ON THE REDUCTION AND CLASSIFICATION OF BINARY CUBICS WHICH HAVE A NEGATIVE DISCRIMINANT

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1. Let f' be a binary cubic, with real integral coefficients, and roots α', β', γ', of which the first is real and the others complex. Then, as proved in a former paper, f' is properly equivalent to one, and (with a certain convention) only one reduced form f, the roots of which satisfy the conditions

$$\beta\gamma-1 \geqslant 0,$$

$$\beta+\gamma-1 \leqslant 0 \leqslant \beta+\gamma+1.$$

It has also been shown that in any limiting case when one at least of the inequalities becomes an equality, the form f, and therefore f', has a rational linear factor. Consequently we consider only the strict inequalities

$$\beta\gamma-1 > 0, \quad \beta+\gamma-1 < 0, \quad \beta+\gamma+1 > 0.$$

Since β, γ are conjugate

$$(\alpha\beta-1)(\alpha\gamma-1) = |\,\alpha\beta-1\,|,$$

a positive quantity; so the first condition is equivalent to

$$\Pi(\beta\gamma-1) > 0,$$

where the product is a symmetrical function of the roots. Let

$$f = ax^3+bx^2y+cxy^2+dy^3 = (a,\ b,\ c,\ d);$$

then the condition last written reduces to

$$C_1 = d(d-b)+a(c-a) > 0. \tag{1}$$

Similarly the other two conditions may be replaced by

$$\Pi(\beta+\gamma-1) < 0, \qquad \Pi(\beta+\gamma+1) > 0;$$

or, in terms of the coefficients,

$$C_2 = ad-(a+b)(a+b+c) < 0, \tag{2}$$

$$C_3 = ad+(a-b)(a-b+c) > 0. \tag{3}$$

These, then, are the necessary and sufficient conditions that (a, b, c, d) may be reduced.

As an example, take the form $(2, -12, 11, -21)$ obtained by a different criterion in a former paper; here

$$C_1 = -21(-9)+2.9 > 0,$$
$$C_2 = -21.2-(-10)1 < 0,$$
$$C_3 = -21.2+14.25 > 0,$$

and the form is reduced.

2. In order to reduce a given form, we proceed as follows. Let S, T denote the substitutions

$$\begin{pmatrix}1, & 1\\ 0, & 1\end{pmatrix}, \quad \begin{pmatrix}0, & -1\\ 1, & 0\end{pmatrix},$$

respectively. Then, if $C_1 < 0$, we apply T, and thus obtain a form with $C_1 > 0$. If now $C_2 > 0$, this means that $\beta+\gamma-1 > 0$, and the application of S will diminish C_2; on the other hand, if $C_3 < 0$, the application of S^{-1} will increase C_3, so there is always a substitution $S^{\pm h}$, which will make the new form satisfy $C_2 < 0$ and $C_3 > 0$. If in this form $C_1 > 0$, it is reduced; if not, we apply T and proceed as before. From the known theory of definite quadratics it follows that the process must end after a finite number of operations; and we shall then have obtained a reduced form properly equivalent to the given one.

For instance, suppose

$$f = (97, 471, 762, 411).$$

Here $C_1 > 0$, $C_2 < 0$, $C_3 < 0$, so we must apply S^{-1}; the result is

$$f_1 = (97, 180, 111, 23),$$

for which $C_1 < 0$; hence we apply T and find

$$f_2 = (23, -111, 180, -97).$$

Proceeding thus, we have successively

$$f_3 = Sf_2 \quad = (23, -42, 27, -5),$$
$$f_4 = Tf_3 \quad = (-5, -27, -42, -23),$$
$$f_5 = S^{-1}f_4 = (-5, -12, -3, -3),$$
$$f_6 = Tf_5 \quad = (-3, 3, -12, 5),$$

the last of which is reduced. Since $f(-x, -y) \sim f(x, y)$ we may replace it by $(3, -3, 12, -5)$, so as to make the first coefficient positive.

3. We now pass on to the classification of cubics for a given discriminant $-\Delta$. Since the discriminant of the Hessian is 3Δ, and this must be of the form $4n$ or $4n+1$, it follows that Δ must be of the form $4n$ or $4n+3$. Further, if $\Delta = 3n^2$, the Hessian breaks up into rational factors, and, as this case requires special treatment, it will be postponed for the present.

Before proceeding to the general theory, it will be well to consider a special case. Suppose, then, that $\Delta = 23$. Then the discriminant of the Hessian is 69, and this quadratic must belong to one of the classes represented by

$$(1, 1, -17), \quad (-1, 1, 17).$$

Now all the cubics which have a Hessian of the form $\mu(x^2+xy-17y^2)$, where μ is an integer, are given by

$$f = mx^3+nx^2y+(51m+n)\,xy^2+(17m+6n)\,y^3,$$

with a Hessian $\quad h = (153m^2+3mn-n^2)(x^2+xy-17y^2)$

$$= G(m, n)\,H(x, y),$$

say; so that all the associated cubics which have a discriminant -23 must be derivable from the integral solutions of

$$153m^2+3mn-n^2 = \pm 1.$$

Since -1 is not a quadratic residue of 3, the upper sign is inadmissible; taking the lower sign, we have solutions $(0, \pm 1)$, and all the others are given by putting

$$P = \begin{pmatrix} 11 & 1 \\ 153 & 14 \end{pmatrix},$$

$$P^h = \begin{pmatrix} \alpha_h & \beta_h \\ \gamma_h & \delta_h \end{pmatrix} \quad (h = \pm 1, \ \pm 2, \ \ldots),$$

and then taking $m = \pm \beta_h$, $n = \pm \delta_h$. In fact, P is the fundamental automorph of $G(m, n)$.

Now the primary automorph of $H(x, y)$ is given by

$$Q = \begin{pmatrix} 11 & 51 \\ 3 & 14 \end{pmatrix},$$

and if this be applied to f the result is found to be the same as if we apply the substitution P^3 to m and n. Hence there cannot be more than three classes of cubics, and it is sufficient to consider the cases

$$(m, n) = (0, 1), \ (1, 14), \ (1, -11),$$

the last of which is derived from P^{-1}. The first values give

$$f_1 = (x^2+xy+6y^2)\,y,$$

the second $$f_2 = x^3+14x^2y+65xy^2+101y^3,$$

and the third $$f_3 = x^3-11x^2y+40xy^2-49y^3.$$

The first cubic is reduced; with regard to the others we find that

$$(1,\ 14,\ 65,\ 101) \sim (1,\ 2,\ 1,\ 1) \sim (1,\ -1,\ 2,\ -1),$$

and
$$\begin{aligned}(1,\ -11,\ 40,\ -49) &\sim (1,\ -2,\ 1,\ -1)\\ &\sim (-1,\ -1,\ -2,\ -1)\\ &\sim (1,\ 1,\ 2,\ 1).\end{aligned}$$

The result, then, is that, for $\Delta = 23$, there are three classes of cubics represented by the reduced forms

$$\begin{aligned}g_1 &= (x^2+xy+6y^2)\,y,\\ g_2 &= x^3-x^2y+2xy^2-y^3,\\ g_3 &= x^3+x^2y+2xy^2+y^3;\end{aligned}$$

all of these are derived from the Hessian $(1,\ 1,\ -17)$.

Similarly, if we start from $(-1,\ 1,\ 17)$, we find

$$f = mx^3+nx^2y+(51m-n)\,xy^2+(-17m+6n)\,y^3,$$

with $$h = (n^2+3mn-153m^2)(-x^2+xy+17y^2).$$

Comparing this with the previous case, we see that the values of $(m,\ n)$ to be considered are given by

$$(m,\ n) = (0,\ 1),\ (-1,\ 14),\ (-1,\ -11),$$

leading to the forms

$$\begin{aligned}f_1 &= (x^2-xy+6y^2)\,y,\\ f_2 &= -x^3+14x^2y-65xy^2+101y^3,\\ f_3 &= -x^3-11x^2y-40xy^2-49y^3.\end{aligned}$$

But these do not give any more reduced forms;* in fact,

$$\begin{aligned}&f_1 \sim (x^2+xy+6y^2)\,y,\\ &f_2 \sim (-1,\ 2,\ -1,\ -1) \sim (1,\ 1,\ 2,\ 1),\\ &f_3 \sim (-1,\ -2,\ -1,\ -1) \sim (-1,\ 1,\ -2,\ 1) \sim (1,\ -1,\ 2,\ -1).\end{aligned}$$

* This might have been foreseen, because $(-1,\ -1,\ 17) \sim (-1,\ 1,\ 17)$.

Finally, then, we see that for $\Delta = 23$, there are three classes represented by

$$(0,\ 1,\ 1,\ 6),\quad (1,\ \pm 1,\ 2,\ \pm 1),$$

the last pair being improperly equivalent.

It may be added that 23 is the least value of Δ for which there exists a cubic form which has no linear factor.

4. For the purpose of explaining the general theory it will be necessary to apply a proposition first stated by Eisenstein. If $(A,\ B,\ C)$ is the Hessian of $(a,\ b,\ c,\ d)$, then

$$-A^2 = Ab^2 - 3Bba + 9Ca^2, \tag{4}$$

identically, in virtue of the relations $A = 3ac - b^2, \ldots$, which define the Hessian. Let L denote the form $(A,\ -3B,\ 9C)$, and suppose that $(A,\ B,\ C)$ is an assigned quadratic H, of determinant 3Δ. Then it follows from (4) that the first two coefficients $(a,\ b)$ of every cubic which has a Hessian H supply an integral solution of

$$L\,(b,\ a) = -A^2.$$

Now it can be proved without difficulty that conversely if $(b,\ a)$ is any integral solution of (4) there is a corresponding cubic $(a,\ b,\ c,\ d)$, and only one, which has a Hessian H. Thus there is a one-one correspondence between the cubics and the representations of $-A^2$ by L.

5. By means of the automorphs of L, these cubics may be arranged in a finite number of sequences as follows. Let $(T,\ U)$ be the primary solution of

$$T^2 - 27\Delta U^2 = 4,$$

and let $(a,\ b,\ c,\ d)$ be a cubic derived from any one solution of (4). The primary automorph of L is given by

$$P = \begin{bmatrix} \tfrac{1}{2}(T+3BU), & -9CU \\ AU, & \tfrac{1}{2}(T-3BU) \end{bmatrix}, \tag{5}$$

and if this be applied to $(a,\ b)$ and the new values of $c,\ d$ calculated it is found, without much difficulty, that

$$P\,(a,\ b,\ c,\ d) = (a',\ b',\ c',\ d'),$$

or, say,

$$Pf = f',$$

where
$$\left.\begin{aligned} a' &= \tfrac{1}{2}(T-3BU)\,a+AUb \\ b' &= -9CUa+\tfrac{1}{2}(T+3BU)\,b \\ c' &= \tfrac{1}{2}(T-3BU)\,c+9AUd \\ d' &= -CUc+\tfrac{1}{2}(T+3BU)d \end{aligned}\right\}. \qquad (6)$$

Hence f is a member of a sequence of forms

$$\ldots f_{-2},\ f_{-1},\ f,\ f_1,\ f_2,\ \ldots,$$

where f_n means $P^n f$. It will now be proved that the forms of this sequence are either all equivalent, or else fall into three distinct classes. Let (t, u) be the primary solution of

$$t^2-3\Delta u^2 = 4\,;$$

then the primary automorph of H is given by

$$Q = \begin{bmatrix} \tfrac{1}{2}(t-Bu), & -Cu \\ Au, & \tfrac{1}{2}(t+Bu) \end{bmatrix},$$

and if this is applied to the variables in f the result is

$$Qf = (a'', b'', c'', d'')$$

where
$$\left.\begin{aligned} a'' &= \tfrac{1}{2}(t_3-Bu_3)\,a+\tfrac{1}{3}Au_3b \\ b'' &= -3Cu_3a+\tfrac{1}{2}(t_3+Bu_3)\,b \\ c'' &= \tfrac{1}{2}(t_3-Bu_3)c+3Au_3d \\ d'' &= -\tfrac{1}{3}Cu_3c+\tfrac{1}{2}(t_3+Bu_3)\,d \end{aligned}\right\}. \qquad (7)$$

Here (t_3, u_3) are found from

$$\frac{t_3+u_3\sqrt{3\Delta}}{2} = \left(\frac{t+u\sqrt{3\Delta}}{2}\right)^3,$$

so that
$$t_3 = \tfrac{1}{4}t(t^2+3\Delta u^2), \qquad u_3 = \tfrac{3}{4}(t^2+\Delta u^2)\,u, \qquad (8)$$

whence we see that u_3 is divisible by 3, and t_3 is not. Putting, therefore, $t_3 = T_h$, $u_3 = 3U_h$, we have (T_h, U_h) a solution of $T^2-27\Delta U^2 = 4$, and

$$Q_{x,y}(f) = P^h_{a,b}(f),$$

where the suffixes are used to emphasise the fact that P affects the coefficients and Q the variables of f. The question now is—what is the value of h? We observe that since

$$T^2-27\Delta U^2 = T^2-3\Delta(3U)^2,$$

it follows that the solutions of $T^2-27\Delta U^2 = 4$ are simply $(t_n, \tfrac{1}{3}u_n)$, where

(t_n, u_n) are those solutions of $t_n^2-3\Delta u_n^2=4$ in which u_n is divisible by 3. There are two, and only two, possible cases. If u_1 is prime to 3 so is u_2, and $(t_3, \frac{1}{3}u_3)$ is the primary solution (T, U); in this case $h=1$, and all the forms f_n are equivalent. But it may happen that u_1 is divisible by 3: in this case $(T, U)=(t_1, \frac{1}{3}u_1)$, and $h=3$, so that any three consecutive forms f_{n-1}, f_n, f_{n+1} are non-equivalent. The case of $\Delta=23$ is one of those in question, for

$$t^2-69u^2=4$$

has the primary solution (25, 3), whence $(T, U)=(25, 1)$; so that every possible Hessian leads to three classes of cubics, as, in fact, was proved in Art. 3.

If the solutions of $L(b, a)=-A^2$ are not all associated with the sequence (f_n), we take one of those that remain and form another sequence of cubics, and so on. From the theory of quadratics, it follows that there is only a finite number of sequences, depending upon the number of solutions of the congruence $x^2\equiv 3\Delta \pmod{4A^2}$.

6. We are now in a position to find a complete set of representative cubics for any possible negative discriminant $-\Delta$. To do so we take a complete set, both primitive and derived, of representative quadratics $H_1, H_2, \ldots$ of determinant 3Δ. Let H, or (A, B, C), be any one of them; then if there are integral solutions (a, b) of

$$-A^2=Ab^2-3Bba+9Ca^2,$$

there will be one or three corresponding classes of cubics, which can be determined by the methods of Art. 5, for each associated set of solutions. If there are no solutions, there will be no cubics associated with H.

7. In practice, it is inconvenient to have to solve the equation

$$L(b, a)=-A^2,$$

but the difficulty can be avoided by the theory explained in my paper on the relations between cubics and their Hessians.* It is there shown that, if H is a prescribed Hessian, all the cubics which have a Hessian of the form μH, where μ is an integer, will be of the form

$$f=m\phi+n\psi,$$

where ϕ, ψ are determinate cubics, and m, n arbitrary integers. The Hessian is now

$$h(f)=G(m, n)\,H(x, y),$$

where $G(m, n)$ is a determinate quadratic in m and n. Suppose now that

* *Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 200.

H has a determinant 3Δ, and we wish to find all the associated cubics of discriminant $-\Delta$: then

$$G(m, n) = 1,$$

and this equation is much simpler to deal with than $L(b, a) = -A^2$, which it replaces. In fact, since $f = m\phi + n\psi$, the coefficients a, b are linear functions of m, n, which convert $L(b, a)$ into $-A^2 . G(m, n)$; conversely m, n are linear functions of a, b which are fractional in form but give integral values for m, n whenever (b, a) is a solution of

$$L(b, a) = -A^2.$$

Thus there is a one-one correspondence between the forms L, G, and it follows that a sequence of forms (f_n) derived from $G(m, n) = 1$ is obtainable by the same linear substitution applied to all the forms of a sequence derived from a solution of $L(b, a) = -A^2$. In particular, each sequence falls into one class, or three, according to the criterion found in Art. 5.

As a check to calculation, it may be remarked that if

$$a = pm + qn,$$
$$b = rm + sn,$$

the determinant of G is $27\Delta (ps - qr)^2 / A^4$.

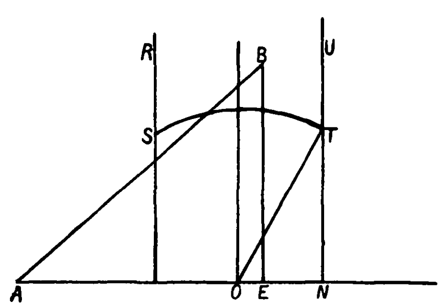
In order that $G(m, n) = 1$ may be possible, it is necessary and sufficient that G should belong to the principal class of its determinant.

8. When, for a given discriminant, a set of representative forms has been obtained, the final process is to put for them the corresponding reduced forms, calculated by the method explained in the earlier part of this paper.

It has been shown, on a previous occasion, that, if (a, b, c, d) is a reduced form,

$$27a^4 < 16\Delta;$$

Mr. Berwick has discovered other inequalities which limit the coefficients when Δ is given, and I take this opportunity of communicating them.



In the figure $RSTU$ is part of the contour of the fundamental triangle; A, B are the points representing the roots α, β of the cubic, and BE is perpendicular to AN. We have

$$AE = \tfrac{1}{2}(\beta+\gamma)-\alpha = \tfrac{3}{2}(\beta+\gamma)+\frac{b}{a}.$$

Now, if C is the point corresponding to the root γ, we have

$$AE^4 < AB^2.AC^2,$$

$$3 < BC^2;$$

and therefore $$3AE^4 < \Delta/a^4;$$

hence $$3\{2b+3a(\beta+\gamma)\}^4 < 16\Delta,$$

and, *a fortiori*, since $|\beta+\gamma| < 1$,

$$3(2|b|-3|a|)^4 < 16\Delta. \qquad (9)$$

Again, if B is kept fixed, the greatest value of OB^2/AB^2 is when A is at E, and the greatest value of OB^2/BE^2 is when B is at S or T; consequently, in general, $$4AB^2 > 3OB^2.$$

Similarly $$4AC^2 > 3OC^2;$$

also $$a^2\alpha^2 < (|a|+|b|)^2;$$

hence $$9d^2 < (|a|+|b|)^2.16AB^2.AC^2,$$

and since $3 < BC^2$, we obtain by multiplication

$$27d^2 < \frac{16(|a|+|b|)^2\Delta}{a^4}. \qquad (10)$$

The inequalities (9) and (10) impose limits on b, d in addition to the limits previously found for a. Now when a, b, d, Δ are assigned, there is a cubic to find possible values of c. All we have to do then is to find the sets (a, b, c, d) which give the proper value to Δ, and also satisfy the three conditions of inequality; if we reduce the resulting forms, we are certain to find representatives of every class, and we see that the number of distinct classes is limited.*

9.† Since writing the greater part of the present paper, I have looked at the two memoirs on binary cubics by Pepin, which are contained in the *Atti dell' Accad. Pontif. dei Nuovi Lincei*, Vols. 37, 39 (1884, 1886). In

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* [*Added May 4th*, 1911.—Mr. Berwick has since found the additional inequality
$$27(|c|-|a|-|b|)^2a^2 < 16\Delta.]$$

† §§ 9 and 10 and the table added May 4th, 1911.

the first of these he develops the theory on the lines laid down by Eisenstein, making use of the equation $L(b, a) = -A^2$, or rather an equivalent one, as he takes the standard form to be $(a, 3b, 3c, d)$. Of course, some of my results practically coincide with his; but he does not go into the question of reduction, nor into the geometrical theory. In the second paper he discusses classification, but confines himself to the case of a positive discriminant.

10. The following table, calculated by Mr. Berwick, gives all the non-composite reduced cubics for values of Δ which do not exceed 999. It will be understood that each entry (a, b, c, d) really stands for $(a, \pm b, c, \pm d)$, and represents two classes, improperly equivalent to each other.

Δ	f	Δ	f	Δ	f
23	(1, 1, 2, 1)	244	(2, 2, 3, 2)	432	(1, 3, 3, 5)
31	(1, 0, 1, 1)	247	(1, 3, 4, 5)	436	(1, 3, 4, 6)
44	(1, 2, 2, 2)	255	(1, 1, 0, 3)	439	(1, 2, −1, 3)
59	(1, 0, 2, 1)	268	(2, 4, 4, 3)	440	(2, 0, 1, 2)
76	(1, 1, 3, 1)	279	(1, 1, 4, 3)	451	(2, 3, 5, 3)
83	(1, 1, 1, 2)	283	(1, 0, 4, 1)	459	(2, 3, 3, 3)
87	(1, 2, 3, 3)	300	(2, 2, 4, 1)	460	(1, 1, 5, 3)
104	(2, 2, 3, 1)	304	(1, 4, 4, 4)	464	(1, 3, 5, 7)
107	(1, 1, 3, 2)	307	(1, 2, 4, 5)		(2, 1, 4, 1)
108	(1, 3, 3, 3)	324	(2, 0, 3, 1)	472	(2, 4, 5, 4)
116	(1, 1, 0, 2)	327	(3, 3, 4, 1)	484	(1, 2, 5, 6)
135	(1, 0, 3, 1)	331	(1, 1, 3, 4)	491	(1, 2, 2, 5)
139	(1, 2, 2, 3)	332	(1, 1, 2, 4)	492	(1, 2, 4, 6)
140	(1, 0, 2, 2)	335	(1, 2, 5, 5)	496	(2, 0, 2, 2)
152	(2, 3, 4, 2)	339	(1, 2, 0, 3)	499	(1, 0, 4, 3)
172	(2, 0, 2, 1)	351	(1, 0, 3, 3)	503	(2, 5, 5, 4)
175	(1, 1, 2, 3)	356	(2, 1, 2, 2)	515	(1, 4, 4, 5)
176	(1, 2, 4, 4)	364	(1, 0, 4, 2)	516	(3, 3, 4, 2)
199	(1, 1, 4, 1)	367	(1, 2, 3, 5)	519	(3, 5, 6, 3)
200	(1, 2, 3, 4)	368	(2, 2, 4, 2)	524	(1, 1, 3, 5)
204	(1, 1, 1, 3)	379	(1, 1, 1, 4)	527	(1, 0, 5, 1)
211	(2, 1, 3, 1)	411	(1, 1, 5, 2)	543	(1, 1, 2, 5)
212	(1, 1, 4, 2)	416	(1, 1, 5, 1)	547	(3, 2, 4, 1)
216	(1, 0, 3, 2)		(1, 2, 1, 4)	556	(1, 4, 3, 4)
231	(1, 2, 1, 3)	419	(2, −1, 3, 1)	560	(1, 2, 0, 4)
236	(2, −1, 2, 1)	424	(3, 4, 5, 2)	563	(1, 1, 5, 4)
239	(1, 3, 2, 3)	428	(1, 3, 2, 4)	567	(3, 0, 3, 1)
243	(1, 3, 3, 4)	431	(2, 1, 3, 2)	575	(1, 1, 4, 5)

(*Continued on p. 138.*)

Δ	f	Δ	f	Δ	f
588	(1, 2, 6, 6)	751	(1, 1, 6, 1)	876	(3, 2, 4, 2)
608	(1, 0, 5, 2)	755	(1, 2, 6, 7)	883	(2, 1, 1, 3)
	(1, 1, 1, 5)	756	(2, 3, 6, 3)	888	(2, 2, 5, 3)
620	(2, 0, 4, 1)	759	(1, 1, 6, 3)	891	(1, 0, 6, 1)
628	(2, 5, 6, 5)	771	(1, 1, 3, 6)	907	(2, 1, 5, 1)
643	(1, 3, 1, 4)	780	(1, 4, 4, 6)	908	(3, 4, 6, 2)
648	(2, 0, 3, 2)	783	(3, 1, 4, 1)	931	(1, 1, 5, 6)
652	(2, 2, 4, 3)		(1, 3, 6, 9)	932	(1, 0, 5, 4)
655	(1, 2, 1, 5)	800	(2, 3, 4, 4)	940	(1, 3, 1, 5)
671	(1, 3, 2, 5)		(1, 5, 5, 5)	944	(1, 2, 5, 8)
675	(1, 3, 3, 6)	804	(1, 1, 4, 6)		(2, 0, 4, 2)
676	(2, 2, 5, 2)	808	(1, 1, 2, 6)	948	(1, 2, 1, 6)
679	(1, 3, 4, 7)	812	(2, 4, 6, 5)	959	(1, 2, 7, 7)
680	(2, 2, 5, 1)	815	(3, 4, 5, 3)	964	(4, 5, 6, 2)
684	(2, 2, 2, 3)	816	(3, 3, 5, 1)	971	(3, 1, 3, 2)
687	(1, 2, 5, 7)	823	(3, −2, 3, 1)	972	(1, 0, 6, 2)
688	(1, 0, 4, 4)	835	(1, 2, 0, 5)		(1, 3, 3, 7)
695	(1, 4, 5, 7)	839	(1, 4, 3, 5)		(2, 3, 6, 4)
696	(1, 2, −1, 4)	843	(3, 3, 5, 2)		(2, 6, 6, 5)
704	(2, 4, 4, 4)	844	(1, 1, 6, 4)	976	(1, 2, 6, 8)
707	(1, 3, 5, 8)	848	(1, 4, 2, 4)		(1, 3, 4, 8)
716	(3, −1, 3, 1)		(4, 4, 5, 1)	980	(2, 4, 5, 5)
728	(1, 1, 6, 2)	856	(2, 2, 1, 3)	983	(1, 1, 6, 5)
731	(1, 2, 4, 7)	863	(1, 2, 3, 7)	984	(2, 1, 0, 3)
743	(1, 0, 5, 3)	864	(1, 3, 0, 4)	996	(1, 4, 5, 8)
744	(2, −1, 4, 1)		(3, 6, 6, 4)	999	(2, 3, 3, 4)
748	(1, 2, 2, 6)	867	(2, 1, 3, 3)		