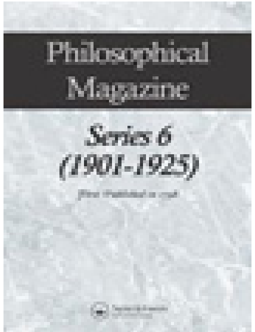

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VII. *On the Behaviour of Pleochroitic Crystals along Directions in the Neighbourhood of an Optic Axis* *. By Professor W. Voigt, *of Göttingen* †.

1. ALL theories regarding the absorption of light in crystals agree in recognizing the fact that on account of absorption there is added to the characteristic triplet of directed quantities corresponding to each colour in transparent crystals a *second* triplet of analogous properties. The first triplet may conveniently, though not quite correctly, be regarded as forming the system of the three mutually perpendicular axes of symmetry for the *conservative* forces, while the second corresponds to the *absorptive* forces. To each of these axes there corresponds a certain number, which in transparent crystals represents, as is well known, the velocity of all those plane waves for which Fresnel's vibration-vector is parallel to the corresponding axis. If we use the term *tensor-triplet*, proposed by myself, to denote a system of three two-sided mutually normal directed quantities, then an absorbing crystal will have to be characterized, for a given colour, by *two* tensor-triplets. These two triplets may be denoted by the symbols a_1, a_2, a_3 and b_1, b_2, b_3, and we shall suppose that

$$a_1 > a_2 > a_3 \quad \text{and} \quad b_1 > b_2 > b_3 . \quad . \quad . \quad . \quad . \quad (1)$$

2. It is known that in the case of transparent crystals most characteristic properties correspond to those two directions A_1 and A_2 which, lying in the plane of the greatest and the least tensors a_1 and a_3, make an angle ϑ with the latter defined by the equations

$$\sin^2 \vartheta = \frac{a_1 - a_2}{a_1 - a_3}, \qquad \cos^2 \vartheta = \frac{a_2 - a_3}{a_1 - a_3}. \quad . \quad . \quad . \quad (2)$$

These directions are also of special importance in absorbing crystals, and are also called *optic axes* in this case, although on account of their altered properties they would be more correctly denoted by some other term (*e. g.* polarization axes).

Corresponding to the optic axes, we may conveniently consider as *absorption axes* the directions B_1, B_2 which, lying in the plane of the greatest and least tensors b_1 and b_3, make an angle δ with the latter defined by the equations

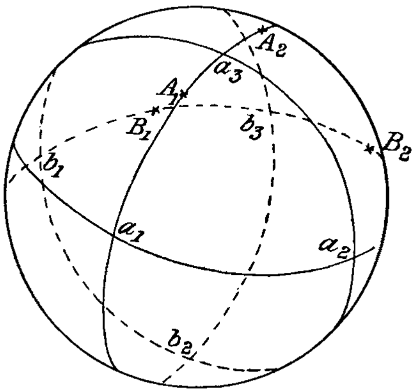
$$\sin^2 \delta = \frac{b_1 - b_2}{b_1 - b_3}, \qquad \cos^2 \delta = \frac{b_2 - b_3}{b_1 - b_3}. \quad . \quad . \quad . \quad (3)$$

* A summary of the results contained in a paper presented on Feb. 8, 1902, to the Kgl. Gesellschaft der Wissenschaften zu Göttingen (*Gött. Nachr.* 1902, Heft 1).

† Communicated by Lord Kelvin.

The relative position of the two pairs A_1, A_2 and B_1, B_2 of axes is in general as impossible to assign as that of the tensor-triplets a_1, a_2, a_3 and b_1, b_2, b_3. It is only when the crystal possesses special symmetry that it becomes possible to say something more or less definite. Fig. 1 gives some idea

Fig. 1.



regarding the general position of these axes; all the characteristic lines are drawn through the centre of a sphere, and their intersections with the spherical surface indicated.

3. The fundamental formulæ in the theory of plane wave propagation are obtained most simply by introducing a system of coordinate axes x, y, z, of which z coincides with the direction of propagation. If, then, we denote the components of the two tensor-triplets along these axes by a_{11}, a_{22}, a_{33}, a_{23}, a_{31}, a_{12} and b_{11}, b_{22}, b_{33}, b_{23}, b_{31}, b_{12}*, and if we write shortly

$$a_{hk}+ib_{hk}=c_{hk}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $i=\sqrt{-1}$, then

$$(c_{11}-v^2)\,(c_{22}-v^2) = c_{12}^2, \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{g^2}{f^2}+\frac{c_{11}-c_{22}}{c_{12}}\cdot\frac{g}{f}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where v stands for the so-called complex velocity, g/f for the ratio of the complex amplitudes for Neumann's vibration-vector along the y and x axes respectively. Expressed in

* If α_h, β_h, γ_h are the direction-cosines of the triplet a_1, a_2, a_3 relatively to the axes x, y, z, then

$$a_{11}=a_1\alpha_1{}^2+a_2\alpha_2{}^2+a_3\alpha_3{}^2, \;\ldots\,.$$
$$a_{23}=a_1\beta_1\gamma_1+a_2\beta_2\gamma_2+a_3\beta_3\gamma_3, \;\ldots\,.$$

terms of the real velocity ω and the absorption-index κ,

$$v=\frac{\omega}{1-i\kappa}, \quad \ldots \ldots \quad (7)$$

and if G, F denote the real amplitudes, and R their relative retardation,

$$\frac{g}{f}=\frac{G}{F}e^{-iR}. \quad \ldots \ldots \quad (8)$$

The formulæ (5) and (6) express the fact that in every direction there are propagated two elliptically polarized waves with, in general, different velocities and different rates of damping.

4. Of special interest are those crystals in which absorption is so weak that κ^2 may be neglected in comparison with unity, so that we may write

$$v^2=\omega^2(1+2i\kappa). \quad \ldots \ldots \quad (9)$$

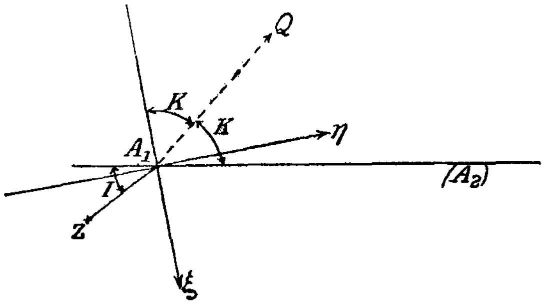
Such crystals produce no appreciable absorption in a thickness corresponding to a few wave-lengths, and are the only ones exhibiting the remarkable phenomena whose explanation forms the subject of the present paper. We have in the first place to consider the effects obtained within a small region in the neighbourhood of the optic axes defined by equations (2) ; in this region the variation of ω is, as may be shown, very slight, and the quantity $2\kappa\omega^2=k$ has a variation corresponding almost exactly to that of the analogous function κ/ω characteristic for absorption.

The two complex equations (5) and (6) thus define, within the region now considered, the real and imaginary parts of the two unknown quantities v and g/f, as well as the real velocity of propagation ω, the parameter of absorption k, the ratio G/F of the real amplitudes, and the relative retardation R.

5. Regarding these four quantities it must here suffice to state four propositions which, though only qualitative, are easily understood and extremely helpful towards an explanation of the effects considered. These propositions follow from the equations (5) and (6). In order to arrive at them, consider once more all the directions as passing through the centre of a sphere, and defined by their intersections with the spherical surface. The region surrounding an optic axis A_1 may then be approximately represented by a plane, as shown in fig. 2. We here have, besides the direction of the optic axis, represented by the point A_1, and that of the wave normal, represented by the point Z, also the plane $A_1 A_2$ of the optic axes ; A_2 being inclosed in brackets in order to

indicate the fact that the direction A_2 is not capable of being represented in the figure itself. The straight line $A_1 Q$ further represents a plane through the optic axis A_1 obtained

Fig. 2.



as follows:—Let a plane be drawn through the axes $A_1 B_1$, and another through the axes $A_1 B_2$; let the angle between them be J_1. Then $A_1 Q$ is the trace of the plane bisecting the angle J_1; let the angle made by this plane with the direction $A_1 A_2$ be denoted by K.

6. From the formulæ (5) and (6) it follows that the phenomena under consideration in the neighbourhood of the axis A_1 are symmetrical with respect to the $\xi\eta$ system of coordinates (also shown in the figure), the angle made by the $-\xi$-axis with the direction $A_1 A_2$ being equal to 2K.

Such being the case, the plane $A_1\eta$ contains two directions, C_1 and C_1' (see fig. 3), making the same angle θ with A_1, and which are highly characteristic of the behaviour of the crystal, and may be termed *singular axes*. The angle θ is determined by the tensors a_1, a_2, a_3 and b_1, b_2, b_3 which are characteristic of the crystal, and is most simply expressed in the form

$$\theta = \frac{(b_1 - b_3) \sin V_{11} \sin V_{21}}{2\sqrt{(a_1 - a_2)(a_2 - a_3)}}, \quad . \quad . \quad . \quad . \quad (10)$$

wherein V_{11} stands for the angle between A_1 and B_1, and V_{21} for that between A_1 and B_2. The angle θ thus determined is in all practically important cases *extraordinarily small.*

7. The four propositions referred to above follow from the formulæ (5) and (6), and, using the representation in the $\xi\eta$ plane, may be stated as follows:—

(*a*) The difference of the squares of the velocities of propagation, $\omega_1^2 - \omega_2^2$, of the two waves (corresponding to each direction Z)—and hence also, on account of the small difference between ω_1 and ω_2 in the region considered, the difference itself $\omega_1-\omega_2$ approximately—is constant over ellipses having the points C_1 and C_1' for foci. This difference vanishes along the straight line C_1C_1', and increases as the ellipse opens out. (The mean square of the velocity, $\frac{1}{2}(\omega_1^2+\omega_2^2)$, on the other hand, is constant along straight lines perpendicular to the direction $A_1 A_2$.)

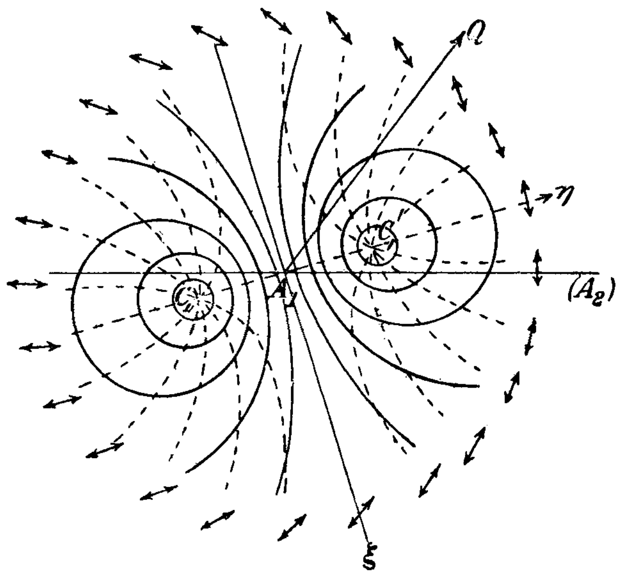
(*b*) The parameters k_1 and k_2, which determine the absorption of the two waves (corresponding to the same direction Z), are constant along hyperbolas having their foci at C_1 and C_1'. They have the same value k_0 along the straight lines obtained by producing $C_1 C_1'$ both ways, and along any hyperbola have values which differ from k_0 by equal amounts of opposite sign, the maximum difference occurring along the ξ-axis.

(*c*) The vibration-ellipses of the two waves (corresponding to the same direction Z) have constant ratios of axes along circles whose centres lie on the straight lines obtained by producing $C_1 C_1'$ both ways, and whose radii are such that all the circles cut the circle described on $C_1 C_1'$ as diameter orthogonally. The ellipses degenerate into circles at the points C_1 and C_1', and become straight lines in the ξ-axis. The direction of vibration is of opposite sense on the two sides of the ξ-axis, but is everywhere the same for the two waves.

(*d*) The principal axes of the vibration-ellipses of the two waves are crossed; their position is constant along equilateral hyperbolas whose vertices lie on the lemniscate having $C_1 C_1'$ for axis, and which pass through the points C_1 and C_1'. The coordinate-axes ξ and η are special cases of these hyperbolas. In order to determine the positions of the axes of the ellipses corresponding to these hyperbolas, it is most convenient to use the well-known theoretical result according to which in the case of *weak* absorption the vibrations, within a certain distance from the optic axes, differ only imperceptibly from those taking place in transparent crystals, and are therefore nearly rectilinear along directions determined by the famous construction due to Fresnel. According to this construction we have to draw planes through the axes Z and A_1, and Z and A_2, and to bisect the angle I between them; the ordinary wave is then polarized in a direction parallel, and the extraordinary one in a direction normal, to this bisecting plane.

In fig. 3 are shown the curves corresponding to propositions (*c*) and (*d*); in addition to this the double-headed arrows arranged round the circumference of a circle indicate the directions of polarization of the ordinary waves when the absorption is vanishingly small. In accordance with what has been said above, these arrows also give the directions of the axes *majores* for the ordinary, and of the axes *minores* for

Fig. 3.



the extraordinary waves on the neighbouring branches of the hyperbolas. It will be seen that these directions are not constant along the entire hyperbolic branches, but that they become rotated through 45° during the passage through the points C_1 and C_1' of circular polarization.

It is evident that in all these propositions the singular axes play an extremely important part—they uniquely determine, from the qualitative point of view, the effects considered.

8. Now as regards the appearances which may be noticed in the neighbourhood of an optic axis, these present themselves in looking through a plate which has been cut in a direction approximately normal to an optic axis. It is easy to deduce the formulæ for the intensities with a degree of approximation corresponding to that of the experimental observations—for a given incident light—made either with the naked eye or by means of an inserted analyser. The formulæ are without exception very complicated, but may be simplified for distances from the singular axes C_1 and C_1' such that the terms containing the squares of the ratios ϵ of

the minor to the major axis of the vibration-ellipse as factors may be neglected. The region within which the simplified formulæ are not applicable is in general extraordinarily small.

In this way an explanation is very easily obtained of the dark pencils—first noticed by Brewster—which make their appearance when a natural source of light of sufficient dimensions is viewed through the crystal, without the use of any polarizer. The distribution of intensity is such that the hyperbolas mentioned in proposition (*b*) represent curves of constant absorption. The ellipticity of the vibrations in the plate need not in the case of this phenomenon—to the degree of approximation considered—be taken into account. According to the *strictly correct* formulæ the distribution of the intensity is somewhat modified in the neighbourhood of the singular axes.

9. The ellipticity of the vibrations has, on the other hand, a marked effect on the appearances presented when a *single* polarizer is used—whether it be that natural light is allowed to fall on the plate and is examined by means of an analyser, or that plane polarized light is after transmission examined by the unaided eye. Here an opportunity presents itself for testing in a simple manner one of the most remarkable consequences of theory.

If the experiment be carried out in the manner indicated with a plate of an ordinary or an active transparent crystal, whereby two linear or two elliptic similar but crossed vibrations of *oppositely directed* sense are propagated through the crystal, then according to theory there should be no interference-rings visible around the optic axis. Should, on the other hand, the vibrations be propagated through a pleochroitic crystal whereby the two elliptic vibrations have the *same sense* of rotation, then theory demands the presence of such rings. Experiment is completely in accord with this, and thus proves the existence, hitherto not established, of these pairs of waves with vibrations of the kind described.

The existence of the rings is easily proved in many pleochroitic crystals. Frequently it is possible to see them by looking with the unaided eye through the crystal plate at the sky when the light from the latter is strongly polarized; they have in this case been known for some time, but only now has a theoretical explanation been furnished. That the explanation given above, which is based on the peculiar nature of the vibrations in the waves traversing the plate, is the correct one, follows especially from the changes which come over the appearance presented when the plane of polarization of the incident light is rotated—changes which take place precisely in the manner indicated by the formulæ. When natural incident light is used, then according to the strictly correct formulæ, the rings are not entirely absent, but, depending as they do on ϵ^2, are faintly outlined only in the immediate neighbourhood of the singular axes. As a matter of fact there is in this case visible nothing beyond a mere trace, which appears as a dark spot, of the first minimum within the Brewster's pencils.

10. Although, in accordance with what has been said above, the interference rings observable with only a single polarizer (idiophanic rings) prove the propagation of two similarly rotating elliptic waves, they do not exhibit the change in the direction of rotation during the passage through the ξ-axis, referred to in the fourth proposition. A supplementary experiment thus appears desirable, and the following arrangement would appear to meet all requirements.

If instead of plane polarized we use elliptically polarized incident light, and view it with the naked eye, then according to theory, and as is otherwise evident, the phenomenon to be observed must assume a certain dissymmetry, which was non-existent with plane polarized incident light. Now this want of symmetry is, as a matter of fact, very marked.

If we use, say, a plate of rhombic andalusite (whereby the plane $A_1\xi$ coincides with the plane of optic axes $A_1 A_2$) in white or, perhaps better, blue light, and if the plane of the polarizer be normal to that of the optic axes of the crystal, then we obtain a nearly circular system of rings, crossed by a dark band which lies in the plane of the optic axes, a phenomenon with two mutually normal lines of symmetry. If between the polarizer and the crystal we insert a quarter-wave plate, then as the latter is rotated from the position of vanishing effect, the appearance becomes entirely asymmetric, and in a manner which agrees with the predictions of theory.

The dissymmetry is particularly striking when the light incident on the plate is right or left-handed *circularly* polarized light. In this case there appears on one side of the plane of the optic axis (the ξ-axis in fig. 3) a strongly marked dark spot, and on the other a bright spot, and these spots change places if the direction of rotation is reversed.

This observation proves in the simplest manner possible that the waves propagated through the crystal on the two sides of the ξ-axis have opposite directions of rotation, thus furnishing the desired and necessary completion of the proof of the theory.

Göttingen, March 1902.