that in general a necessary condition that a function $f(s)$ may be expressed in the form (1) is that it should satisfy all the linear relations that are satisfied by $\kappa(s, t)$ for all values of t. This condition is only necessary so long as the function $\phi(t)$ is restricted to be such that the operation of forming the linear relation may be interchanged with that of integration in equation (1). We cannot say at present whether it is sufficient or not, because the conditions which are to be laid on $\phi(t)$ have not been determined, and it has been found that in certain cases a function represented by equation (1) satisfies linear conditions (such as continuity*) which the function $\kappa(s, t)$ does not.

(2) If $m = n$, there is in general a unique set of quantities ϕ_t and these are determined by a set of linear equations of the form

$$\phi_t = \sum_{s=1}^{m} \bar{\kappa}_{ts} f_s \quad (t = 1, \ldots, n), \tag{3}$$

provided no relation of the form

$$\Sigma a_s \kappa_{st} = 0$$

is satisfied for all values of t.

(3) If $m < n$, there are an infinite number of sets of solutions, but we may single out one set by imposing $n-m$ linear conditions on the quantities ϕ_t.

We conclude from this, that, in general, the solution of equation (1) will not be unique, but that it may be rendered unique by imposing a number of linear conditions upon the function $\phi(t)$. The whole question depends, of course, upon the range of values for which $f(s)$ is given:†

* The condition of continuity must be regarded as being equivalent to a number of linear conditions.

† In some cases $f(s)$ may only be given for an enumerable set of values of s; but, by properly choosing the conditions to be satisfied by ϕ, we can make the solution unique, as, for instance, in Stieltjes' problem of the moments. (*Annales de la Faculté des Sciences de Toulouse*, t. VIII., 1894.)

for one range of values the function $\phi(t)$ may be uniquely determinate, while for a smaller range this will, in general, not be the case, and it may be necessary to restrict $\phi(t)$ to be zero for a certain portion of the range of integration in order to render the solution unique under the new conditions.

Supposing, then, that conditions have been chosen which will make the solution of (1) unique, we shall expect, in analogy to (3), to obtain an expression for $\phi(t)$ of the form

$$\phi(t) = L_t f(t) \tag{4}$$

where L_t denotes a linear operator which may be built up of terms of the types

$$p(t)\frac{d^r f}{dt^r}, \quad f(t+a), \quad \int_c^d F(t, s) f(s)\, ds.$$

It frequently happens that the inversion formula takes the simple form

$$\phi(t) = \int_L \bar{\kappa}(t, s) f(s)\, ds; \tag{5}$$

but it is to be remarked that this integral is in general of a different character from that which occurs in equation (1).

This may be seen by considering a particular example in which the limits a and b are finite and the function $\kappa(s, t)$ remains finite and continuous within the range. By choosing a particular function $\phi(t)$ which experiences a sudden change of value at a point x within the range of integration, we obtain a continuous function $f(s)$. The integral in equation (5), on the other hand, must represent a discontinuous function, and this can only be the case if the integral is an improper one.

The exceptional character of the integral in equation (5) may either be due to the limits being infinite or to a discontinuity in the function $\bar{\kappa}(t, s)$.

When the solution of the equation

$$f(s) = \int_a^b \kappa(s, t)\, \phi(t)\, dt$$

is given by a formula of the type

$$\phi(t) = \int_L \bar{\kappa}(t, s) f(s)\, ds,$$

in which the path L does not depend upon the value of t, we can, in general, assert that the solution of the associated equation

$$\chi(t) = \int_L \kappa(s, t)\, \psi(s)\, ds$$

is, under suitable conditions, given by the formula

$$\psi(s) = \int_a^b \bar{\kappa}(t, s)\,\chi(t)\,dt.$$

For we have

$$\int_L f(s)\,\psi(s)\,ds = \int_L \int_a^b \kappa(s, t)\,\psi(s)\,\phi(t)\,ds\,dt$$

$$= \int_a^b \chi(t)\,\phi(t)\,dt = \int_a^b \int_L \bar{\kappa}(t, s)\,\chi(t) f(s)\,dt\,ds$$

and $f(s)$ is arbitrary; hence we must have

$$\psi(s) = \int_a^b \bar{\kappa}(t, s)\,\chi(t)\,dt.$$

2. *The Classification of Integral Equations of the First Kind.*

Integral equations of the first kind may be classified according to the principles on which their inversion formulæ depend.*

The method which we adopt to ascertain the nature of an equation is as follows:—

Let $\kappa(s, r)$ be substituted for $f(s)$ in the inversion formula with the view of obtaining, if possible, a function $\phi(t) = h(t, r)$ which will give a representation of $\kappa(s, r)$ in the form

$$\kappa(s, r) = \int_a^b \kappa(s, t)\,h(t, r)\,dt \tag{6}$$

for values of r lying between a and b.

Now, although the function $\kappa(s, r)$ satisfies all the linear relations that are satisfied by $\kappa(s, t)$, it can, in general, only be expressed in this form if the integral is an improper one. For we know that, in the case of a proper integral, the equation †

$$\psi(r) = \int_a^b \psi(t)\,h(t, r)\,dt$$

is only satisfied for a finite number of functions ψ if at all; whereas, by giving different values to s in equation (6), we shall obtain an infinite number of linearly independent functions ψ, unless it happens that the function $\kappa(s, t)$ can be expressed as a finite sum of the form

$$\kappa(s, t) = \sum_{n=1}^{m} \phi_n(s)\,\psi_n(t).$$

When $\kappa(s, r)$ is substituted for $f(s)$ in the inversion formula it frequently happens that the result takes the form of a divergent definite

* An integral equation is regarded here as consisting of the equation itself *plus* a number of conditions which will render the solution unique.

† Fredholm, *Acta Math.*, Vol. XXVII. (1903).

integral or series, in this case we associate a value with it by applying one of the known methods of summation such as Borel's exponential method.*

If this method is applied to all the equations for which the formulæ for inversion are known, it will be found that the function $h(t, r)$ takes one of a limited number of distinct forms; and so the equations may be classified accordingly.

In equations of type 1 the function $h(t, r)$ is zero, except in the vicinity of $t = r$. As an example of an equation of this type, we may take Hilbert's equation

$$f(s) = \int_0^1 \kappa(s, t)\, \phi(t)\, dt$$

where $\kappa(s, t) = s(1-t),\ s \leqslant t; \ = t(1-s),\ s \geqslant t,$

the solution of which is given by

$$\phi(t) = -\frac{d^2}{dt^2} f(t).$$

Now
$$\frac{d}{dt}\{\kappa(t, r)\} = 1-r \quad \text{for } t \leqslant r$$
$$= -r \quad \text{for } t \geqslant r,$$

and so is discontinuous at the point $t = r$; but for any other point we may differentiate again, obtaining

$$h(t, r) = -\frac{d^2}{dt^2}\{\kappa(t, r)\} = 0.†$$

The equations of the first type form a very large class; other types depend on the following forms of formula (6):—

(2)
$$\kappa(s, r) = \frac{1}{2\pi i}\int_C \frac{\kappa(s, t)}{t-r}\, dt,$$

(3a)
$$\kappa(s, r) = \frac{1}{\pi}\int_{-\infty}^{\infty} \kappa(s, t)\, \frac{\sin(t-r)}{t-r}\, dt.$$

(4a)
$$\kappa(s, r) = \int_0^\infty \kappa(s, t)\, h(t, r)\, dt$$

where
$$h(t, r) = t\int_0^1 J_0(tx)\, J_0(rx)\, x\, dx,$$

the numbers being chosen as above because the types (1), (2), (3), and (4)

* This method is applied to definite integrals in a paper by G. H. Hardy, *Quarterly Journal*, Vol. xxxv., p. 22.

† [*Note added December 11th.*—When, however, the solution of the integral equation is not unique the function $h(t, r)$ may differ from one of the forms given below by a multiple of a function $\phi(t)$ which makes the definite integral zero.]

are connected in some way with the differential equations

$$\frac{d^2V}{dx^2} = 0, \quad \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0, \quad \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + V = 0,$$

and
$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + \frac{d^2V}{dw^2} + V = 0$$

respectively.

It is exceedingly probable that the equations of other types may be made to depend eventually upon what may be called the fundamental formula of type 1, and so we proceed to consider this formula in detail.

3. *The Fundamental Formula for Integral Equations of the First Type.*

The equation which I have referred to as the fundamental formula for integral equations of the first type is strictly not a mathematical equation at all, and cannot be used in a rigorous demonstration until it has been rendered more precise by a determination of the class of functions to which it is applicable and of the necessary and sufficient conditions to be satisfied in order that operations such as differentiation and integration under the integral sign may be performed upon it.

These the present writer does not feel competent to give, but some justification of our employing it to attain the ends we have in view may be derived from the following considerations:—

Let $f(t)$ be a function which possesses a continuous derivative for all values of t within the range (a, b), and let $F(x, t)$ be a function which is defined as follows:—

$$\left.\begin{aligned} F(x, t) &= -1 \quad (t < x) \\ &= +1 \quad (t > x) \end{aligned}\right\}; \tag{7}$$

then
$$\int_a^b F(x, t)\, f'(t)\, dt = f(b) + f(a) - 2f(x).$$

Now let us suppose, for the moment, that we can integrate this equation by parts; then we shall have, by the ordinary rule,

$$\int_a^b F(x, t)\, f'(t)\, dt = f(b) + f(a) - \int_a^b \frac{\partial}{\partial t} F(x, t)\, f(t)\, dt. \tag{8}$$

Accordingly, if the improper integral

$$\int_a^b \frac{\partial}{\partial t} F(x, t)\, f(t)\, dt$$

be defined by this equation, we have

$$f(x) = \tfrac{1}{2} \int_a^b \frac{\partial}{\partial t} F(x, t)\, f(t)\, dt, \tag{9}$$

and this is the fundamental formula to which I have referred. It is clear that the limits a and b can be made infinite and the function $F(x, t)$ replaced by $F(x, t)+\psi(x)$ without altering the argument; but it is not clear whether this formula can be considered to hold when $f(x)$ does not possess a continuous derivative.

A geometrical interpretation of the formula may be obtained by writing $F(x, t) = \frac{2\theta}{\pi}$ where θ is the angle which the radius vector, from the point x to the point t, makes with the line through x perpendicular to the axis. If the point x is excluded from the range of integration by a small semi-circle of radius ϵ, the formula may be written

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\partial\theta}{\partial t} f(t)\,dt = \frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} f\left[x+\epsilon e^{i(\theta-\frac{1}{2}\pi)}\right] d\theta,$$

and this is easily seen to be true if $f(x+h)$ can be expanded in a Taylor's series.

Now let us consider an integral equation of the first type

$$f(s) = \int_a^b \kappa(s, t)\,\phi(t)\,dt,$$

for which the inversion formula is

$$\phi(t) = \int_L \bar{\kappa}(t, s) f(s)\,ds.$$

Then, by hypothesis, the function

$$h(t, r) = \tfrac{1}{2}\frac{\partial}{\partial t} F(r, t)$$

is obtained when $\kappa(s, r)$ is substituted for $f(s)$, *i.e.*,

$$\tfrac{1}{2}\frac{\partial}{\partial t} F(r, t) = \int_L \bar{\kappa}(t, s)\,\kappa(s, r)\,ds\ ?, \tag{10}$$

the sign $=?$ being used to denote that the integral may be divergent, in which case $\frac{1}{2}\frac{\partial}{\partial t} F(r, t)$ is the value associated with the divergent integral.

Accordingly, if we wish to find the inversion formula, we must look for a relation of the form (10) : *i.e., if the path L is known, we must solve the integral equation*

$$\tfrac{1}{2}\frac{\partial}{\partial t} F(r, t) = \int_L \chi(s)\,\kappa(s, r)\,ds,$$

for $\chi(s)$.

I shall indicate later a method depending on the theory of linear differential equations by which relations of the form (10) may be constructed. For the present I shall content myself by showing that the relation (9) may be considered to be connected with many of the well known representations of a function.

In the case of Fourier's double integral

$$f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty \cos s(x-t)\, f(t)\, ds\, dt$$

the equation (10) has the form

$$\tfrac{1}{2}\frac{\partial}{\partial t}F(x,\, t) = \frac{1}{\pi}\int_0^\infty \cos s(x-t)\, ds\ ?,$$

the representation for $F(x, t)$ being the exact equation

$$F(x,\, t) = -\frac{2}{\pi}\int_0^\infty \frac{\sin s(x-t)}{s}\, ds$$

and the limits a and b being $\pm\infty$ respectively.

The representation of a function as a series of Legendre polynomials may be considered to be based on the exact equation *

$$F(x,\, t) = P_1(t)\, P_0(x) + \sum_1^\infty \left[P_{n+1}(t) - P_{n-1}(t)\right] P_n(x)\, ;$$

the relation corresponding to (10) is

$$\tfrac{1}{2}\frac{\partial}{\partial t}F(x,\, t) = \sum_1^\infty \frac{2n+1}{2}\, P_n(x)\, P_n(t)\ ?\, ;$$

and the limits are $a = +1,\ b = -1$.

4. *Study of a particular Equation.*

We shall now consider the integral equation

$$f(s) = \int_{-1}^{+1} \frac{\phi(t)\, dt}{(1-2ts+s^2)^\nu} \quad (\nu > 0). \tag{11}$$

If $|s| < 1$ and $f(s)$ satisfies certain conditions, it will be shown that the inversion formula is

$$\phi(s) = \frac{(1-s^2)^{\nu-\frac{1}{2}}}{\pi}\int_0^\pi \left[\nu f(x) + x f'(x)\right] \sin^{2\nu-1} a\, da \tag{12}$$

where $$x = s + i\sqrt{1-s^2}\cos a.$$

* The necessary and sufficient conditions that a function satisfying the conditions laid down in Dirichlet's proof of Fourier's theorem may be expanded in a series of Legendre polynomials are given by Darboux, "Approximation des Fonctions de grands Nombres," *Liouville's Journal* (3e série), t. IV., p. 393 (1878). The function $F(x, t)$ satisfies these conditions, and so may be expanded in the above form.

Substituting $f(x) = \dfrac{1}{(1-2xr+x^2)^\nu}$ according to the rule, we obtain

$$\phi(t) = h(t, r) = \frac{(1-t^2)^{\nu-\frac{1}{2}}}{\pi} \int_0^\pi \frac{\nu(1-x^2)}{(1-2xr+x^2)^{\nu+1}} \sin^{2\nu-1} a\, da$$

where
$$x = t+i\sqrt{1-t^2}\cos a$$

or
$$h(t, r) = \frac{\nu}{i\pi} \int_{t-i\sqrt{(1-t^2)}}^{t+i\sqrt{(1-t^2)}} \frac{(1-x^2)(1-2tx+x^2)^{\nu-1}}{(1-2tx+x^2)^{\nu+1}}\, dx$$

$$= \frac{1}{2i\pi} \int_{t-i\sqrt{(1-t^2)}}^{t+i\sqrt{(1-t^2)}} \frac{1}{r-t} \frac{d}{dx} \left\{\left(\frac{1-2tx+x^2}{1-2rx+x^2}\right)^\nu\right\} dx$$

$$= 0, \quad \text{unless } r = t.$$

The integral equation is thus seen to be of the first type.

We can also show that $h(t, r)$ is the derivative of a discontinuous function of the type required, for

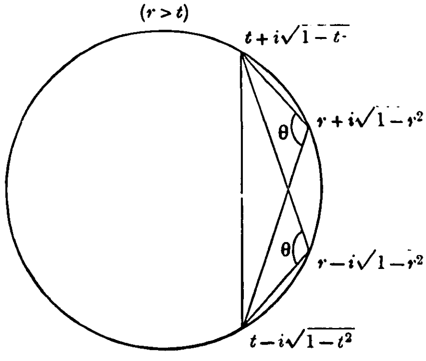
$$\int h(t, r)\, dr = \frac{1}{2i\pi} \int_{t-i\sqrt{(1-t^2)}}^{t+i\sqrt{(1-t^2)}} \frac{1-x^2}{x} \frac{(1-2tx+x^2)^{\nu-1}}{(1-2rx+x^2)^\nu}\, dx,$$

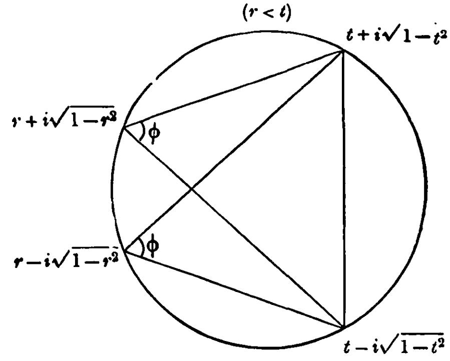
and this integral is discontinuous at $t = r$. To determine the change of value, we write it in the form

$$I = \frac{1}{2\pi i} \int \left\{ \frac{2(r-x)}{1-2rx+x^2} + \frac{1}{x} + \frac{A(t-r)}{(1-2rx+x^2)^2} + \text{higher powers of } (t-r) \right\} dx$$

$$= -\frac{1}{2\pi i} \Big[\log(x-r-i\sqrt{1-r^2})(x-r+i\sqrt{1-r^2})\Big]_{t-i\sqrt{(1-t^2)}}^{t+i\sqrt{(1-t^2)}}.$$

Let the path of integration be the straight line joining the two points $t \pm i\sqrt{1-t^2}$, then we have the two figures





The only part of I which is discontinuous at $t = r$ is the first term, and this is equal to $+\theta/\pi$ if $r > t$ and $-\phi/\pi$ if $r < t$. Hence, since $\theta + \phi = \pi$, I will suddenly increase by $+1$ as r increases through the value t; accordingly, if we write

$$F(t, r) = 2\int h(t, r)\, dr,$$

the formula (9) will give

$$\phi(t) = \int_{-1}^{+1} h(t, r)\, \phi(r)\, dr$$

$$= \frac{\nu}{i\pi}\int_{-1}^{+1}\int_{t-i\sqrt{(1-t^2)}}^{t+i\sqrt{(1-t^2)}} \frac{(1-x^2)\,\phi(r)}{(1-2rx+x^2)^{\nu+1}}\,(1-2tx+x^2)^{\nu-1}\, dx\, dr,$$

and, if we assume that the order of integration can be changed, we may write this :

$$\phi(t) = \frac{\nu}{i\pi}\int_{t-i\sqrt{(1-t^2)}}^{t+i\sqrt{(1-t^2)}} (1-2tx+x^2)^{\nu-1}\,\chi(x)\, dx, \qquad (12)'$$

$$\chi(x) = \int_{-1}^{+1} \frac{(1-x^2)\,\phi(r)\, dr}{(1-2rx+x^2)^{\nu+1}} = f(x) + \frac{x}{\nu} f'(x)$$

where
$$f(x) = \int_{-1}^{+1} \frac{\phi(r)\, dr}{(1-2rx+x^2)^{\nu}}.$$

The formula (12)′ is easily seen to be equivalent to (12) when the substitution

$$x = t + i\sqrt{1-t^2}\cos\alpha$$

is made; accordingly the inversion formula can actually be constructed by means of equation (9), and so the integral equation is proved to be of the first type.

We may find a *sufficient* set of conditions to be satisfied by $f(s)$ in order that it may be represented in the form (11) by using Dini's method of expansion, that is, by representing the function $\kappa(s, t)$ in the form

$$\kappa(s, t) = \Sigma a_n \psi_n(s)\, \theta_n(t).$$

In the present case we use the expansion

$$(1-2st+s^2)^{-\nu} = \sum_{n=0}^{\infty} s^n C_n^{\nu}(t) \quad (|s| < 1). \qquad (13)$$

The properties of the polynomials $C_n^{\nu}(t)$ are fairly well known, but it will be convenient to furnish proofs of the different relations that will be required, as I do not know exactly where some of them are to be found.

LEMMA I.—The functions $C_n^\nu(t)$ satisfy the integral relations

$$\int_{-1}^{+1}(1-t^2)^{\nu-\frac{1}{2}}C_n^\nu(t)C_m^\nu(t)\,dt = 0 \qquad (m\neq n)$$

$$= \frac{\pi}{2^{2\nu-1}(\nu+n)}\,\frac{\Gamma(n+2\nu)}{\Gamma^2(\nu)\,n!} \qquad (m=n). \quad (14)$$

To prove these, we observe first of all that the function

$$V = (1-2ts-s^2)^{-\nu}$$

satisfies the partial differential equation

$$(t^2-1)\frac{\partial^2 V}{\partial t^2}+(2\nu+1)t\frac{\partial V}{\partial t} = s^{1-2\nu}\frac{\partial}{\partial s}\left\{s^{2\nu+1}\frac{\partial V}{\partial s}\right\}.$$

Replacing V by the series and equating coefficients of s^n, we find that $C_n^\nu(t)$ satisfies the differential equation

$$(t^2-1)\frac{d^2y}{dt^2}+(2\nu+1)t\frac{dy}{dt}-n(n+2\nu)y = 0.$$

Writing the equations satisfied by the two functions $C_n^\nu(t)$, $C_m^\nu(t)$ in the form

$$\frac{d}{dt}\left[(t^2-1)^{\nu+\frac{1}{2}}\frac{d}{dt}C_n^\nu(t)\right] = n(n+2\nu)(t^2-1)^{\nu-\frac{1}{2}}C_n^\nu(t),$$

$$\frac{d}{dt}\left[(t^2-1)^{\nu+\frac{1}{2}}\frac{d}{dt}C_m^\nu(t)\right] = m(m+2\nu)(t^2-1)^{\nu-\frac{1}{2}}C_m^\nu(t),$$

multiplying the first by $C_m^\nu(t)$, the second by $C_n^\nu(t)$, and subtracting, we obtain

$$\frac{d}{dt}\left[(t^2-1)^{\nu+\frac{1}{2}}\left\{C_m^\nu(t)\frac{d}{dt}C_n^\nu(t)-C_n^\nu(t)\frac{d}{dt}C_m^\nu(t)\right\}\right]$$

$$= (n-m)(n+m+2\nu)(t^2-1)^{\nu-\frac{1}{2}}C_n^\nu(t)\,.\,C_m^\nu(t).$$

Now $C_m^\nu(t)$ is a polynomial in t, and so remains finite when $t=\pm 1$; accordingly the quantity inside the square bracket vanishes for these values of t, and so, if $n\neq m$, we have

$$\int_{-1}^{+1}(1-t^2)^{\nu-\frac{1}{2}}C_n^\nu(t)\,C_m^\nu(t)\,dt = 0.$$

To find the value of the integral when $m=n$, we require the recurrence formula satisfied by $C_n^\nu(t)$, viz.,

$$(n+1)C_{n+1}-2(n+\nu)tC_n+(n+2\nu-1)C_{n-1} = 0. \quad (15)$$

To prove this, we differentiate equation (13) with regard to s, obtaining

$$2\nu(t-s)(1-2ts+s^2)^{-\nu-1} = \Sigma n s^{n-1}C_n^\nu(t),$$

and compare with the former expansion, whence we get the relation

$$(1-2st+s^2)\Sigma n s^{n-1} C_n^\nu(t) = 2\nu(t-s)\,\Sigma s^n C_n^\nu(t),$$

from which the recurrence formula is obtained by equating coefficients of s^n. Multiplying (15) by $(1-t^2)^{\nu-\frac{1}{2}} C_{n+1}^\nu(t)\,dt$ and integrating, we obtain the relation

$$(n+1)\int_{-1}^{+1} (1-t^2)^{\nu-\frac{1}{2}}\{C_{n+1}^\nu(t)\}^2\,dt = 2(n+\nu)\int_{-1}^{+1} t(1-t^2)^{\nu-\frac{1}{2}} C_n^\nu(t)\, C_{n+1}^\nu(t)\,dt.$$

Similarly, multiplying by $(1-t^2)^{\nu-\frac{1}{2}} C_{n-1}^\nu(t)\,dt$ and integrating, we obtain

$$(n+2\nu-1)\int_{-1}^{+1} (1-t^2)^{\nu-\frac{1}{2}}\{C_{n-1}^\nu(t)\}^2\,dt$$
$$= 2(n+\nu)\int_{-1}^{+1} t(1-t^2)^{\nu-\frac{1}{2}} C_n^\nu(t)\, C_{n-1}^\nu(t)\,dt.$$

Changing n into $n+1$ and comparing with the last equation, we find that

$$I_{n+1} = \int_{-1}^{+1} (1-t^2)^{\nu-\frac{1}{2}}\{C_{n+1}^\nu(t)\}^2\,dt = \frac{(n+\nu)(n+2\nu)}{(n+\nu+1)(n+1)}\, I_n.$$

Now $C_0^\nu(t) = 1$, and, if we put $x = \dfrac{1-t}{2}$ in the integral

$$I_0 = \int_{-1}^{+1} (1-t^2)^{\nu-\frac{1}{2}}\,dt,$$

it becomes $$I_0 = 2^{2\nu}\int_0^1 x^{\nu-\frac{1}{2}}(1-x)^{\nu-\frac{1}{2}}\,dx = 2^{2\nu}\frac{\Gamma^2(\nu+\frac{1}{2})}{\Gamma(2\nu+1)}.$$

Hence the relation $$I_{n+1} = \frac{(n+\nu)(n+2\nu)}{(n+\nu+1)(n+1)}\, I_n$$

will give $$I_n = 2^{2\nu}\,\frac{\nu}{n+\nu}\,\frac{1}{n!}\,\frac{\Gamma^2(\nu+\frac{1}{2})\,\Gamma(n+2\nu)}{\Gamma(2\nu+1)\,\Gamma(2\nu)}.$$

This expression may be simplified by using the relation*

$$\Gamma(\nu)\,\Gamma(\nu+\tfrac{1}{2}) = 2^{1-2\nu}\sqrt{\pi}\,.\,\Gamma(2\nu),$$

and we finally obtain $$I_n = \frac{\pi}{2^{2\nu-1}(n+\nu)}\,\frac{\Gamma(n+2\nu)}{\Gamma^2(\nu)\,n!}.$$

The solution of the integral equation for $f(s) = s^n$ may now be obtained, for, if we multiply the expansion (13) by $(1-t^2)^{\nu-\frac{1}{2}} C_n^\nu(t)\,dt$ and integrate between -1 and $+1$, we obtain

$$\int_{-1}^{+1} \frac{(1-t^2)^{\nu-\frac{1}{2}} C_n^\nu(t)\,dt}{(1-2st+s^2)^\nu} = \frac{\pi s^n}{2^{2\nu-1}(n+\nu)}\,\frac{\Gamma(n+2\nu)}{\Gamma^2(\nu)\,n!};$$

* Whittaker's *Analysis*, p. 180.

so that the corresponding function $\phi_n(t)$ is given by

$$\phi_n(t) = \frac{2^{2\nu-1}}{\pi}(n+\nu)\frac{\Gamma^2(\nu)\,n!}{\Gamma(n+2\nu)}(1-t^2)^{\nu-\frac{1}{2}}C_n^\nu(t).$$

By giving n different values, we see that the solution for

$$f(s) = \Sigma a_n s^n$$

should be

$$\phi(t) = \Sigma a_n \phi_n(t).$$

Now, in order to establish the inversion formula, we must prove first of all that

$$C_n^\nu(t) = \frac{\Gamma(n+2\nu)}{2^{2\nu-1}\Gamma^2(\nu)\,n!}\int_0^\pi \{t+i\sqrt{1-t^2}\cos a\}^n \sin^{2\nu-1} a\,da. \qquad (16)$$

If we call the right-hand side F_n and substitute in the recurrence formula (15), we find that

$$(n+1)F_{n+1}-2(n+\nu)\,t\,F_n+(n+2\nu-1)F_{n-1}$$

$$= \frac{\Gamma(n+2\nu)\sqrt{t^2-1}}{2^{2\nu-1}\Gamma^2(\nu)\,n!}\left[\int_0^\pi (t+i\sqrt{1-t^2}\cos a)^n(2\nu\sin^{2\nu-1}a\cos a\,da) - \int_0^\pi n(t+i\sqrt{1-t^2}\cos a)^{n-1}i\sqrt{1-t^2}\sin^{2\nu+1}a\,da\right].$$

But, on integrating the first integral by parts, we find that the quantity under the square brackets is zero; hence F_n satisfies the same recurrence formula as $C_n^\nu(t)$. Also, when $n = 0$, we have

$$F_0 = \frac{\Gamma(2\nu)}{2^{2\nu-1}\Gamma^2(\nu)}\int_0^\pi \sin^{2\nu-1}a\,da = 1,$$

$$F_1 = \frac{\Gamma(2\nu+1)}{2^{2\nu-1}\Gamma^2(\nu)}\int_0^\pi (t+i\sqrt{1-t^2}\cos a)\sin^{2\nu-1}a\,da = 2\nu t,$$

and so the first two values of F_n coincide with those of $C_n^\nu(t)$; therefore F_n is equal to $C_n^\nu(t)$ for all positive integral values of n.

We are now in a position to prove the following theorem :—

THEOREM.—*If $f(s)$ is a function which can be expanded in a power series*

$$f(s) = \Sigma a_n s^n,$$

which converges within the unit circle in such a way that the series

$$\Sigma|(n+\nu)a_n|$$

is convergent, then $f(s)$ can be expressed by means of the definite integral

$$f(s) = \int_{-1}^{+1} \frac{\phi(t)\,dt}{(1-2st+s^2)^\nu} \quad (|s| < 1),$$

and the function ϕ is given by the formula

$$\phi(t) = \frac{(1-t^2)^{\nu-\frac{1}{2}}}{\pi} \int_0^\pi [\nu f(x) + x f'(x)] \sin^{2\nu-1} a\, da,$$

where $$x = t + i\sqrt{1-t^2} \cos a.$$

Since $$(1-2ts+s^2)^{-\nu} = (1-se^{i\phi})^{-\nu}(1-se^{-i\phi})^{-\nu},$$

we find, on expanding both factors and calculating the coefficient of s^n, that

$$C_n^\nu(t) = A_n \cos n\phi + A_{n-2} \cos(n-2)\phi + \dots$$

where the coefficients A_r are all positive; for all the coefficients in $(1-x)^{-\nu}$ are all positive.

The modulus of $C_n^\nu(t)$ will therefore certainly be less than its value when $\phi = 0$, *i.e.*, when $t = 1$, and then we have

$$C_n^\nu(1) = \frac{\Gamma(n+2\nu)}{\Gamma(2\nu)\, n!}.$$

Now we saw that for $f(s) = \sum_0^\infty a_n s^n$ we had the formal solution

$$\phi(t) = \sum_0^\infty \frac{2^{2\nu-1}}{\pi} (n+\nu) \frac{\Gamma^2(\nu)\, n!}{\Gamma(n+2\nu)} (1-t^2)^{\nu-\frac{1}{2}} a_n C_n^\nu(t),$$

and, if we take out the factor $(1-t^2)^{\nu-\frac{1}{2}}$, we shall have a series which is absolutely and uniformly convergent for $|C_n^\nu(t)| \leqslant \dfrac{\Gamma(n+2\nu)}{\Gamma(2\nu)\, n!}$, and by hypothesis the series $\Sigma |(n+\nu) a_n|$ is convergent.

To show that when this series is substituted for $\phi(t)$ in the integral it can be integrated term by term, we employ the rule given by G. H. Hardy,* viz.: If $\phi = \Sigma \phi(t)$ is uniformly convergent throughout $(a, A-\epsilon)$, however small be the positive quantity ϵ, $f(t)$ is continuous throughout (a, A), and $\int_a^A \bar{\phi}(t)$ is convergent where

$$\bar{\phi}(x) = \Sigma |\phi_n(t)|,$$

then $$\int_a^A \phi f(t) = \Sigma \int_a^A \phi_n f\, dt.$$

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* *Mess. of Math.*, Vol. xxxv., No. 8, p. 126.

Taking $f = \frac{1}{(1-2st+s^2)^\nu}$, it is easily seen that, if $|s| < 1$, the condition is satisfied; hence, on integration, we shall obtain the series for f, and the first part of the theorem is proved.

Next, to prove that the function ϕ is actually given by the inversion formula, we again apply Hardy's rule; observing that

$$\nu f(x) + x f'(x) = \Sigma (n+\nu) a_n x^n,$$

$$|x| = \sqrt{\cos^2 a + t^2 \sin^2 a},$$

we see that, if $$\bar{\phi} = \Sigma |(n+\nu) a_n x^n \sin^{2\nu-1} a|,$$

then $\int_0^\pi \bar{\phi}\, da$ is convergent, and so the integration term by term may be effected.

The formula (12) is thus proved for any function which satisfies the conditions laid down; these conditions are sufficient, but not necessary. It is important to notice that they are satisfied in the case of a function $f(s)$ which can be expanded in a power series whose radius of convergence is greater than unity.

The solution of the integral equation for the case in which $|s| > 1$ may be deduced from the inversion formula by writing $s = \frac{1}{s'}$.

5. *Applications of the preceding Formula.*

If in equations (11) and (12) we write $\nu = 1$, the results may be expressed in a simpler form; for we have

$$\phi(t) = \frac{i}{\pi} \Big[x f(x) \Big]_0^\pi$$

$$= \frac{1}{i\pi} \Big[(t+i\sqrt{1-t^2}) f(t+i\sqrt{1-t^2}) - (t-i\sqrt{1-t^2}) f(t-i\sqrt{1-t^2}) \Big], \tag{17}$$

$$f(s) = \int_{-1}^{+1} \frac{\phi(t)\, dt}{1-2ts+s^2}. \tag{18}$$

Writing $$s = \mu + i\sqrt{1-\mu^2},$$

we have $$s f(s) = \psi(\mu) = \tfrac{1}{2} \int_{-1}^{+1} \frac{\phi(t) dt}{\mu - t}. \tag{19}$$

The function $\psi(\mu)$ is, in general, a many-valued function of μ, and there are two values of s corresponding to each value of μ. The inversion formula may be written*

$$\phi(t) = \frac{1}{i\pi} \operatorname*{Lt}_{\epsilon=0} [\psi(\mu - i\epsilon) - \psi(\mu + i\epsilon)], \tag{20}$$

thus allowing for the multiformity of the function ψ.

In particular, if $\phi(t) = P_n(t)$, we have $\psi(\mu) = Q_n(\mu)$, and the formula

$$Q_n(\mu) = \tfrac{1}{2} P_n(\mu) \log\left(\frac{1+\mu}{1-\mu}\right) + R_n(\mu) \tag{21}$$

where $R_n(\mu)$ is single valued shows at once once how formula (20) will give the correct value of ϕ.

Again, if $f(s) = \dfrac{\sin ns}{s}$, we find that $\phi(t) = \dfrac{2}{\pi} \cos nt \sinh n\sqrt{1-t^2}$; and so we have the formula

$$\frac{\sin ns}{s} = \frac{2}{\pi} \int_{-1}^{+1} \frac{\sinh n\sqrt{1-t^2} \cos nt}{1-2ts+s^2}\, dt, \tag{22}$$

which gives for $s = 0$

$$\int_{-1}^{+1} \sinh(n\sqrt{1-t^2}) \cos nt\, dt = \frac{n\pi}{2}.$$

Next let us consider the equation associated with equation (18), viz.,

$$f(s) = \int_{-1}^{+1} \frac{\phi(t)\, dt}{1-2st+t^2}. \tag{23}$$

To obtain a solution of this, we determine a function $h(r, s)$ such that

$$\frac{1}{1-2tr+r^2} = \int_{-1}^{+1} \frac{h(r, s)\, ds}{1-2st+t^2} \quad (|r| < 1);$$

then we shall have

$$\int_{-1}^{+1} \frac{\phi(t)\, dt}{1-2tr+r^2} = \int_{-1}^{+1} \int_{-1}^{+1} \frac{h(r, s)\, \phi(t)\, ds\, dt}{1-2st+t^2}$$

$$= \int_{-1}^{+1} h(r, s) f(s)\, ds = \psi(r) \quad \text{(say)},$$

and so the function $\phi(t)$ will be determined by solving the equation†

$$\psi(r) = \int_{-1}^{+1} \frac{\phi(t)\, dt}{1-2tr+r^2}.$$

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* The sign to be chosen is obtained by considering a particular function. This formula is similar to the one given by Stieltjes for the integral $F(z) = \int_0^\infty \frac{f(u)\, du}{z+u}$. See Borel's *Leçons sur les Séries divergentes*, p. 69.

† The method adopted here is capable of more general application.

Applying the inversion formula (17), we find

$$h(r, s) = \frac{1}{i\pi}\left[\frac{s+i\sqrt{1-s^2}}{1+r^2-2r(s+i\sqrt{1-s^2})} - \frac{s-i\sqrt{1-s^2}}{1+r^2-2r(s-i\sqrt{1-s^2})}\right]$$

$$= \frac{2}{\pi}\,\frac{\sqrt{1-s^2}\,(1+r^2)}{(1-2rs+r^2)^2+4r^2(1-s^2)}.$$

Hence the inversion formula for the equation

$$f(s) = \int_{-1}^{+1} \frac{\phi(t)\,dt}{1-2st+t^2} \quad (|s| < 1)$$

is
$$\phi(t) = \frac{1}{i\pi}[F(t+i\sqrt{1-t^2})-F(t-i\sqrt{1-t^2})]$$

where
$$F(r) = \frac{2}{\pi}\,r(1+r^2)\int_{-1}^{+1} \frac{f(s)\sqrt{1-s^2}\,ds}{1+6r^2+r^4-4rs(1+r^2)}.$$

If we substitute
$$f(s) = \frac{1}{1-2sz+z^2} \tag{24}$$

in this, we find
$$\phi(t) = 0 \quad (t \neq z);$$

accordingly, this integral equation is also of the first type.

When $\nu = \frac{1}{2}$, we may write equations (11) and (12) in the form

$$\chi(s) = f(s)+2sf'(s) = \int_{-1}^{+1} \frac{(1-s^2)\,\phi(t)\,dt}{(1-2ts+s^2)^{\frac{3}{2}}}, \tag{25}$$

$$\phi(t) = \frac{1}{2\pi}\int_0^{\pi} \chi(t+i\sqrt{1-t^2}\cos a)\,da, \tag{26}$$

which shows at once that they arise from a potential problem.

For we know that

$$V = \frac{1}{\pi}\int_0^{\pi} \chi(z+i\sqrt{x^2+y^2}\cos a)\,da \tag{27}$$

represents a potential function symmetrical about the axis, and, since (26) gives us the values of this function at points on the sphere $x^2+y^2+z^2 = 1$, the values at all other points, and in particular those on the axis, may be determined by Green's formula

$$V = \frac{1}{4\pi}\iint V\frac{\partial G}{\partial n}\,ds,$$

which is equivalent to (25), but formula (27) gives $V = \chi(z)$ on the axis.

6. *The Determination of the Inversion Formula by means of a Linear Differential Equation.*

We shall now consider the case in which the function $\kappa(s, t)$ in an integral equation of the first type satisfies a linear differential equation of the form

$$P_t(u)+\psi(s)\,Q_t(u) = 0, \tag{28}$$

where $P_t(u)$ and $Q_t(u)$ are used to denote the expressions

$$p_0(t)\,\frac{d^n u}{dt^n}+p_1(t)\,\frac{d^{n-1}u}{dt^{n-1}}+\dots$$

and
$$q_0(t)\,\frac{d^n u}{dt^n}+p_1(t)\,\frac{d^{n-1}u}{dt^{n-1}}+\dots$$

respectively.

The adjoint linear differential equation will clearly be of the same form, and may be written

$$\bar{P}_t(u)+\psi(s)\,\bar{Q}_t(u) = 0. \tag{29}$$

The quantity $\kappa(s, t)$, being a solution of the original equation, will be an integrating factor of the above expression.

Now let $u(r, t)$ be a solution of the equation

$$\bar{P}_t(u)+\psi(r)\,\bar{Q}_t(u) = 0\,;$$

then we have

$$[\psi(s)-\psi(r)]\,\kappa(s, t)\,\bar{Q}_t[u(r, t)] = \kappa(s, t)\,[\bar{P}_t(u)+\psi(s)\,\bar{Q}_t(u)] = \frac{dW}{dt}\,;$$

therefore*
$$\int_{t_0}^{t_1}\kappa(s, t)\,\bar{Q}_t[u(r, t)]\,\psi'(r)\,dt = \frac{\psi'(r)}{\psi(s)-\psi(r)}\,[W]_{t_0}^{t_1}. \tag{30}$$

Now, let the limits be chosen so that W takes the same value (usually zero) at both; then we shall have

$$\psi'(r)\int_{t_0}^{t_1}\kappa(s, t)\,\bar{Q}_t\,[u(r, t)]\,dt = 0 \quad (r \neq s). \tag{31}$$

If the limits cannot be thus chosen, we try to determine them so that the quantity W oscillates very rapidly in their vicinity; we can then write

$$[W]_{t_0}^{t_1} = 0\,?$$

and
$$\psi'(r)\int_{t_0}^{t_1}\kappa(s, t)\,\bar{Q}_t\,[u(r, t)]\,dt = 0\,? \quad (r \neq s). \tag{31$'$}$$

* The factor $\psi'(r)$ is inserted purposely, as it appears later in the inversion formula.

In either case we have a function of the type required for formula (9), provided that we can show that the function obtained by integrating with regard to r has a discontinuity at the point $r = s$. The above equation will then give us the inversion formula for one or both of the integral equations

$$\left.\begin{aligned} f(s) &= \int_{t_0}^{t_1} \kappa(s, t)\, \phi(t)\, dt \\ \phi(t) &= A \int_m^n \bar{Q}_t[u(r, t)]\, \psi'(r)\, f(r)\, dr \end{aligned}\right\}. \tag{32}$$

To illustrate the method we shall take the following examples :—

(1) If $\kappa(s, t) = J_m(st)$, the differential equation is

$$\frac{d}{dt}\left(t\frac{dv}{dt}\right) - \frac{m^2}{t}v + s^2 tv = 0,$$

which is self-adjoint. Hence, if $u = J_m(rt)$, we have

$$\frac{d}{dt}\left[t\left(v\frac{du}{dt} - u\frac{dv}{dt}\right)\right] = (s^2 - r^2)\, tuv.$$

Now the quantity within the square brackets is zero when $t = 0$ and oscillates very rapidly near $t = \infty$; for we have approximately

$$J_m(z) = \sqrt{\frac{2}{\pi z}} \cos\left\{z - (m + \tfrac{1}{2})\frac{\pi}{2}\right\};$$

hence we may write

$$h(s, r) = r\int_0^\infty J_m(st)\, J_m(rt)\, t\, dt = 0\,? \quad (r \neq s),$$

and we are led to consider the possibility of an equation of the form

$$f(s) = \int_0^\infty \int_0^\infty J_m(st)\, J_m(rt)\, tr\, f(r)\, dr\, dt. \tag{33}$$

The actual formula is well known and was given by Hankel* in 1875. We shall not stop to verify that $h(s, r)$ is the derivative of a discontinuous function.

(2) If $V = (1 - 2st + t^2)^{\nu - 1} = \kappa(s, t)$, we have the differential equation

$$(1 - 2st + t^2)\frac{dV}{dt} - 2(\nu - 1)(t - s)\, V = 0,$$

which is adjoint to

$$(1 + t^2)\frac{du}{dt} + 2\nu tu - 2s\left[t\frac{du}{dt} + \nu u\right] = 0.$$

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* *Math. Ann.*, Bd. VIII., p. 482.

Here
$$u(r, t) = (1-2rt+t^2)^{-\nu},$$
$$Q_t\,[u(r, t)] = t\frac{du}{dt}+\nu u = \frac{(1-t^2)\,\nu}{(1-2rt+t^2)^{\nu+1}},$$
$$2(r-s)\,Q_t[u(r, t)]\,\kappa(s, t) = \frac{d}{dt}\,[(1-2st+t^2)^{\nu}\,u(r, t)],$$

and the quantity under the square brackets vanishes if
$$t = s \pm i\sqrt{1-s^2}.$$

Hence we have
$$\int_{s-i\sqrt{(1-s^2)}}^{s+i\sqrt{(1-s^2)}} (1-2st+t^2)^{\nu-1}\,\frac{(1-t^2)}{(1-2rt+t^2)^{\nu+1}}\,dt = 0 \quad (r \neq s), \tag{34}$$

which leads us to the equation considered in § 4.

(3) Consider the equation
$$f(s) = \int_0^{\infty} J_0(st)\,K_0(st)\,t\,\phi(t)\,dt, \tag{35}$$

where $J_0(z)$ and $K_0(z)$, the two solutions of Bessel's equation of order zero, are so defined that their approximate values for large positive values of z are
$$\sqrt{\frac{2}{\pi z}}\cos\left(\frac{\pi}{4}-z\right) \quad \text{and} \quad \sqrt{\frac{2}{\pi z}}\sin\left(\frac{\pi}{4}-z\right)$$
respectively.

The function $\kappa(s, t) = tJ_0(st)\,K_0(s, t)$ satisfies the self-adjoint differential equation
$$\frac{d^3v}{dt^3}+\left(4s^2+\frac{1}{t^2}\right)\frac{dv}{dt}-\frac{v}{t^3} = 0,$$

of which another solution is $tJ_0^2(st)$.

If, then, $u(r, t) = tJ_0^2(rt)$, we have
$$4(s^2-r^2)\,v\frac{du}{dt} = \frac{d}{dt}\left[v\frac{d^2u}{dt^2}-\frac{dv}{dt}\,\frac{du}{dt}+u\left\{\frac{d^2v}{dt^2}+\left(4s^2+\frac{1}{t^2}\right)v\right\}\right],$$

v being written for $\kappa(s, t)$.

The quantity inside the square brackets is zero when $t = 0$, and oscillates very rapidly when $t = \infty$; hence, since in this case
$$Q_t(u) = \frac{du}{dt},$$

we have
$$r\int_0^{\infty} J_0(st)\,K_0(st)\,\frac{d}{dt}\,\{tJ_0^2(rt)\}\,t\,dt = 0\,? \quad (r \neq s),$$

and we are led to consider an inversion formula of the form

$$\phi(t) = A \int_0^\infty \frac{d}{dt} \{tJ_0^2(rt)\} rf(r)\, dr.$$

We may verify by means of the equations

$$J_0(2x) = \int_0^{\frac{1}{2}\pi} \frac{d}{dx} \{xJ_0^2(x \sin\theta)\} \sin\theta\, d\theta,$$

$$\frac{\pi}{2} J_0(zt) K_0(zt) = \int_0^\infty J_0(2zt \cosh u)\, du,$$

and Hankel's inversion formula given in (1) that this inversion formula is correct if $A = 2\pi$: but the formula was discovered originally from the differential equation.

The formula that we have just obtained, viz.,

$$\left.\begin{aligned} f(s) &= \int_0^\infty J_0(st) K_0(st)\, t\phi(t)\, dt \\ \phi(t) &= 2\pi \int_0^\infty \frac{d}{dt} \{tJ_0^2(rt)\} rf(r)\, dr \end{aligned}\right\}, \tag{36}$$

is analogous to Hankel's formula

$$\left.\begin{aligned} \psi(s) &= \int_0^\infty J_m(st)\, t\, \phi(t)\, dt \\ \phi(t) &= \int_0^\infty J_m(rt)\, r\psi(r)\, dr \end{aligned}\right\}, \tag{37}$$

in one respect: either of the functions $f(s)$ or $\phi(t)$ may be taken to be zero for values of s or t greater than a given quantity a. The solution of the integral equation with finite limits is then given by a definite integral with an infinite limit.

The method which we have just explained does not apply to all functions, because, in general, it is not possible to construct a linear differential equation of the required form which is satisfied by $\kappa(s, t)$ for all values of t. It can, however, be extended a little by the introduction of mixed linear equations in which definite integrals and finite differences can also occur.

For this purpose we require the equation adjoint to a mixed linear equation, and this may be obtained by writing down the adjoint expressions of its various constituents, the expression adjoint to

$$I(\psi) \equiv \int_a^b \kappa(s, t)\, \psi(t)\, dt - \kappa\lambda(s)\, \psi(s)$$

being understood to be

$$J(\chi) \equiv \int_a^b \kappa(s, t)\, \chi(s)\, ds - \kappa\lambda(t)\, \chi(t)\,;$$

for, if $\chi(s)$ is a function satisfying the equation $J(\chi) = 0$, we have

$$\int_a^b \chi(s)\, I(\psi)\, ds \equiv \int_a^b \chi(s)\, ds \int_a^b \kappa(s, t)\, \psi(t)\, dt - \int_a^b \lambda(s)\, \psi(s)\, \chi(s)\, ds \equiv 0.$$

The theory of mixed linear equations is rather difficult; so we shall not pursue these enquiries any further in the present paper; we may mention, however, that one of the chief difficulties we are faced with is that of determining whether our equations can be satisfied for a continuum of values of the arbitrary parameter or only for an enumerable set of values.

7. *Equations of Type* 3*a*.

The fundamental formula upon which the inversion formulæ of integral equations of this type depend is the following:—

$$f(r) = \frac{1}{\pi} \int_{-}^{\infty} \frac{\sin(r-t)}{r-t}\, f(t)\, dt. \tag{38}$$

It is at once evident that this formula is not satisfied by a perfectly arbitrary function: we must therefore find a convenient description of a class of functions to which the formula is applicable.

Now the function $\dfrac{\sin(r-t)}{r-t}$ can be written in the form

$$\tfrac{1}{2} \int_{-1}^{+1} e^{ir\mu - it}\; d\mu.$$

Accordingly, if the order of integration can be changed, we shall have

$$f(r) = \int_{-1}^{+1} e^{ir\mu}\, \psi(\mu)\, d\mu \tag{A}$$

where

$$\psi(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\mu} f(t)\, dt. \tag{B}$$

We shall therefore assume that $f(r)$ is a function defined by means of a definite integral of the form (A), and is such that the integral (B) is uniformly convergent for the range $\mu = (-1, 1)$; so that the order of integration in our double integral can be changed.

Now, in order to use this formula to solve integral equations, we must express the function $\frac{\sin(r-t)}{r-t}$ as a definite integral of the required type. The method to be adopted is similar to that used in § 6, and is best illustrated by means of an example.

The function $J_0(x-t)$ satisfies the differential equation

$$(x-t)\frac{d^2u}{dx^2}+\frac{du}{dx}+(x-t)u=0;$$

also $v=J_0(x-s)$ is an integrating factor of the equation with s written instead of t; therefore we have

$$\frac{d}{dx}\left[(x-s)\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)\right]=(t-s)v\left[\frac{d^2u}{dx^2}+u\right].$$

Now for large real values of x we have approximately

$$J_0(x-t)=\frac{2\cos\left(\frac{\pi}{4}-x+t\right)}{\sqrt{2\pi(x-t)}}$$

and

$$\frac{d}{dx}J_0(x-t)=\frac{2\sin\left(\frac{\pi}{4}-x+t\right)}{\sqrt{2\pi(x-t)}};$$

hence for $x=\infty$ the quantity under the square brackets becomes equal to

$$\frac{2}{\pi}\sin(t-s),$$

and for $x=s$ it is zero.

Further, we have

$$u+\frac{d^2u}{dx^2}=-\frac{1}{x-t}\frac{d}{dx}J_0(x-t)=\frac{J_1(x-t)}{x-t};$$

therefore the equation gives

$$\int_s^\infty J_0(x-s)\frac{J_1(x-t)}{x-t}\,dx=\frac{2}{\pi}\frac{\sin(s-t)}{s-t}. \tag{39}$$

Combining this with the fundamental formula, we obtain

$$\int_{-\infty}^\infty\int_s^\infty J_0(x-s)\frac{J_1(x-t)}{x-t}f(t)\,dx\,dt=2f(s), \tag{40}$$

an equation which can, in general, be written in the form

$$\left.\begin{aligned}\phi(x)&=\int_{-\infty}^\infty\frac{J_1(x-t)}{x-t}f(t)\,dt,\\ f(s)&=\tfrac{1}{2}\int_s^\infty J_0(s-x)\,\phi(x)\,dx\end{aligned}\right\}. \tag{41}$$

This relation can be verified directly if we assume for $\phi(x)$ an expression of the form

$$\phi(x) = \int_{-1}^{+1} e^{ix\mu} \psi(\mu) \, d\mu,$$

in which $\psi(\mu)$ is finite and continuous along the path of integration.

The method which we have just explained is only applicable to functions $\kappa(s, t)$ which oscillate near $s = \infty$ and which admit an asymptotic representation involving a circular function, so that the term $\sin(t-r)$ can appear. A similar method can be used for equations of type 2; for it is easy to see that, if $[W]$ is a constant instead of zero, we shall obtain a definite integral of the required form equal to $\frac{1}{t-r}$.

8. *The Problem of solving a Linear Differential Equation by means of a Definite Integral of a given Type.*

We shall now consider the case in which the function $f(s)$ is not explicitly given, but is to be derived from a given linear differential or integral equation

$$L_s(f) = 0. \tag{42}$$

We shall suppose that $f(s)$ satisfies a set of linear conditions which are sufficient to distinguish it from other solutions of equation (42), and that these conditions are included in or are the same as those to be satisfied by $f(s)$ in order that it may be represented in the form

$$f(s) = \int_a^b \kappa(s, t) \, \phi(t) \, dt. \tag{43}$$

The success of the method to be adopted depends upon the possibility of finding a relation of the form

$$L_s\{\kappa(s, t)\} = M_t\{h(s, t)\} \tag{44}$$

where M_t is an operator of the form

$$p_0(t) \frac{d^n}{dt^n} + p_1(t) \frac{d^{n-1}}{dt^{n-1}} + \ldots.$$

For, if $\phi(t)$ is an integrating factor of the expression $M_t(v)$, we shall have

$$L_s(f) = \int_a^b L_s\{\kappa(s, t)\} \, \phi(t) \, dt = \int_a^b M_t\{h(s, t)\} \, \phi(t) \, dt = \int_a^b dR, \tag{45}$$

provided the interchange of operations in the first line is permissible.

There will, in general, be more than one integrating factor of $M_t(v)$: we choose one so that R takes the same value (usually zero) at a and b or at two points α and β within the range (a, b). The function ϕ is thus a solution of the linear differential equation adjoint to $M_t(v) = 0$, and the above requirement may suffice to distinguish it from other solutions of this equation. If the points α and β (which must be independent of s) are different from a and b, we reduce the range of integration to (α, β) by defining the function $\phi(t)$ to be zero outside this range and equal to the given integrating factor inside.

We have seen that the solution of the integral equation is often given by a formula of the same type

$$\phi(t) = \int_c^d \bar{\kappa}(s, t) f(s) ds.$$

Now, according to the previous work, $\phi(t)$ should satisfy the equation adjoint to $M_t(v) = 0$, which we may write

$$\overline{M}_t(v) = 0.$$

Consequently, if the same kind of analysis applies for this integral equation as for the previous one, we shall expect to have an identical relation of the form

$$\overline{M}_t\{\bar{\kappa}(t, s)\} \equiv \overline{L}_s\{\bar{h}(t, s)\}. \tag{46}$$

We shall now simplify matters by assuming that the functions $\kappa(s, t)$ and $h(s, t)$ are the same and that the corresponding functions $\bar{\kappa}$ and $\bar{h}$ are also the same; in this way we may lose a certain amount of generality, but the analysis becomes more manageable.

The functions $\kappa(s, t)$ and $\bar{\kappa}(s, t)$ then satisfy the partial differential equations

$$L_s(u) = M_t(u)$$

and

$$\overline{L}_s(v) = \overline{M}_t(v) \tag{47}$$

respectively.

Now this is an important fact, because, if we know the solutions of the integral equation

$$f(s) = \int_a^b \kappa(s, t)\, \phi(t)\, dt$$

corresponding to a few particular forms of $f(s)$, we may be able to determine a number of partial differential equations of the form

$$L_s(u) = M_t(u)$$

which are satisfied by $\kappa(s, t)$. If, then, the integral equation is amenable to this treatment, the corresponding partial differential equations

$$\overline{L}_s(v) = \overline{M}_t(v)$$

will all be satisfied by the function $\bar{\kappa}(s, t)$; *and so the problem is reduced to that of finding the common solution of a number of partial differential equations.*

Thus, for example,* if the quantities a_{nm} are quite arbitrary and

$$L_s(u) \equiv \sum_m \sum_n a_{nm} s^n \frac{d^m u}{ds^m},$$

$$M_t(u) \equiv \sum_m \sum_n a_{nm} t^m \frac{d^n u}{dt^n},$$

the partial differential equation

$$L_s(u) = M_t(u)$$

is always satisfied by $\quad u = \kappa(s, t) = e^{st}.$

The adjoint expressions are

$$\bar{L}_s(v) = \Sigma\Sigma(-1)^m a_{nm} \frac{d^m}{ds^m} (s^n v),$$

$$\bar{M}_t(v) = \Sigma\Sigma(-1)^n a_{nm} \frac{d^n}{dt^n} (t^m v),$$

and, since $\quad (-1)^m \frac{d^m}{ds^m} \quad {}^n e^{-st}) \equiv (-1)^n \frac{d^n}{dt^n} (t^m e^{-st}),$

the corresponding partial differential equation

$$\bar{L}_s(v) = \bar{M}_t(v)$$

is always satisfied by $\quad v = \bar{\kappa}(s, t) = e^{-st}.$

This example corresponds to Pincherle's well known formula

$$f(s) = \frac{1}{2\pi i} \int_c e^{st} \phi(t)\, dt,$$

$$\phi(t) = \int_0^\infty e^{-st} f(s)\, ds.$$

Another fact which is worth noticing is the reciprocal nature of the pair of equations

$$L_s(u) = M_t(u),$$

$$\bar{L}_s(v) = \bar{M}_t(v).$$

If $\psi(t)$ is a solution of the equation $M_t(w) = 0$, we shall expect a function $\chi(s)$ for which

$$\psi(t) = \int_c^d \kappa(s, t)\, \chi(s)\, ds,$$

* This example is deduced from some work of Petzval's (*Integration der linearen Differentialgleichungen*, pp. 472–473).

to be given by $\bar{L}_s(\chi) = 0$, and the second equation suggests that it should be given by a definite integral of the type

$$\chi(s) = \int_a^b \bar{\kappa}(s, t)\, \psi(t)\, dt.$$

This corresponds exactly to the result which is predicted in § 1.

9. *The Partial Differential Equation* $L_s(u) = M_t(u)$.

We have seen that a system of partial differential equations of the form

$$L_s(u) = M_t(u) \tag{48}$$

may be connected with the two integral equations

$$\left.\begin{aligned} f(s) &= \int_a^b \kappa(s, t)\, \phi(t)\, dt \\ \psi(t) &= \int_c^d \kappa(s, t)\, \chi(s)\, ds \end{aligned}\right\}. \tag{49}$$

Now we assumed in § 1 that the function $\kappa(s, t)$ satisfied a number of linear conditions in s independently of t: this assumption was made to make the work perfectly general. Accordingly, when we consider both equations, we must admit functions $\kappa(s, t)$ which satisfy a number of linear conditions in t independently of s. The system of partial differential equations will then possess a solution which satisfies a number of linear conditions in both s and t.

Let

$$L_s[\kappa(s, t)] \equiv F(s, t) = M_t[\kappa(s, t)];$$

then $\kappa(s, t)$ is the solution of the ordinary linear differential equations

$$L_s(u) = F(s, t),$$

$$M_t(u) = F(s, t)$$

which satisfies certain linear conditions in both s and t.

Now, if these differential equations possess Green's functions* $G(s, x)$

* The characteristic property of the Green's function for a linear differential equation $L_s(w) = 0$ and a set of linear conditions is that the solution of

$$L_s(u) + f(s) = 0$$

which satisfies the given linear conditions can be expressed in the form of a definite integral

$$u = \int_c^d G(s, x)\, f(x)\, dx.$$

If the differential equation is of the n-th degree, the function $G(s, x)$, which is called the Green's function, will be a continuous function of s, x satisfying the given linear conditions for all values of x, but its $(n-1)$-th derivative will experience a sudden change of value at the point $x = s$. The linear conditions usually take the form of relations between the value of the function and

and $H(x, t)$, corresponding to the given linear conditions, we shall have

$$u = \kappa(s, t) = -\int_c^d G(s, x)\, F(x, t)\, dx$$

and

$$u = \kappa(s, t) = -\int_a^b H(x, t)\, F(s, x)\, dx;$$

whence

$$\int_c^d G(s, x)\, F(x, t)\, dx = \int_a^b H(x, t)\, F(s, x)\, dx. \tag{50}$$

We have shown elsewhere* that an integral relation of this type implies that the numbers λ for which the equations

$$\left.\begin{aligned} \phi(s)-\lambda\int_c^d G(s, x)\,\phi(x)\,dx &= 0 \\ \text{and}\qquad \chi(t)-\lambda\int_c^b H(x, t)\,\chi(x)\,dx &= 0 \end{aligned}\right\} \tag{51}$$

can possess solutions different from zero are, in general, the same.

In the demonstration it is necessary to assume that the function $F(s, x)$ is such that no functions $a(s)$ and $b(x)$ exist for which

$$\int_c^d a(s)\, F(s, x)\, ds = 0,$$

for all values of x,

and

$$\int_a^b F(s, x)\, b(x)\, dx = 0,$$

for all values of s.

Suppose, then, that λ is a quantity such that the homogeneous equation

$$\phi(s)-\lambda\int_c^d G(s, x)\,\phi(x)\,dx = 0$$

possesses a solution $\phi(s)$ which is not identically zero.

The equation†

$$\theta(x)-\lambda\int_c^d G(s, x)\,\theta(s)\,ds = 0$$

its first $(n-1)$ derivatives at the points a and b, but Hilbert has shown that, when these points are singularities of the differential equation, conditions of remaining finite or becoming infinite in a specified way may be introduced. It is probable that linear conditions expressed by definite integrals can be added to these to complete the generality of the theory.

The one-dimensional Green's function is, in many respects, analogous to the function used by Green in electrostatics. It was discovered by Burkhardt, and its properties have been developed by the following writers :—Burkhardt, *Bull. Soc. Math.*, Bd. XXII. (1894) ; Böcher, *Amer. Bull.* (1901), p. 297 ; Dunkel, *Amer. Bull.* (1902), p. 288 ; Mason, Diss. Gött. (1903), *Trans. Amer. Math. Soc.*, Vol. v., No. 2, pp. 220-225 ; Hilbert, *Gött. Nachr.* (1904), Heft 3.

* *Trans. Camb. Phil. Soc.*, Vol. XX., No. 10, p. 234.

† Fredholm, *Acta Math.*, p. 27 (1903).

will also possess a solution $\theta(x)$ different from zero for the same value of λ; accordingly, if

$$\chi(t) = \int_c^d F(x, t)\,\theta(x)\,dx,$$

$\chi(t)$ is not identically zero, and we shall have

$$\chi(t) = \lambda \int_c^d \int_c^d G(s, x)\,F(x, t)\,\theta(s)\,ds\,dx$$

$$= \lambda \int_c^d ds \int_a^b H(x, t)\,F(s, x)\,\theta(s)\,dx$$

$$= \lambda \int_a^b H(x, t)\,\chi(x)\,dx.$$

Conversely, if λ is a quantity for which this equation holds, a function ψ will also exist for which

$$\psi(x) = \lambda \int_a^b H(x, t)\,\psi(t)\,dt$$

and, if

$$\phi(s) = \int_a^b F(s, x)\,\psi(x)\,dx,$$

we have

$$\phi(s) = \lambda \int_a^b \int_a^b F(s, x)\,H(x, t)\,\psi(t)\,dt$$

$$= \lambda \int_a^b \int_c^d G(s, x)\,F(x, t)\,\psi(t)\,dt$$

$$= \lambda \int_c^d G(s, x)\,\phi(x)\,dx.$$

Now, in the present case, a function $\phi(s)$ which satisfies the homogeneous integral equation

$$\phi(s) = \lambda \int_c^d G(s, x)\,\phi(x)\,dx$$

will satisfy the differential equation

$$L_s\phi + \lambda\phi = 0,$$

and will also satisfy the linear conditions associated with the function G. Hence the values of λ for which a solution of the homogeneous integral equation exists are the values of λ for which the above differential equation can possess a solution satisfying the given linear conditions.

Similarly, the values of λ for which a solution of

$$\chi(t) = \lambda \int_a^b H(x, t)\,\chi(x)\,dx$$

exists are the values of λ for which a solution of

$$M_t \chi(t) + \lambda \chi(t) = 0$$

can satisfy the linear conditions associated with the function H.

Hence, since the values of λ for the two integral equations are the same, the values of λ for the two differential equations are also the same.

We conclude from this that the partial differential equation will, in general, only possess a solution satisfying the linear conditions identically both in s and t, when the equations

$$L_s(u) + \lambda u = 0,$$

$$M_t(v) + \lambda v = 0$$

can possess solutions of the required type for the same values of λ.

We can also show that, in general, any function which satisfies the relation

$$\int_c^d G(s, x) f(x, t) dx = \int_a^b f(s, x) H(x, t) dx$$

must be a solution of the partial differential equation

$$L_s(w) = M_t(w).$$

Let $$g(s, t) = \int_c^d G(s, x) f(x, t) dx = \int_a^b f(s, x) H(x, t) dx;$$

then $g(s, t)$ will, in general, satisfy the linear conditions associated with both G and H; and so we shall have

$$f(s, t) = -L_s(g),$$

$$f(s, t) = -M_t(g),$$

whence $$L_s(f) = -L_s M_t(g) = M_t(f).$$

The partial integral equation

$$\int_c^d G(s, x) f(x, t) dx = \int_a^b f(s, x) H(x, t) dx \tag{52}$$

is thus satisfied by a certain group of solutions of the partial differential equation, but we cannot say that it is satisfied by every solution of the partial differential equation. If the equation be written in the symbolical form

$$G_s\{f(s, t)\} = H_t\{f(s, t)\}$$

where G and H are linear distributive operators, it is possible to regard the operation $G_s - H_t$ as a factor of the operation $L_s - M_t$.

A particular function which satisfies the partial integral equation is

$f(s, t) = \kappa(s, t)$, for $\kappa(s, t)$ satisfies the given conditions in t; and so the function

$$g(s, t) = \int_c^d G(s, x)\, \kappa(x, t)\, dx$$

satisfies them also. $g(s, t)$ is therefore a solution of the equation

$$L_s(u) + \kappa(s, t) = 0$$

which satisfies the given linear conditions in both s and t.

Operating on this equation with M_t, we have

$$M_t L_s(u) = -M_t\, \kappa(s, t) = -L_s\, \kappa(s, t);$$

therefore

$$L_s[M_t(u) + \kappa(s, t)] = 0.$$

Now $M_t(u) + \kappa(s, t)$ satisfies the given linear conditions in s, since both u and $\kappa(s, t)$ do so, and we see from the above that it also satisfies the equation $L_s(v) = 0$; accordingly, it must be identically zero; for we know from the Green's formula that the solution of the equation

$$L_s(v) + f(s) = 0$$

is given by

$$v = \int_c^d G(s, x)\, f(x)\, dx,$$

and, if $f(x)$ is zero, v is also zero.

Putting, then,

$$M_t(u) + \kappa(s, t) = 0,$$

we have, since u is a function which satisfies the given linear conditions in t,

$$g(s, t) = u = \int_a^b H(x, t)\, \kappa(s, x)\, dx,$$

which gives the required relation

$$\int_c^d G(s, x)\, \kappa(x, t)\, dx = \int_a^b H(x, t)\, \kappa(s, x)\, dx.$$

Now this relation is of some interest in connection with the original integral equations

$$f(s) = \int_a^b \kappa(s, t)\, \phi(t)\, dt,$$

$$\psi(t) = \int_c^d \kappa(s, t)\, \chi(s)\, ds;$$

for we can show that, if f and ϕ are two functions connected by the first relation,

$$f_1(s) = \int_c^d G(s, x)\, f(x)\, dx$$

and

$$\phi_1(s) = \int_a^b H(s, t)\, \phi(t)\, dt$$

is another pair.

Substituting the given value of $f(x)$, we have

$$f_1(s) = \int_c^d \int_a^b G(s, x)\, \kappa(x, t)\, \phi(t)\, dx\, dt$$

$$= \int_a^b \int_a^b \kappa(s, x)\, H(x, t)\, \phi(t)\, dx\, dt$$

$$= \int_a^b \kappa(s, x)\, \phi_1(x)\, dx.$$

Similarly, if ψ and χ are one pair of functions connected by the second relation, the functions

$$\psi_1(t) = \int_a^b H(x, t)\, \psi(x)\, dx$$

and

$$\chi_1(t) = \int_c^d G(x, t)\, \chi(x)\, dx$$

are another pair; for, on substitution, we get

$$\psi_1(t) = \int_a^b \int_c^d H(x, t)\, \kappa(s, x)\, \chi(s)\, ds\, dx$$

$$= \int_c^d \int_c^d G(s, x)\, \kappa(x, t)\, \chi(s)\, ds\, dx$$

$$= \int_c^d \kappa(x, t)\, \chi_1(x)\, dx.$$

We shall complete this series of propositions concerning the partial integral equation (52) by remarking that the equation corresponding to the adjoint partial differential equation

$$\overline{L}_s(u) = \overline{M}_t(u)$$

is no other than

$$\int_c^d f(s, x)\, G(x, t)\, dx = \int_a^b H(s, x)\, f(x, t)\, dx.$$

This result follows at once from the fact that, when we interchange the arguments in a Green's function of a linear differential equation, we obtain the corresponding Green's function for the adjoint equation.

10. *Investigations on the Existence Theorem.*

In a former paper* we attempted to define a class of functions $f(s)$ which could be represented by definite integrals of the form

$$\int_a^b \kappa(s, t)\, \phi(t)\, dt$$

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* *Supra*, pp. 103–106.

to any required degree of approximation. The conditions laid down, however, were not sufficient to ensure that the function $\phi(t)$ would always tend to a finite limit when the approximation was used to obtain an exact representation. It is clear that, if the function $\phi(t)$ is restricted to remain finite within the range of integration, the definite integral is only capable of representing a much narrower class of functions. The investigation can, however, be made more satisfactory when this assumption is made and an existence theorem stated more precisely.

The method which we adopted is analogous to that used in solving a linear differential equation by means of a definite integral and depends upon the possibility of constructing a relation of the form

$$\int_a^b \kappa(s, t) F(t, x)\, dt = \frac{d}{dx} H(s, x). \tag{53}$$

A relation of this type may be constructed in many ways; the one, however, which adapts itself best to our requirements is obtained as follows :—

Let (c, d) be the range of values of s for which the representation is required, and $h(s, t)$ a convenient function which is finite and integrable for values of s and t within the ranges (c, d) and (a, b) respectively. Further, let

$$\left.\begin{aligned} g_0(t) &= \int_c^d h(s, t) f(s)\, ds \\ f_n(s) &= \int_a^b \kappa(s, t)\, g_{n-1}(t)\, dt \\ g_n(t) &= \int_c^d h(s, t)\, f_n(s)\, ds \\ F(t, x) &= x g_1(t) - \frac{x^3}{1\,!}\, g_3(t) + \frac{x^5}{2\,!}\, g_5(t) - \dots \\ 2H(s, x) &= -f(s) + \frac{x^2}{1\,!}\, f_2(s) - \frac{x^4}{2\,!}\, f_4(s) + \dots \end{aligned}\right\}; \tag{54}$$

then it is easily seen that we have the relation

$$\int_a^b \kappa(s, t) F(t, x)\, dt = \frac{d}{dx} H(s, x).$$

If now we write $$\phi(t) = 2\int_0^\infty F(t, x)\, dx,$$

we shall have $\displaystyle\int_a^b \kappa(s, t)\, \phi(t)\, dt = 2\int_0^\infty \frac{d}{dx} H(s, x)\, dx = f(s),$

provided

(1) The integral $\int_0^\infty F(t, x)\,dx$ has a meaning;

(2) We can change the order of integration in the double integral

$$\int_a^b \int_0^\infty F(t, x)\,\kappa(s, t)\,dt\,dx\,;$$

(3)
$$H(s, \infty) = 0.$$

The function $h(s, t)$ is at our disposal. In the previous account of the method we took it to be the same as $\kappa(s, t)$, but it is clearly advantageous to leave it undefined, as this adds to the elasticity of the method.

We shall assume that all the functions we are dealing with are finite and integrable for the given range of values of s and t. The series which represent the functions $F(t, x)$ and $H(s, x)$ will then be absolutely and uniformly convergent for all finite values of x; for, if h, κ, and f are the maximum values of the moduli of $h(s, t)$, $\kappa(s, t)$, and f respectively, it is easy to see that

$$|g_0(s)| \leqslant |d-c|\,hf, \qquad |f_n(s)| \leqslant |b-a|^n\,|d-c|^{n-1}\kappa^n h^{n-1} f,$$

$$|g_n(s)| \leqslant |b-a|^n\,|d-c|^n\,\kappa^n h^n f\,;$$

so that the series can be compared with exponential series.

Now write

$$\left.\begin{aligned} P(s, t) &= \int_a^b \kappa(s, r)\,h(t, r)\,dr \\ Q(s, t) &= \int_c^d h(r, s)\,\kappa(r, t)\,dr \end{aligned}\right\}, \tag{55}$$

and let $\psi_m(s)$, $\chi_m(t)$ be the series of functions for which the homogeneous integral equations

$$\left.\begin{aligned} \psi_m(s) &= \lambda_m \int_c^d P(s, t)\,\psi_m(t)\,dt \\ \chi_m(s) &= \lambda_m \int_a^b Q(s, t)\,\chi_m(t)\,dt \end{aligned}\right\} \tag{56}$$

can be satisfied. It should be noticed that the values of λ_m for which these equations possess solutions different from zero are the same; for, if we calculate the determinantal equations of which the quantities λ_m are the roots, we shall find that they are identical.

[*Note added December 26th.*—In what follows it will be supposed that these values of λ_m are all real. This is certainly true if $h(s, t)$ and $\kappa(s, t)$ are the same; for then $P(s, t)$ and $Q(s, t)$ are symmetrical functions. The choice of the function $h(s, t)$ is thus not entirely arbitrary.]

It is easy to see that, if

$$\theta_m(s) = \int_a^b \kappa(s, t)\, \chi_m(t)\, dt,$$

then
$$\theta_m(s) = \lambda_m \int_c^d P(s, t)\, \theta_m(t)\, dt;$$

so that $\psi_m(s)$ may be taken to be equal to $\theta_m(s)$ and can be defined by the above equation. We then have the further relation

$$\chi_m(t) = \lambda_m \int_c^d h(s, t)\, \psi_m(s)\, ds.$$

The existence theorem which we shall now prove is that, *if the function $f(s)$ can be expanded in a convergent series of the form $\Sigma a_m \psi_m(s)$ which is such that the derived series $\Sigma |\lambda_m a_m \psi_m|$ is convergent, ψ_m being the maximum value of $|\psi_m(s)|$ within the range (c, d), then a function $\phi(t)$ exists for which*

$$f(s) = \int^b \kappa(s, t)\, \phi(t)\, dt,$$

and this function may be determined by the formula

$$\phi(t) = 2\int_0^\infty F(t, x)\, dx.$$

If we write
$$\omega(s) = \Sigma \lambda_m a_m \psi_m(s), \tag{57}$$

we have
$$\phi(t) = \int_c^d h(s, t)\, \omega(s)\, ds = \Sigma a_m \chi_m(t),$$

and this series is absolutely and uniformly convergent for

$$|a_m \chi_m(t)| \leqslant |\lambda_m a_m (d-c)\, h \psi_m|$$

where h and ψ_m are the maximum values of the moduli of $h(s, t)$ and $\psi_m(s)$ within the given ranges, and the series $|\lambda_m a_m \psi_m|$ is convergent by hypothesis.

This series, for $\phi(t)$, may be integrated term by term, and we obtain

$$\int_a^b \kappa(s, t)\, \phi(t) = \Sigma a_m \psi_m(s) = f(s)$$

as required.

We have now to prove that this series for $\phi(t)$ may be obtained from the formula

$$\phi(t) = 2\int_0^\infty F(t, x)\, dx. \tag{58}$$

Calculating the functions $g_n(t)$ and $f_{n+1}(s)$ in turn, we have *

$$\left.\begin{aligned} g_0(t) &= \Sigma \frac{a_m}{\lambda_m} \chi_m(t) \\ f_n(s) &= \Sigma \frac{a_m}{\lambda_m^n} \psi_m(s) \\ g_n(t) &= \Sigma \frac{a_m}{\lambda_m^{n+1}} \chi_m(t) \end{aligned}\right\}. \tag{59}$$

The series for $F(t, x)$ may now be transformed into

$$F(t, x) = \sum_0^\infty x \frac{a_m}{\lambda_m^2} e^{-x^2\lambda_m^2} \chi_m(t),$$

and it is clear that the integral $2\int_0^\infty F(t, x)\, dx$ will give the series for $\phi(t)$, provided the integration term by term is legitimate.

Now a sufficient set of conditions for the integration term by term of a series

$$s(x) = \sum_1^\infty u_n(x)$$

is the following:—

(1) The series $\sum_1^\infty u_n(x)$ should be uniformly convergent in an arbitrary interval;

(2) $\int u_n(x)\, dx$ should exist for all values of n;

(3) $\sum_{n=1}^\infty \int_a^\infty u_n\, dx$ should converge for all values of a between 0 and ∞;

(4) A number p independent of r should exist for which

$$\left| \sum_{n=1}^r \int_\kappa^\infty u_n\, dx \right| < \epsilon \text{ for all } \kappa\text{'s} > p.$$

The first and third conditions are clearly satisfied, since the series $\Sigma a_m \chi_m(t)$ is absolutely and uniformly convergent; the fourth condition will be satisfied if p can be chosen so that

$$\left| \sum_1^r a_m e^{-\kappa^2/\lambda_m^2} \chi_m(t) \right| < \epsilon \text{ for } \kappa > p.$$

Now this can clearly be done; for, if m_1 is a number such that

$$\sum_{m_1}^r |a_m \chi_m(t)| < \frac{\epsilon}{2} \text{ for all } r\text{'s} > m_1,$$

* These series will all be absolutely and uniformly convergent, since the quantities λ_m increase indefinitely in magnitude, being the zeroes of a whole function.

and p is chosen so large that

$$\left|\sum_1^{m_1} a_m e^{-\kappa^2/\lambda_m^2} \psi_m(t)\right| < \frac{\epsilon}{2} \text{ for } \kappa > p,$$

we have

$$\left|\sum_1^r (\ldots)\right| \leqslant \left|\sum_1^{m_1} (\ldots)\right| + \left|\sum_{m_1}^r (\ldots)\right| \leqslant \frac{\epsilon}{2} + \sum_{m_1}^r \left|a_m \chi_m(t)\right| < \epsilon.$$

The theorem we have just proved does not tell us anything about the uniqueness of the solution of an integral equation, and it does not give a value of $\phi(t)$ different from zero for which

$$0 = \int_a^b \kappa(s, t)\, \phi(t)\, dt$$

when such a value exists. It is by no means certain, however, that the solution which is obtained by using one function $h(s, t)$ is the same as that which would be obtained if we used another. If the two values of $\phi(t)$ thus obtained were different, their difference would be a solution corresponding to $f(s) = 0$.

In general, the function $\phi(t)$ will take a simpler form when the range (c, d) is bounded by two points at which the function $\kappa(s, t)$ is discontinuous than if it is taken arbitrarily. It often happens that the solution in the first case is unique, but not so in the second case, unless we impose additional restrictions upon the function ϕ. Examples of this phenomenon may be obtained by considering the problems in which we require to find the distribution of electricity over a closed surface when the value of the potential function is given (1) over the whole surface, (2) over a portion of the surface.