On the Composition of Group-Characteristics. By W. BURNSIDE.

Received and communicated March 14th, 1901.

In my paper "On Group-Characteristics" (*Proc. Lond. Math. Soc.*, Vol. XXXIII., p. 146) I have already given a short account of what Herr Frobenius had called their "composition."*

In the present communication I consider in greater detail the system of relations of the form

$$G_i G_j = \Sigma g_{ijk} G_k,$$

which indicate how the various irreducible representations of a group combine among themselves. The main result arrived at, which is, I believe, new, indicates how from the complete system of relations of the above form the existence of each self-conjugate sub-group which the group possesses may be deduced.

If G is a group with r sets of conjugate operations, the r distinct representations of the group as an irreducible group of linear substitutions will be denoted by $G_1, G_2, \ldots, G_r$. Of these G_1 will always be used to denote that representation in which each operation corresponds to identity. In G_i the characteristics of the conjugate sets are

$$\chi_1^i, \chi_2^i, \ldots, \chi_r^i,$$

and the number of variables is χ_1^i.

I recall briefly the process on which the so-called composition depends.

Let G_i and G_j be actually set up as groups of linear substitutions in the two distinct sets of variables

$$x_1, x_2, \ldots, x_{\chi_1^i};$$

and

$$y_1, y_2, \ldots, y_{\chi_1^j}.$$

To every operation of G there will then correspond a definite linear substitution on the $\chi_1^i \chi_1^j$ products of the x's and y's, so that G is thus

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* The process made use of by Herr Frobenius to obtain the composition of characteristics is given in a slightly different connection by M. Jordan (*Traité des Substitutions*, p. 221).

represented as a group of linear substitutions in $\chi_1^i\chi_1^j$ variables. This representation will not, in general, be irreducible. Suppose it resolved into its irreducible components, and denote by g_{ijk} the number of times that G_k occurs (for each k from 1 to r). Each symbol g_{ijk} is either zero or a positive integer. The sum of the multipliers of any operation of the p-th conjugate set in this representation will then be $\sum_{k=1}^{k=r} g_{ijk}\chi_p^k$. On the other hand, the sum of these multipliers is $\chi_p^i\chi_p^j$, in consequence of the manner in which the representation has been constructed from G_i and G_j.

Hence for each p we have the equation

$$\chi_p^i\chi_p^j = \sum_1^r g_{ijk}\chi_p^k.$$

This system of r equations among the χ's will now be represented by the single symbolical equation

$$G_i G_j = \sum_1^r g_{ijk} G_k = G_j G_i. \tag{i}$$

The complete system of r^2 equations of this form, for $i, j = 1, 2, \ldots, r$, may be primarily regarded as giving in a succinct form the result of combining any two of the irreducible representations of G by the process used above. From this point of view $G_i G_j$ may be regarded as a symbol for the group of linear substitutions on $\chi_1^i\chi_1^j$ variables constructed as above; and the notation may be extended to give a definite meaning to any symbol of the form $G_i^a G_j^b \ldots G_k^c$, or to the sum of any number of such symbols.

Equations (i) may, however, be looked at from another point of view; viz., as giving the multiplication table of a set of complex commutative numbers. In fact, if G_i, G_j, G_k be set up on the sets of variables

$$x_1, x_2, \ldots, x_{\chi_1^i},$$

$$y_1, y_2, \ldots, y_{\chi_1},$$

$$z_1, z_2, \ldots, z_{\chi_1},$$

and the resulting group of linear substitutions on the $\chi_1^i\chi_1^j\chi_1^k$ products of the x's, y's and z's be resolved into its irreducible components, the number of times that G_l occurs may be represented either by $\sum_p g_{ijp}g_{pkl}$ or by $\sum_p g_{ikp}g_{pjl}$ or by $\sum_p g_{jkp}g_{pil}$. These three numbers must therefore be the same whatever i, j, k, and l may be. But these conditions are sufficient to ensure that when $G_i G_j . G_k$ and $G_i . G_j G_k$ are calculated from equations (i) they shall have the same value.

Equations (i) are therefore a consistent system for defining the multiplication of a set of r complex commutative numbers as stated.

The analogy in form between equations (i) and the equations*

$$C_i C_j = \sum_1^r c_{ijk} C_k = C_j C_i, \tag{ii}$$

which express the way in which the conjugate sets of G combine among themselves, is complete. Moreover the latter set may be regarded as giving the multiplication table of a set of commutative complex numbers.†

Now with the system of equations (ii) what is ordinarily called the composition of the group is intimately connected. In fact, the necessary and sufficient condition that G may have a self-conjugate sub-group is that it may be possible to select a set of the C's (less than the whole) which combine by multiplication among themselves. The totality of the operations belonging to such a set then constitute a self-conjugate sub-group.

It is natural to ask whether the system of equations among the G's have not a similar connexion with the composition of the group. Let Γ be a self-conjugate sub-group of G. Among the irreducible representations of G there must be a certain number in which every operation of Γ corresponds to the identical operation (*Proc. Lond. Math. Soc.*, Vol. XXIX., pp. 563, 564). If G_i and G_j be two of these, then in the group formed from G_i and G_j by the above process of composition, every operation of Γ corresponds to the identical operations; and this is therefore true for every irreducible component that occurs in $\sum_k g_{ijk} G_k$ with a non-zero coefficient.

Hence the totality of the G's, in which the operations of Γ correspond to the identical operation, combine among themselves by multiplication. The converse of this result will now be shown to be true. The result to be proved may be stated as follows :—

THEOREM.—If a number of the irreducible representations of G (less than the whole) combine among themselves by multiplication, then G has a self-conjugate sub-group, whose operations correspond, in these representations and in these only, to the identical operation.

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* "Group-Characteristics" (p. 148).
† *Loc. cit.* (p. 148).

The proof of this theorem is facilitated by the following lemma:—
Let $X_1, X_2, \ldots, X_m$ be m linear functions of the n variables

$$x_1, x_2, \ldots, x_n;$$

subject to the sole condition that no one X is a multiple of any other. Then, if these functions be formed with s distinct sets of variables

$$\begin{array}{l} x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}; \\ x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}; \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ x_1^{(s)}, x_2^{(s)}, \ldots, x_n^{(s)}; \end{array}$$

it is always possible to ensure, by taking s sufficiently large, that the m products

$$X_1^{(1)} X_1^{(2)} \ldots X_1^{(s)}, \quad X_2^{(1)} X_2^{(2)} \ldots X_2^{(s)}, \quad \ldots, \quad X_m^{(1)} X_m^{(2)} \ldots X_m^{(s)}$$

are linearly independent.

Suppose that, when $s = t-1$, just k of the products are linearly independent; so that for each suffix i

$$X_i^{(1)} X_i^{(2)} \ldots X_i^{(t-1)} = \sum_{j=1}^{j=k} a_{ij} P_j,$$

where $P_1, P_2, \ldots, P_k$ are linearly independent. When $s = t$, the m products can be expressed linearly in terms of

$$P_a X_b^{(t)} \quad (a = 1, 2, \ldots, k;\ b = 1, 2, \ldots, m).$$

If only k of these products were linearly independent, the various $X^{(t)}$'s which multiply any one P must be multiples of each other. Also, since by supposition $k < m$, there must be at least one P which is multiplied by two different $X^{(t)}$'s. Hence the supposition that the number of linearly independent t-products is equal to the number of independent $(t-1)$-products involves that some one $X^{(t)}$ is a multiple of some other, contrary to the supposition made. The number of linearly independent t-products is therefore greater than the number of independent $(t-1)$-products; and hence by increasing t sufficiently a finite integer s may be found such that the m s-products are linearly independent.

Suppose now that G' is an irreducible group of linear substitutions in m variables, of order n'. Unless G' contains operations which multiply each variable by the same root of unity (*i.e.*, self-conjugate operations), it is always possible to choose a linear function

$$a_1 x_1 + a_2 x_2 + \dots + a_m x_m$$

of the variables which is not changed into a multiple of itself by any operation of G'. In fact, in order that this linear function may be changed into a multiple of itself by some operation S, one or more relations must hold among the a's; and it is therefore only necessary to choose the a's so that no one of a finite number of linear equations connecting them is satisfied.

Hence, if G' has no self-conjugate operations, n' linear functions of the variables

$$X_1, X_2, \dots, X_{n'}$$

may be found which are regularly permuted among themselves when the variables undergo the n' operations of the group, while also no one X is a multiple of another. These n' functions will not be linearly independent; but, by the lemma, it is possible to choose s so that the s-products

$$X_1^{(1)} X_1^{(2)} \dots X_1^{(s)}, \quad X_2^{(1)} X_2^{(2)} \dots X_2^{(s)}, \quad \dots, \quad X_{n'}^{(1)} X_{n'}^{(2)} \dots X_{n'}^{(s)}$$

formed from s distinct sets of variables are linearly independent, while they are regularly permuted among themselves when each set of variables undergoes simultaneously the substitutions of G'. Hence, among the component groups that occur in G'^s, the representation of G' as a group of regular permutations is one. But this when reduced to its irreducible components contains every irreducible representation of G'. Among the groups which arise from G' by successive multiplications by itself, every possible irreducible representation of G' will therefore occur.

Next suppose that G' has self-conjugate operations. Since they multiply each variable by the same root of unity, they must constitute a cyclical sub-group. Let p be the order of this sub-group, ω a primitive p-th root of unity, and S a self-conjugate operation which generates the sub-group. Then in this case $n'' \; (= n'/p)$ linear functions of the variables

$$X_1, X_2, \dots, X_{n''}$$

may be found, no one of which is a multiple of another, with the following properties. Each is changed into ω times itself by S, and every operation of G' which is not self-conjugate permutes the p-th powers of these functions regularly. Suppose now s chosen so large that

$$X_1^{(1)} X_1^{(2)} \dots X_1^{(s)}, \quad X_2^{(1)} X_2^{(2)} \dots X_2^{(s)}, \quad \dots, \quad X_{n''}^{(1)} X_{n''}^{(2)} \dots X_{n''}^{(s)}$$

are linearly independent; and further take s to be a multiple of p. Then the n' functions

$$X_i^{(1)} X_i^{(2)} \dots X_i^{(s)} \left[x_0^{p-1} + \omega x_0^{p-2} X_i^{(s+1)} + \omega^2 x_0^{p-3} X_i^{(s+1)} X_i^{(s+2)} + \dots \right.$$
$$\left. \dots + \omega^{p-1} X_i^{(s+1)} X_i^{(s+2)} \dots X_i^{(s+p-1)}\right]$$
$$\left(\begin{array}{l} i = 1, 2, \dots, n''; \\ \omega = \text{each } p\text{-th root of unity, including unity} \end{array}\right)$$

are linearly independent; and they are regularly permuted among themselves when the $s+p-1$ sets of variables simultaneously undergo the operations of G', while x_0 is unaltered, *i.e.*, undergoes the identical operation only. Hence, among the components of

$$G'^s G_1^{p-1} + G'^{s+1} G_1^{p-2} + \dots + G'^{s+p-1},$$

the representation of G' as a group of regular permutations occurs. Now G_1 must arise at some stage when G' is repeatedly multiplied by itself. Hence in this case again, among the groups which arise from G' by successive multiplication by itself, every possible irreducible representation of G' must occur.

Suppose now that G_i and G_j are any two of the irreducible representations of G; and that H_i and H_j are the self-conjugate sub-groups of G whose operations are represented by identity in G_i and G_j respectively. Construct, by the process which has just been investigated, a function of s (sufficiently large) sets of variables which takes n_i linearly independent values for the operations of G_i, n_i being the order of G/H_i; and construct a similar function in t sets for G_j. Call these functions P and Q. The product PQ will then remain unaltered only for those operations of G which are common to H_i and H_j. If H_{ij} is the group common to H_i and H_j, PQ will take n_{ij} distinct values for the operations of G, n_{ij} being the order of G/H_{ij}; and, since the P's and Q's are linearly independent, these n_{ij} products are so also. Hence, among the components of $G_i^s G_j^t$, there occurs a group of regular permutations, simply isomorphic with G/H_{ij}; and therefore, by the repeated multiplication of G_i and G_j, every irreducible representation of G will arise in which the operations of H_{ij} correspond to identity, and no others.

From these results the truth of the theorem stated above follows at once. Suppose in fact that, of the irreducible representations $G_1, G_2, \dots, G_r$ of a group G, a number less than r, say

$$G_1, G_2, \dots, G_s,$$

combine by multiplication among themselves. No one of these can be simply isomorphic with G, since from such a representation every other one arises by multiplication. Moreover, if $H_1, H_2, \ldots, H_s$ are the self-conjugate sub-groups of G which correspond to the identical operation in $G_1, G_2, \ldots, G_s$, then $H_1, H_2, \ldots, H_s$ must have a common sub-group H, as otherwise, by the last result, s would be equal to r. Finally, if H is the greatest common sub-group of $H_1, H_2, \ldots, H_s$, then from $G_1, G_2, \ldots, G_s$ all possible irreducible representations of G/H arise; and therefore the whole of the operations of H cannot correspond to identity in any of the remaining representations $G_{s+1}, G_{s+2}, \ldots, G_r$. The theorem is therefore completely proved.

In illustration of the foregoing the multiplication tables of the G's for the octahedral and icosahedral group have been calculated.* Each of these groups has five sets of conjugate operations, and therefore five distinct irreducible representations. For the octahedral group the numbers of variables in the distinct forms are 1, 1, 2, 3, 3. If the corresponding representations are denoted by G_1, G_2, G_3, G_4, and G_5, then, for each t,

$$G_1 G_t = G_t,$$

and the remaining products are given by the table

	G_2	G_3	G_4	G_5
G_2	G_1	G_3	G_5	G_4
G_3		$G_1+G_2+G_3$	G_4+G_5	G_4+G_5
G_4			$G_1+G_3+G_4+G_5$	$G_2+G_3+G_4+G_5$
G_5				$G_1+G_3+G_4+G_5$

For the icosahedral groups the numbers of the variables are 1, 3, 3, 4, 5, and the corresponding table is

	G_2	G_3	G_4	G_5
G_2	G_1+2G_4	G_4+G_5	$G_3+G_4+G_5$	$G_2+G_3+G_4+G_5$
G_3		G_1+2G_4	$G_2+G_4+G_5$	$G_2+G_3+G_4+G_5$
G_4			$G_1+G_2+G_3+G_4+G_5$	$G_2+G_3+G_4+2G_5$
G_5				$G_1+G_2+G_3+2G_4+2G_5$

The table for the octahedral group shows that G_1 and G_2 combine

* The calculation of these tables involves only the solution of simple equations when the characteristics are known. The determination of the latter present no difficulties in the present cases. They are given (p. 1012) in Herr Frobenius' memoir "Ueber Gruppencharaktere" (*Berliner Sitzungsberichte*, 1896).

among themselves, as also do G_1, G_2, and G_3. Every operation of the tetrahedral group which is contained as a self-conjugate sub-group is represented by identity in G_1 and G_2; and every operation of the quadratic self-conjugate sub-group of order 4 is represented by identity in G_1, G_2, and G_3.

From the table for the icosahedral group it is clear that no set of G's can be chosen which combine among themselves, and this agrees with the fact that the icosahedral group is simple.

In conclusion it may be noticed that a single relation among the G's will sometimes be sufficient to indicate the existence of a self-conjugate sub-group. Thus, if such a relation as

$$G_i G_j = \chi_1^i G_j$$

holds, those G's which arise from the repeated multiplication of G_i by itself must combine among themselves. In fact they all, when multiplying G_j, reproduce G_j, and they therefore cannot constitute the complete system of G's. Rather more generally, if a, β, γ, &c., are positive integers, the totality of G_i's for which such a relation as

$$G_i(aG_a+\beta G_b+\gamma G_c+\ldots) = \chi_1^i(aG_a+\beta G_b+\gamma G_c+\ldots)$$

holds combine among themselves. Thus, for the octahedral group G_4+G_5 is such a combination, which is reproduced on multiplication by G_1, G_2, and G_3.

Thursday, April 11th, 1901.

Dr. HOBSON, F.R.S., President, in the Chair.

Twelve members present.

Mr. Martin Adlard, B.A., Mathematical Master, King's School, Ely, and Mr. James Hopwood Jeans, B.A., Scholar of Trinity College, Cambridge, were elected members.

Mr. Basset made a short communication "On the Projective Properties of Cubic and Quartic Curves." Mr. Love also spoke on the subject.

Lt.-Col. Cunningham announced the factorisation of the algebraic prime factors of $5^{75}-1$ and $5^{105}-1$. The former

$$= 151 . 3301 . 183794551 . 99244414459501,$$

and the latter

$$= 21226783250214361 . 207468970805907721.$$

The composition of the three large factors has not been determined.

A paper by Dr. F. Morley, entitled "Summation of the Series $\Sigma_0^{\infty} \Gamma^3 (a+n)/\Gamma^3 (1+n)$," was communicated from the Chair.

The following presents were made to the Library :—

"Educational Times," April, 1901.

"Indian Engineering," Vol. xxix., Nos. 8–10, Feb. 23–March 9, 1901.

"Nautical Almanac for 1904," 8vo ; Edinburgh, 1901.

"Periodico di Matematica," Serie 2, Vol. iii., Fasc. 5 ; "Supplemento," Anno iv., Fasc. 5 ; Livorno, 1901.

"Proceedings of the American Philosophical Society," Vol. xxxix., No. 164, Oct.–Dec., 1900.

"Le Matematiche, pure ed applicate," Vol. i., Num. 1 ; 1901.

Various papers by Carl Størmer :—

"Quelques théorèmes sur l'équation de Pell

$$x^2 - Dy^2 = \pm 1$$

et leurs applications" ; Christiania, 1897.

"Om en generalisation af integralet

$$\int_0^{\infty} \frac{\sin ax}{x}\, dx = \frac{\pi}{2}\text{''};$$

Christiania, 1895.

"Sur une propriété arithmétique des logarithmes des nombres algébriques," Paris, 1900.

"Sur une équation indéterminée" ; Paris, 1898.

"Solution complète en nombres entiers de l'équation

$$m \operatorname{arc} \tan \frac{1}{x} + n \operatorname{arc} \tan \frac{1}{y} = k\, \frac{\pi}{4}\text{''};$$

Paris, 1899.

Various papers by N. J. Hatzidakis :—

"Displacements depending on One, Two, ..., k Parameters in a Space of n Dimensions," 4to.

"Sur les équations cinématiques fondamentales des variétés dans l'espace à n dimensions" (*Comptes Rendus*, 1900).

"Συμβόλη εἰς τὴν διαφοφίκην γεωμέτριαν τῶν n διαστασέων," Tom i.b, and βιβλιοκρισιαί ; Athens, 1900.

"Remarque sur une formule de M. Pirondini" ; 1899.

"Sur quelques points de la terminologie mathématique."

"Deux démonstrations nouvelles des théorèmes d'Euler et de Meusnier" ; 1899.

"Electricité et Optique: la Lumière et ses Théories électrodynamiques," par H. Poincaré (2nd edition, L. Blondin et E. Néculcéa, Paris, 1901).

The following exchanges were received:-

"Proceedings of the Royal Society," Vol. LXVIII., No. 443; 1901.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXV., Hefte 2, 3; Leipzig, 1901.

"Bulletin de la Société Mathématique de France," Tome XXIX., Fasc. 1; Paris, 1901.

"Bulletin of the American Mathematical Society," Vol. VII., No. 6; New York, March, 1901.

"Bulletin des Sciences Mathématiques," Tome XXIV., Dec.; Paris, 1900.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. X., Fasc. 5; Roma, 1901.

"Proceedings of the Physical Society," Vol. XVII., Pt. 5; March, 1901.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Math.-Phys. Klasse, Heft 4; 1900.

Thursday, May 9th, 1901.

Dr. HOBSON, F.R.S., President, in the Chair.

Fifteen members present.

At the President's request Prof. Elliott spoke briefly on the loss the Society had sustained by the death of Mr. C. E. Bickmore (elected February 11th, 1875).

Dr. John Alexander Third was elected a member of the Society, and Prof. Steggall was admitted into the Society.

Major MacMahon communicated two notes: (i.) "On the Series whose Terms are the Cubes and Higher Powers of the Binomial Coefficients," and (ii.) "A Case of Algebraic Partitionment."

Mr. J. B. Dale read a paper on "The Product of Two Spherical Surface Harmonic Functions."

Mr. Macdonald communicated a "Note on the Zeros of the Spherical Harmonic $P_n^{-m}(\mu)$.

A note on "A Property of Recurring Series," by Mr. G. B. Mathews, was read by title.

The following presents were made to the Library:—

"Educational Times," May, 1901.

"Indian Engineering," Vol. XXIX., Nos. 11-15, March 16-April 13, 1901.

"Wiadomósci Matematyczne," Tom V., Zeszyt 1-3; Warsaw, 1901.

Valentin, G. (offprints from "Bibliotheca Mathematica"):—

"Eine Seltene schrift über Winkeldreitheilung."—"Die Beiden Euclid Ausgaben des Jahres 1482"; Berlin, 1893.

"Die Frauen in den exakten Wissenschaften"; Berlin, 1895.

"Beitrag zur Bibliographie der Euler'schen Schriften"; Berlin, 1898.

"Die Vorarbeiten für die allgemeine mathematische Bibliographie": Leipzig.

Valentin, G.—"De aequatione algebraica quae est inter duas variabiles, in quandam formam canonicam transformata," pamphlet, 4to; Berolini. (Dissertation for Ph.D. degree at Berlin, July 5th, 1879.)

"Mathematical Gazette," Vol. I., No. 27; May, 1901.

"Supplemento al Periodico di Matematica," Anno IV., Fasc. 6, Aprile 1901: Livorno.

"Annals of Mathematics," Series 2, Vol. II., No. 3; Harvard University, 1901.

"Annales de la Faculté des Sciences," Série 2, Tome II.; Toulouse, 1900.

"Washington Observations, 1891–1892," 4to; Washington, 1899.

"A Binary Canon, showing Residues of Powers of 2 for Divisors under 1000, and Indices to Residues," compiled by Lt.-Col. Allan Cunningham, R.E., under the auspices of a British Association Committee; London, 1900. From the author.

The following exchanges were received:—

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXV., Heft 4: Leipzig, 1901.

"Bulletin of the American Mathematical Society," Series 2, Vol. VII., No. 7: New York, April, 1901.

"Monatshefte für Mathematik und Physik," Jahrgang XII., Parts 2 and 3; Wien, 1901.

"Bulletin des Sciences Mathématiques," Tome XXV., Jan., Fév.; Paris, 1901.

"Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. VII., Fasc. 3; Napoli, 1901.

"Atti dell'Accademia delle Scienze Fisiche e Matematiche," Vol. X.; Napoli, 1901.

"Journal für die reine und angewandte Mathematik," Bd. CXXIII., Heft 2; Berlin, 1901.

"Archives Néerlandaises," Serie 2, Tome IV., Liv. 2; La Haye, 1901.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. X., Fasc. 6, 7; Roma, 1901.

"Journal of the Institute of Actuaries," Vol. XXXVI., Pt. 1, No. 201; 1901.

"Proceedings of the Cambridge Philosophical Society," Vol. XI., Pt. 2; 1901.