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XLI. *On Steady Motion in an Incompressible Viscous Fluid.*
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IN the following paper an attempt has been made to give the theory of steady motion in an ordinary viscous fluid, and the results of the general investigation applied to the case when there is a solid body of given form immersed in the fluid, and the fluid moves subject to the condition that at infinity all the flow is parallel to the axis of z .

The general equations of motion of an incompressible viscous fluid are well known to be

$$\left. \begin{aligned} \rho \frac{Du}{dt} &= \rho X - \frac{dp}{dx} + \mu \nabla^2 u, \\ \rho \frac{Dv}{dt} &= \rho Y - \frac{dp}{dy} + \mu \nabla^2 v, \\ \rho \frac{Dw}{dt} &= \rho Z - \frac{dp}{dz} + \mu \nabla^2 w, \end{aligned} \right\} \dots \dots (1)$$

maintained, and actually *is* maintained in the universe, would become strictly true, in so far as the limits within which such uniformity of temperature is non-existent would be (relatively speaking) indefinitely small.

The late Prof. Clerk Maxwell remarks ('Theory of Heat,' p. 163) :— "The transference of heat, therefore, from one body of the system to another always increases the 'entropy' of the system. Clausius expresses this by saying that the entropy of the system always tends towards a maximum value." It appears, therefore, that, on the above view that the universe is already in a state of equilibrium of temperature, the "entropy" of the universe would already have reached a maximum value, which continually tends to be maintained. The theory of finality in the universe seems to have been discussed with considerable interest in Germany. A critical notice of this subject may be found in Lange's *Geschichte des Materialismus* (of which I believe an English translation now exists).

It should be mentioned [as also noticed in my previous essays] that Dr. Croll, in the *Phil. Mag.* for May 1868 and July 1878, also Mr. Johnstone Stoney (Proceedings of the Royal Society 1868), have published papers dealing with the eventuality of encounters among the stars, and suggested views which may, as far as they go, be regarded as consistent with the development of an encircling theory suggested in this and my former essays. Also the fact of the present theory having been arrived at before seeing the papers above cited, naturally tended rather to afford some confidence in the result.

The entertaining of the above theory would be no more than admitting the possible application of the principle of evolution to universal changes as well as to planetary changes: the birth and death of worlds comparable to the birth and death of individuals. Or the secular fluctuations of life in the cosmical units of the universe would be paralleled by those of the individuals of a planet—while the conservation of life has its correlative in the conservation of energy and of matter.

* Communicated by A. G. Greenhill, Esq.

in which u, v, w denote the component velocities of a fluid particle in the directions of x, y, z respectively, ρ is the density of the fluid, μ the coefficient of viscosity, and

$$\frac{D}{dt} = \frac{d}{dt} + u \frac{d}{dx} + v \frac{d}{dy} + w \frac{d}{dz},$$

$$\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}.$$

Denoting the components of the spin by ξ, η, ζ , we have

$$\left. \begin{aligned} \xi &= \frac{1}{2} \left(\frac{dw}{dy} - \frac{dv}{dz} \right), \\ \eta &= \frac{1}{2} \left(\frac{du}{dz} - \frac{dw}{dx} \right), \\ \zeta &= \frac{1}{2} \left(\frac{dv}{dx} - \frac{du}{dy} \right). \end{aligned} \right\} \dots \dots \dots (2)$$

We may observe just here, that from equations (2) we can obtain

$$\left. \begin{aligned} \nabla^2 u &= \frac{d\eta}{dz} - \frac{d\zeta}{dy}, \\ \nabla^2 v &= \frac{d\zeta}{dx} - \frac{d\xi}{dz}, \\ \nabla^2 w &= \frac{d\xi}{dy} - \frac{d\eta}{dx}. \end{aligned} \right\} \dots \dots \dots (3)$$

In the case of steady motion we have

$$\frac{du}{dt} = \frac{dv}{dt} = \frac{dw}{dt} = 0;$$

and equations (1) thus assume the form

$$\left. \begin{aligned} \rho \left\{ u \frac{du}{dx} + v \frac{du}{dy} + w \frac{du}{dz} - X \right\} + \frac{dp}{dx} &= \mu \nabla^2 u, \\ \rho \left\{ u \frac{dv}{dx} + v \frac{dv}{dy} + w \frac{dv}{dz} - Y \right\} + \frac{dp}{dy} &= \mu \nabla^2 v, \\ \rho \left\{ u \frac{dw}{dx} + v \frac{dw}{dy} + w \frac{dw}{dz} - Z \right\} + \frac{dp}{dz} &= \mu \nabla^2 w. \end{aligned} \right\} \dots (4)$$

Assume that the forces X, Y, Z possess a potential V , and write

$$q^2 = u^2 + v^2 + w^2:$$

these equations can now be transformed into the following:—

$$\left. \begin{aligned} \frac{dP}{dx} &= 2(v\xi - w\eta) + \tau \nabla^2 u, \\ \frac{dP}{dy} &= 2(w\xi - u\eta) + \tau \nabla^2 v, \\ \frac{dP}{dz} &= 2(u\eta - v\xi) + \tau \nabla^2 w, \end{aligned} \right\} \quad . \quad . \quad . \quad (5)$$

where $\tau = \frac{\mu}{\rho}$ is the “kinematic coefficient of viscosity,” and

$$P = \int \frac{dp}{\rho} + V + \frac{1}{2}q^2.$$

For $\mu=0$, these become the equations given by Lamb in the ‘Proceedings’ of the London Mathematical Society. In the case of $\mu=0$, we know (*vide* Lamb’s ‘Treatise on Fluid Motion,’ page 173)

“that

$$\begin{aligned} u \frac{dP}{dx} + v \frac{dP}{dy} + w \frac{dP}{dz} &= 0, \\ \xi \frac{dP}{dx} + \eta \frac{dP}{dy} + \zeta \frac{dP}{dz} &= 0; \end{aligned}$$

“so that the surfaces

$$P = \text{constant}$$

“contain both stream- and vortex-lines. Further, denoting by dn an element of the normal to such a surface at any point, we have

$$\frac{dP}{dn} = q\omega \sin \theta;$$

where ω is the spin, and θ is the angle between the stream-line and vortex-line at the point where the normal is drawn.

“The conditions, then, that a given state of motion of a perfect fluid may be a possible state of steady motion are as follows:—It must be possible to draw in the fluid an infinite number of surfaces each of which is covered by a network of stream-lines and vortex-lines; and the product $q\omega \sin \theta dn$ must be constant over each such surface, dn being the length of the normal drawn to a consecutive surface of the system.”

For the case of a viscous fluid, the reductions are as simple as those which lead to the above results. To the equations of motion (5) it is of course necessary to add the equation of continuity,

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0;$$

we have also the analogous equation for ξ, η, ζ ,

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0.$$

From equations (5) we obtain

$$\left. \begin{aligned} u \left\{ \frac{dP}{dx} - 2\tau \left(\frac{d\eta}{dz} - \frac{d\zeta}{dy} \right) \right\} + v \left\{ \frac{dP}{dy} - 2\tau \left(\frac{d\xi}{dz} - \frac{d\zeta}{dx} \right) \right\} \\ + w \left\{ \frac{dP}{dz} - 2\tau \left(\frac{d\xi}{dy} - \frac{d\eta}{dx} \right) \right\} = 0, \\ \xi \left\{ \frac{dP}{dx} - 2\tau \left(\frac{d\eta}{dz} - \frac{d\zeta}{dy} \right) \right\} + \eta \left\{ \frac{dP}{dy} - 2\tau \left(\frac{d\xi}{dz} - \frac{d\zeta}{dx} \right) \right\} \\ + \zeta \left\{ \frac{dP}{dz} - 2\tau \left(\frac{d\xi}{dy} - \frac{d\eta}{dx} \right) \right\} = 0. \end{aligned} \right\} \quad (6)$$

If we have a surface $\Theta = \text{const.}$ defined by the differential equations

$$\left. \begin{aligned} \frac{d\Theta}{dx} &= \frac{dP}{dx} - 2\tau \left(\frac{d\eta}{dz} - \frac{d\zeta}{dy} \right), \\ \frac{d\Theta}{dy} &= \frac{dP}{dy} - 2\tau \left(\frac{d\xi}{dz} - \frac{d\zeta}{dx} \right), \\ \frac{d\Theta}{dz} &= \frac{dP}{dz} - 2\tau \left(\frac{d\xi}{dy} - \frac{d\eta}{dx} \right), \end{aligned} \right\} \quad \dots \quad (7)$$

these equations may be written

$$\left. \begin{aligned} u \frac{d\Theta}{dx} + v \frac{d\Theta}{dy} + w \frac{d\Theta}{dz} &= 0, \\ \xi \frac{d\Theta}{dx} + \eta \frac{d\Theta}{dy} + \zeta \frac{d\Theta}{dz} &= 0, \end{aligned} \right\} \quad \dots \quad (8)$$

and we have, as a result, that there exist in the fluid an infinite number of surfaces $\Theta = \text{constant}$, each of which is covered by a network of stream-lines and vortex-lines.

The expression

$$\left\{ \frac{dP}{dx} - 2\tau \left(\frac{d\eta}{dz} - \frac{d\zeta}{dy} \right) \right\} dx + \&c.$$

is not an exact differential unless $\nabla^2 \xi = 0, \nabla^2 \eta = 0, \nabla^2 \zeta = 0$; and consequently equations (7) do not always hold. Equations (6), however, must always hold, as they are obtained independently of the supposition contained in (7). Equations (6) may be written, for brevity, in the form

$$\left. \begin{aligned} u\Phi_1 + v\Phi_2 + w\Phi_3 &= 0, \\ \xi\Phi_1 + \eta\Phi_2 + \zeta\Phi_3 &= 0. \end{aligned} \right\} \quad \dots \quad (9)$$

Write

$$\Omega = \sqrt{\Phi_1^2 + \Phi_2^2 + \Phi_3^2};$$

then we see from (9) that the stream-lines and vortex-lines always lie on a surface the direction-cosines of a normal to which are

$$\frac{\Phi_1}{\Omega}, \quad \frac{\Phi_2}{\Omega}, \quad \frac{\Phi_3}{\Omega}.$$

We shall confine ourselves at present to the case where

$$\Phi_1 = \frac{d\Theta}{dx}, \text{ \&c.,}$$

i. e.

$$\Phi_1 dx + \Phi_2 dy + \Phi_3 dz$$

is an exact differential $d\Theta$. It is clear that, in order that this quantity may be an exact differential, it is necessary only that

$$2\tau \left\{ \left(\frac{d\eta}{dz} - \frac{d\xi}{dy} \right) dx + \left(\frac{d\xi}{dx} - \frac{d\xi}{dz} \right) dy + \left(\frac{d\xi}{dy} - \frac{d\xi}{dx} \right) dz \right\}$$

or, briefly,

$$A dx + B dy + C dz$$

shall be such a quantity. The conditions which have to be fulfilled on this supposition are well known to be

$$\frac{dA}{dy} - \frac{dB}{dx} = 0, \text{ \&c.,}$$

or, substituting for A, B, C their values,

$$\nabla^2 \xi = 0, \quad \nabla^2 \eta = 0, \quad \nabla^2 \zeta = 0. \quad . . . \quad (10)$$

Since, now,

$$\left. \begin{aligned} \frac{d\Theta}{dx} &= 2(v\xi - w\eta), \\ \frac{d\Theta}{dy} &= 2(w\xi - u\zeta), \\ \frac{d\Theta}{dz} &= 2(u\eta - v\xi), \end{aligned} \right\} (11)$$

equations (5) become

$$\left. \begin{aligned} \frac{d(P - \Theta)}{dx} &= 2\tau \left(\frac{d\eta}{dz} - \frac{d\xi}{dy} \right), \\ \frac{d(P - \Theta)}{dy} &= 2\tau \left(\frac{d\xi}{dx} - \frac{d\xi}{dz} \right), \\ \frac{d(P - \Theta)}{dz} &= 2\tau \left(\frac{d\xi}{dy} - \frac{d\eta}{dx} \right). \end{aligned} \right\} . . . \quad (12)$$

Differentiating these for x, y , and z respectively and adding, we have

$$\nabla^2(P - \Theta) = 0, \quad \dots \quad (13)$$

or

$$\nabla^2 P = \nabla^2 \Theta. \quad \dots \quad (14)$$

Denoting by θ the angle between the stream-line and the vortex-line at the point where the normal n is drawn to the surface $\Theta = \text{const.}$, we have, since

$$\omega = \sqrt{\xi^2 + \eta^2 + \zeta^2}.$$

and

$$q\omega \sin \theta = \sqrt{(v\zeta - w\eta)^2 + (w\xi - u\zeta)^2 + (u\eta - v\xi)^2},$$

$$\frac{d\Theta}{dn} = q\omega \sin \theta. \quad \dots \quad (15)$$

From this, as in the case of no viscosity, follows that the product

$$q\omega \sin \theta \, dn$$

must be constant over each of the surfaces $\Theta = \text{const.}$, dn denoting the normal drawn to the consecutive surface.

For the determination of the pressure p it will be convenient to resume equations (4). Since $u = q \frac{dx}{ds}$, &c., these become

$$\left. \begin{aligned} \frac{1}{\rho} \frac{dp}{dx} &= -\frac{dV}{dx} - q \frac{du}{ds} + \tau \nabla^2 u, \\ \frac{1}{\rho} \frac{dp}{dy} &= -\frac{dV}{dy} - q \frac{dv}{ds} + \tau \nabla^2 v, \\ \frac{1}{\rho} \frac{dp}{dz} &= -\frac{dV}{dz} - q \frac{dw}{ds} + \tau \nabla^2 w. \end{aligned} \right\} \quad \dots \quad (16)$$

Multiplying these by $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ respectively and adding these results,

$$\frac{1}{\rho} \frac{dp}{ds} = \frac{d(P - \Theta)}{ds} - \frac{dV}{ds} - \frac{1}{2} \frac{d(q^2)}{ds};$$

from which, since ρ is constant,

$$p = \rho \left\{ (P - \Theta) - V - \frac{1}{2} q^2 \right\} + C. \quad \dots \quad (17)$$

As we have integrated along the stream-line $\int ds$, the quantity C is only constant for this particular line.

Let us assume that a solid sphere is immersed in the fluid, the latter moving past the sphere in such a manner that the

motion is steady. If we write

$$\left. \begin{aligned} u &= \frac{d\phi}{dx} + \frac{dW}{dy} - \frac{dV}{dz}, \\ v &= \frac{d\phi}{dy} + \frac{dU}{dz} - \frac{dW}{dx}, \\ w &= \frac{d\phi}{dz} + \frac{dV}{dx} - \frac{dU}{dy}, \end{aligned} \right\} \dots \dots \dots (18)$$

the equation of continuity

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$$

will be satisfied, provided ϕ satisfies the condition

$$\nabla^2 \phi = 0. \dots \dots \dots (19)$$

The remaining functions U, V, W must, as is well known, satisfy the following conditions:—

$$\left. \begin{aligned} \nabla^2 U &= -2\xi, \quad \nabla^2 V = -2\eta, \quad \nabla^2 W = -2\zeta, \\ \frac{dU}{dx} + \frac{dV}{dy} + \frac{dW}{dz} &= 0. \end{aligned} \right\} \dots (20)$$

The origin of coordinates is to be taken at the centre of the sphere; then, if r denote the distance from the centre to any point x, y, z , we have for the transformation to polar coordinates,

$$\begin{aligned} x &= r \sin \chi \cos \psi, \\ y &= r \sin \chi \sin \psi, \\ z &= r \cos \chi; \end{aligned}$$

the equation $\nabla^2 \phi = 0$ now becomes

$$r \frac{d^2(r\phi)}{dr^2} + \frac{1}{\sin \chi} \frac{d}{d\chi} \left(\sin \chi \frac{d\phi}{d\chi} \right) + \frac{1}{\sin^2 \chi} \frac{d^2 \phi}{d\psi^2} = 0.$$

The most general solution of this in spherical harmonics is

$$\begin{aligned} \phi &= \alpha_0 Y_0 + \alpha_1 r Y_1 + \alpha_2 r^2 Y_2 + \dots \\ &+ \beta_0 \frac{Y_0}{r} + \beta_1 \frac{Y_1}{r^2} + \beta_2 \frac{Y_2}{r^3} + \dots, \end{aligned}$$

or

$$\phi = \sum_0^\infty \left(\alpha_i r^i + \frac{\beta_i}{r^{i+1}} \right) Y_i,$$

Y_i being a surface spherical harmonic of degree i . Write $\phi_i = \alpha_i r^i Y_i$; ϕ_i is a solid spherical harmonic of degree i ; this gives

$$\phi = \sum_0^\infty \left(1 + \frac{\beta_i}{\alpha_i} r^{-(2i+1)} \right) \phi_i. \dots \dots \dots (21)$$

For the determination of the constants α and β we have only one condition; so that at most we can only find one of them in terms of the other. This condition is obtained by equating to zero the normal velocity at the surface of the sphere; if the radius of the sphere is a , we must have

$$\frac{d\phi}{dr} = 0 \text{ for } r = a.$$

Differentiating ϕ with respect to r gives

$$\frac{d\psi}{dr} = \sum_0^n (i\alpha_i r^{i-1} - (i+1)\beta_i r^{-(i+2)}) Y_i;$$

equating this to zero and making $r=a$, we find

$$i\alpha_i a^{i-1} = (i+1)\beta_i a^{-(i+2)},$$

from which

$$\beta_i = \frac{i}{i+1} \alpha_i a^{2i+1}. \quad \dots \dots \dots (22)$$

Substituting this value of β_i in the last form of ϕ , we have

$$\phi = \sum_0^n \left\{ 1 + \frac{i}{i+1} \left(\frac{a}{r} \right)^{2i+1} \right\} \phi_i. \quad \dots \dots (23)$$

For the determination of the functions U, V, W , we have

$$\nabla^2 U = -2\xi, \quad \nabla^2 V = -2\eta, \quad \nabla^2 W = -2\zeta,$$

with the condition

$$\frac{dU}{dx} + \frac{dV}{dy} + \frac{dW}{dz} = 0.$$

Equations (10) hold for the particular case of motion that we are studying; and therefore

$$\nabla^2(\nabla^2 U) = 0, \quad \nabla^2(\nabla^2 V) = 0, \quad \nabla^2(\nabla^2 W) = 0.$$

The functions U, V, W can be expressed in a series the terms of which depend upon the general spherical harmonics. The following solution is merely a generalization of one given by Mr. J. G. Butcher in the 'Proceedings' of the London Mathematical Society. Mr. Butcher says, in his article "On Viscous Fluids in Motion," that the method is due to Stokes, to whose article, however, I am not able to refer.

The equations to be solved are of the form

$$\left. \begin{aligned} \nabla^2 S &= -s, \\ \nabla^2 s &= 0. \end{aligned} \right\} \quad \dots \dots \dots (24)$$

In the problem solved by Mr. Butcher the solution is made to
Phil. Mag. S. 5. Vol. 10. No. 63. Nov. 1880. 2C

depend upon zonal harmonics ; merely generalizing his solution, I am led to write for S the series

$$S = \frac{a_0 S_0}{r} + \frac{a_1 S_1}{r^3} + \frac{a_2 S_2}{r^5} + \dots \\ + \frac{b_1 S_1}{r} + \frac{b_2 S_2}{r^3} + \frac{b_3 S_3}{r^5} + \dots,$$

or simply

$$S = \sum_0^{\infty} \frac{a_i + b_i r^2}{r^{2i+1}} S_i, \quad \dots \quad (25)$$

S_i being a solid spherical harmonic of degree i . For brevity, make

$$R_i = \frac{a_i + b_i r^2}{r^{2i+1}};$$

then

$$S = \sum_0^{\infty} R_i S_i. \quad \dots \quad (26)$$

In order to find the values of the constants, it is necessary to take into account the conditions to be fulfilled at the surface of the sphere, *i. e.* at the finite bounding surface of the fluid. These are

$$\left. \begin{aligned} u &= 0, \\ v &= 0, \\ w &= 0, \end{aligned} \right\} \text{ for } r = a. \quad \dots \quad (27)$$

We must first determine the function U, V, W , however, before these boundary conditions can be introduced. The following method is due to Borchardt, and is given in the *Monatsber. der Berlin. Akad.* for 1873. If we have a function s of x, y, z satisfying the equation

$$\nabla^2 s = 0, \quad \dots \quad (28)$$

and four other functions connected with s by the relations

$$\left. \begin{aligned} s_0 &= s + x \frac{ds}{dx} + y \frac{ds}{dy} + z \frac{ds}{dz}, \\ s_1 &= z \frac{ds}{dy} - y \frac{ds}{dz}, \\ s_2 &= x \frac{ds}{dz} - z \frac{ds}{dx}, \\ s_3 &= y \frac{ds}{dx} - x \frac{ds}{dy}, \end{aligned} \right\} \quad \dots \quad (29)$$

then there results

$$\left. \begin{aligned} \nabla^2 s_0 = \nabla^2 s_1 = \nabla^2 s_2 = \nabla^2 s_3 = 0, \\ \frac{ds_1}{dx} + \frac{ds_2}{dx} + \frac{ds_3}{dx} = 0, \end{aligned} \right\} \dots \dots (30)$$

and also

$$\left. \begin{aligned} \frac{ds_3}{dy} - \frac{ds_2}{dz} &= \frac{ds_0}{dx}, \\ \frac{ds_1}{dz} - \frac{ds_3}{dx} &= \frac{ds_0}{dy}, \\ \frac{ds_2}{dx} - \frac{ds_1}{dy} &= \frac{ds_0}{dz}. \end{aligned} \right\} \dots \dots \dots (31)$$

If $s = \phi_i$, then

$$s_0 = (i+1)\phi_i \dots \dots \dots (32)$$

Since the functions s_1, s_2, s_3 fulfil all the conditions to which ξ, η, ζ are subject, we may write

$$\xi = s_1, \quad \eta = s_2, \quad \zeta = s_3, \quad \dots \dots \dots (33)$$

s itself being still an arbitrary function. We can consequently write

$$\left. \begin{aligned} U &= z \frac{dS}{dy} - y \frac{dS}{dz}, & \xi &= z \frac{ds}{dy} - y \frac{ds}{dz}, \\ V &= x \frac{dS}{dz} - z \frac{dS}{dx}, & \eta &= x \frac{ds}{dz} - z \frac{ds}{dx}, \\ W &= y \frac{dS}{dx} - x \frac{dS}{dy}, & \zeta &= y \frac{ds}{dx} - x \frac{ds}{dy}, \end{aligned} \right\} \dots \dots (34)$$

when all the conditions to which these quantities are subject will be fulfilled, provided

$$\nabla^2 S = -s \text{ and } \nabla^2 s = 0.$$

Using the solution already given for the determination of S , we can now write

$$U = \sum_0^{\infty} R_i \left(z \frac{dS_i}{dy} - y \frac{dS_i}{dz} \right) \&c., \quad \dots \dots (35)$$

since

$$z \frac{dR_i}{dy} - y \frac{dR_i}{dz} = 0 \&c.$$

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Instead of (35), we may write

$$\left. \begin{aligned} U_i &= z \frac{dS_i}{dy} - y \frac{dS_i}{dz}, \\ V_i &= x \frac{dS_i}{dz} - z \frac{dS_i}{dx}, \\ W_i &= y \frac{dS_i}{dx} - x \frac{dS_i}{dy}, \end{aligned} \right\} \dots \dots \dots (36)$$

and

$$U = \sum_0^{\infty} R_i U_i, \quad V = \sum_0^{\infty} R_i V_i, \quad W = \sum_0^{\infty} R_i W_i. \quad (37)$$

We have now for u, v, w

$$\left. \begin{aligned} u &= \frac{d\phi}{dx} + \frac{d}{dy} \sum R_i W_i - \frac{d}{dz} \sum R_i V_i, \\ v &= \frac{d\phi}{dy} + \frac{d}{dz} \sum R_i U_i - \frac{d}{dx} \sum R_i W_i, \\ w &= \frac{d\phi}{dz} + \frac{d}{dx} \sum R_i V_i - \frac{d}{dy} \sum R_i U_i. \end{aligned} \right\} \dots \dots (38)$$

Since ϕ is expressed by a series of the same form as those giving U, V , and W , we may write

$$\phi = \sum_0^{\infty} L_i \phi_i.$$

For the surface conditions we now have i groups of equations, of which the following is the type:—

$$\left. \begin{aligned} \frac{d(L_i \phi_i)}{dx} + \frac{d(R_i W_i)}{dy} - \frac{d(R_i V_i)}{dz} &= 0; \\ \frac{d(L_i \phi_i)}{dy} + \frac{d(R_i U_i)}{dz} - \frac{d(R_i W_i)}{dx} &= 0; \\ \frac{d(L_i \phi_i)}{dz} + \frac{d(R_i V_i)}{dx} - \frac{d(R_i U_i)}{dy} &= 0. \end{aligned} \right\} \dots \dots (39)$$

The first one of these can be written in the form

$$\left. \begin{aligned} L_i \frac{d\phi_i}{dx} + R_i \left(\frac{dW_i}{dy} - \frac{dV_i}{dz} \right) \\ = V_i \frac{dR_i}{dz} - W_i \frac{dR_i}{dy} - \phi_i \frac{dL_i}{dx}. \end{aligned} \right\} \dots \dots (40)$$

Now

$$\begin{aligned} \frac{dW_i}{dy} &= \frac{dS_i}{dx} + y \frac{d^2 S_i}{dx dy} - x \frac{d^2 S_i}{dy dz}; \\ \frac{dV_i}{dz} &= x \frac{d^2 S_i}{dz^2} - z \frac{d^2 S_i}{dx dz} - \frac{dS_i}{dx}; \end{aligned}$$

therefore

$$\frac{dW_i}{dy} - \frac{dV_i}{dz} = -x\nabla^2 S_i + \left(x \frac{d}{dx} + y \frac{d}{dy} + z \frac{d}{dz} + 2\right) \frac{dS_i}{dx}.$$

But S_i is a solid spherical harmonic of degree i , and therefore satisfies the equations

$$\nabla^2 S_i = 0, \quad \left(x \frac{d}{dx} + y \frac{d}{dy} + z \frac{d}{dz}\right) \frac{dS_i}{dx} = (i-1) \frac{dS_i}{dx};$$

and consequently

$$\frac{dW_i}{dy} - \frac{dV_i}{dz} = (i+1) \frac{dS_i}{dx}.$$

Again

$$\begin{aligned} V_i \frac{dR_i}{dz} - W_i \frac{dR_i}{dy} &= \frac{1}{r} \frac{dR_i}{dr} (zV_i - yW_i) \\ &= \frac{1}{r} \frac{dR_i}{dr} \left\{ x \left(x \frac{d}{dx} + y \frac{d}{dy} + z \frac{d}{dz} \right) S_i - r^2 \frac{dS_i}{dx} \right\} \\ &= \frac{1}{r} \frac{dR_i}{dr} \left\{ ixS_i - r^2 \frac{dS_i}{dx} \right\}. \end{aligned}$$

Equation (40) now becomes

$$\begin{aligned} L_i \frac{d\phi_i}{dx} + R_i(i+1) \frac{dS_i}{dx} \\ = \frac{1}{r} \frac{dR_i}{dr} \left\{ ixS_i - r^2 \frac{dS_i}{dx} \right\} - \phi_i \frac{dL_i}{dx}. \quad (41) \end{aligned}$$

The second and third of equations (39) give two others similar to (41). Since

$$\frac{dL_i}{dx} = \frac{dL_i}{dr} \cdot \frac{x}{r}, \quad \frac{dL_i}{dy} = \frac{dL_i}{dr} \cdot \frac{y}{r}, \quad \frac{dL_i}{dz} = \frac{dL_i}{dr} \cdot \frac{z}{r},$$

these may all be written as

$$\left. \begin{aligned} L_i \frac{d\phi_i}{dx} + \frac{x}{r} \cdot \phi_i \frac{dL_i}{dr} &= i \frac{x}{r} \frac{dR_i}{dr} S_i - \left\{ (i+1)S_i + r \frac{dR_i}{dr} \right\} \frac{dS_i}{dx}, \\ L_i \frac{d\phi_i}{dy} + \frac{y}{r} \cdot \phi_i \frac{dL_i}{dr} &= i \frac{y}{r} \frac{dR_i}{dr} S_i - \left\{ (i+1)S_i + r \frac{dR_i}{dr} \right\} \frac{dS_i}{dy}, \\ L_i \frac{d\phi_i}{dz} + \frac{z}{r} \cdot \phi_i \frac{dL_i}{dr} &= i \frac{z}{r} \frac{dR_i}{dr} S_i - \left\{ (i+1)S_i + r \frac{dR_i}{dr} \right\} \frac{dS_i}{dz}. \end{aligned} \right\} \quad (42)$$

These equations are to be satisfied only when $r=a$, and mani-

festly are satisfied only when

$$\left. \begin{aligned} \phi_i &= S_i, \\ L_i &= - \left\{ (i+1)R_i + r \frac{dR_i}{dr} \right\}, \\ \frac{dL_i}{dr} &= i \frac{dR_i}{dr}, \end{aligned} \right\} \quad . \quad . \quad . \quad (43)$$

which are then the conditions to be fulfilled for $u=v=w=0$. Referring now to the values of R_i and L_i , we have

$$\left. \begin{aligned} L_i &= \frac{2i+1}{i+1}, \\ N_i &= \frac{a_i + b_i a^2}{a^{2i+1}}, \end{aligned} \right\} \text{ for } r=a, \quad . \quad . \quad . \quad (44)$$

$$\left. \begin{aligned} r \frac{dR_i}{dr} &= \frac{-(2i+1)a_i + (2i-1)b_i a^3}{a^{2i+1}}, \\ \frac{dL_i}{dr} &= - \frac{i(2i+1)}{i+1} \cdot \frac{1}{a}, \end{aligned} \right\} \text{ for } r=a. \quad . \quad (45)$$

Constructing, now, the last of the condition-equations (43), there result

$$\left. \begin{aligned} (2i+1)a_i + (2i-1)b_i a^2 &= \frac{2i+1}{i+1} a^{2i+1}, \\ ia_i + (i-2)b_i a^2 &= \frac{2i+1}{i+1} a^{2i+1}; \end{aligned} \right\} \quad . \quad (46)$$

from which we derive

$$\left. \begin{aligned} a_i &= \frac{1}{2} \cdot \frac{2i+1}{i+1} a^{2i+1}, \\ b_i &= - \frac{1}{2} \cdot \frac{2i+1}{i+1} a^{2i-1}; \end{aligned} \right\} \quad . \quad . \quad . \quad (47)$$

and consequently

$$S = \frac{1}{2} \sum_0^\infty \frac{2i+1}{i+1} \cdot \left(\frac{a}{r}\right)^{2i-1} \left[\left(\frac{a}{r}\right)^2 - 1 \right] \phi_i \quad . \quad . \quad . \quad (48)$$

From the relation $\nabla^2 S = -s$ we obtain at once

$$s = \frac{1}{2} \sum_0^\infty \frac{4i^2-1}{i+1} \cdot \frac{a^{2i-1}}{r^{2i+1}} \cdot \phi_i \quad . \quad . \quad . \quad (49)$$

Since S and s are thus fully determined, the velocities u, v, w , and ξ, η, ζ are also known, and the problem of the motion of the fluid particles is solved. We had before

$$u = \sum_0^\infty \left\{ L_i \frac{d\phi_i}{dx} + \phi_i \frac{dL_i}{dx} + R_i \left(\frac{dW_i}{dy} - \frac{dV_i}{dz} \right) + W_i \frac{dR_i}{dy} - V_i \frac{dR_i}{dz} \right\},$$

or

$$u = \sum_0^{\infty} \left\{ L_i \frac{d\phi_i}{dx} + \frac{x}{r} \phi_i \frac{dL_i}{dr} + \left[(i+1)R_i + r \frac{dR_i}{dr} \right] \frac{dS_i}{dx} - i \frac{x}{r} S_i \frac{dR_i}{dr} \right\},$$

with two similar expressions for v and w ; these become now

$$\left. \begin{aligned} u &= \sum_0^{\infty} \left\{ \frac{d\phi_i}{dx} \left[L_i + (i+1)R_i + r \frac{dR_i}{dr} \right] + \frac{x}{r} \phi_i \frac{d}{dr} (L_i - iR_i) \right\}, \\ v &= \sum_0^{\infty} \left\{ \frac{d\phi_i}{dy} \left[L_i + (i+1)R_i + r \frac{dR_i}{dr} \right] + \frac{y}{r} \phi_i \frac{d}{dr} (L_i - iR_i) \right\}, \\ w &= \sum_0^{\infty} \left\{ \frac{d\phi_i}{dz} \left[L_i + (i+1)R_i + r \frac{dR_i}{dr} \right] + \frac{z}{r} \phi_i \frac{d}{dr} (L_i - iR_i) \right\}; \end{aligned} \right\} \quad (50)$$

or, since

$$L_i = 1 + \frac{i}{i+1} \left(\frac{a}{r} \right)^{2i+1},$$

$$R_i = \frac{1}{2} \cdot \frac{2i+1}{i+1} \left(\frac{a}{r} \right)^{2i-1} \left(\left(\frac{a}{r} \right)^2 - 1 \right),$$

these can be written

$$\left. \begin{aligned} u &= \sum_0^{\infty} \left\{ \frac{d\phi_i}{dx} \left[1 - \frac{i(2i-1)}{2(i+1)} \left(\frac{a}{r} \right)^{2i-1} \left(\left(\frac{a}{r} \right)^2 - 1 \right) \right] \right. \\ &\quad \left. + x \phi_i \frac{i(2i-1)(2i+1)}{2(i+1)} \left(\frac{a}{r} \right)^{2i-1} \left(\left(\frac{a}{r} \right)^2 - 1 \right) \right\}, \\ v &= \sum_0^{\infty} \left\{ \frac{d\phi_i}{dy} \left[1 - \frac{i(2i-1)}{2(i+1)} \left(\frac{a}{r} \right)^{2i-1} \left(\left(\frac{a}{r} \right)^2 - 1 \right) \right] \right. \\ &\quad \left. + y \phi_i \frac{i(2i-1)(2i+1)}{2(i+1)} \left(\frac{a}{r} \right)^{2i-1} \left(\left(\frac{a}{r} \right)^2 - 1 \right) \right\}, \\ w &= \sum_0^{\infty} \left\{ \frac{d\phi_i}{dz} \left[1 - \frac{i(2i-1)}{2(i+1)} \left(\frac{a}{r} \right)^{2i-1} \left(\left(\frac{a}{r} \right)^2 - 1 \right) \right] \right. \\ &\quad \left. + z \phi_i \frac{i(2i-1)(2i+1)}{2(i+1)} \left(\frac{a}{r} \right)^{2i-1} \left(\left(\frac{a}{r} \right)^2 - 1 \right) \right\}. \end{aligned} \right\} \quad (51)$$

The equations giving ξ , η , ζ are those in the second group of (34); from these, with reference to (49), we obtain at once

$$\left. \begin{aligned} \xi &= \frac{1}{2} \sum_0^{\infty} \frac{(2i+1)(2i-1)}{i+1} \cdot \frac{a^{2i-1}}{r^{2i+1}} \left(z \frac{d\phi_i}{dy} - y \frac{d\phi_i}{dz} \right), \\ \eta &= \frac{1}{2} \sum_0^{\infty} \frac{(2i+1)(2i-1)}{i+1} \cdot \frac{a^{2i-1}}{r^{2i+1}} \left(x \frac{d\phi_i}{dz} - z \frac{d\phi_i}{dx} \right), \\ \zeta &= \frac{1}{2} \sum_0^{\infty} \frac{(2i+1)(2i-1)}{i+1} \cdot \frac{a^{2i-1}}{r^{2i+1}} \left(y \frac{d\phi_i}{dx} - x \frac{d\phi_i}{dy} \right). \end{aligned} \right\} \quad (52)$$

From equations (34) we have

$$\begin{aligned} x\xi + y\eta + z\zeta &= 0, \\ \xi \frac{ds}{dx} + \eta \frac{ds}{dy} + \zeta \frac{ds}{dz} &= 0; \end{aligned}$$

the vortex-lines therefore lie on the surface $s = \text{const.}$, and also on the sphere $r = \text{const.}$ For the particular case when the flow at an infinitely great distance from the sphere is parallel to the axis of x , all of the functions vanish except ϕ_1 ; then writing

$$\phi_1 = -\lambda x,$$

we have

$$\left. \begin{aligned} u &= \lambda \left\{ 1 - \frac{3}{4} \frac{a}{r} - \frac{1}{4} \frac{a^3}{r^3} \right\} + \frac{3\lambda}{4} \frac{a}{r^3} \left(1 - \frac{a^2}{r^2} \right) x^2, \\ v &= \frac{3\lambda}{4} \frac{a}{r^3} \left(1 - \frac{a^2}{r^2} \right) xy, \\ w &= \frac{3\lambda}{4} \frac{a}{r^3} \left(1 - \frac{a^2}{r^2} \right) xz. \end{aligned} \right\} \quad (53)$$

For this case s reduces to

$$s = -\frac{3}{2} \frac{\lambda a}{r^3} x,$$

and consequently

$$\left. \begin{aligned} \xi &= 0, \\ \eta &= \frac{3}{2} \frac{\lambda a}{r^3} z, \\ \zeta &= -\frac{3}{2} \frac{\lambda a}{r^3} y. \end{aligned} \right\} \quad \dots \dots \dots (54)$$

From these last equations we see that the vortex-lines are circles whose centres lie upon the axis of x . The function Θ was defined by the differential equations

$$\frac{d\Theta}{dx} = \frac{dP}{dx} - 2\tau \left(\frac{d\eta}{dz} - \frac{d\zeta}{dy} \right) \&c.;$$

substituting in these the values of ξ, η, ζ , from (34) we obtain

$$\Theta - P = s - r \frac{ds}{dr},$$

which is easily seen to satisfy the equation

$$\nabla^2(\Theta - P) = 0.$$

Substituting this value of $\Theta - P$ in equation (17), we have

for the stream-line $\int ds$,

$$p = \rho \left\{ r \frac{ds}{dr} - s - V - \frac{1}{2} q^2 \right\} + C.$$

It does not seem to me to be possible in this general case to determine the forms of the stream-lines, or the resistance which the friction of the fluid opposes to the motion of the sphere. In the case where the motion is symmetrical to the axis of x , the resistance has been determined by Lamb ('Treatise on the Motion of Fluids,' page 126) by means of the dissipation-function; that process would scarcely be applicable to such a complicated case as the one in hand. I do not see that any other results of importance can be obtained from further study of this problem, unless, indeed, the velocities u, v, w can be given in some other form than that of an infinite series.

Washington, June 12, 1880.

XLII. On the Thermic and Optical Behaviour of Gases under the Influence of the Electric Discharge. By E. WIEDEMANN.*

[Plate IX.]

1. Introduction.

IN two previous experimental investigations (Wied. Ann. v. p. 500, 1878, vi. p. 298, 1879) I have examined the luminosity of gases under the influence of the electric discharge. The result of the first investigation was, that when a mixture of two gases is exposed to the electric discharge, of which the one is a metallic vapour and the other nitrogen or hydrogen, the lines of the metallic vapour are seen in the spectrum, while those of the other gas remain invisible; so that the propagation of the electricity is due entirely to the metallic vapour. The discharge in it is entirely discontinuous. The same result has been recently obtained by H. W. Vogel†, by photographic methods.

The second investigation showed that the temperature of the gas illuminating a Geissler's tube may be below 100°.

This last result has been recently confirmed by Hasselberg‡. Hittorf also has shown, but without accurate measurements, that the same conclusion is throughout in agreement with the phenomena.

* Translated from Wiedemann's *Annalen*, No. 6, 1880, with additions and corrections by the Author.

† H. W. Vogel, *Beibl.* iv. p. 125, 1880.

‡ Hasselberg, *Beibl.* iv. p. 132, 1880.