

THE  
PHYSICAL REVIEW.

---

PRACTICAL APPLICATION OF FOURIER'S SERIES TO  
HARMONIC ANALYSIS.

BY R. W. PRENTISS.

IN many experimental investigations in acoustics, electricity, astronomy, etc., Fourier's theorem may be frequently applied in analyzing compound harmonic curves when the coördinates of certain points on the curve are known or can be measured; and in finding the terms of different periods which enter into a periodic function when certain values of the function are given or derived from observation.

Many, however, are deterred from its use in cases where it would give valuable results by the great labor required in applying it even in a modified form. Attempts have been made from time to time to reduce this labor to a minimum by methods of approximation. But for the most part the schemes presented are not practical in a simple way, or are concealed in little-known monographs or inaccessible papers.

The object of this paper is to present a scheme or algorithm by the use of which curves involving frequencies from 1 to 5, 1 to 11 or 1 to 17 times the fundamental may be analyzed mechanically and with comparatively little labor. The use of this scheme does not require any knowledge of the theorem or consideration of the formulæ. It is altogether mechanical, and, when once mastered by following the directions and explanations given in connection with the special example at the end of this article, may be used with great facility.

At the same time it seems desirable to present the considerations

which have led to the formulation of this scheme in order to indicate the way in which it may be extended to other cases.

Fourier's theorem is usually stated mathematically in the following form :

$$(1) \quad y = f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots \\ + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

or

$$y = \sum a_r \cos rx + \sum b_r \sin rx \quad (r=0, 1, 2 \dots \infty).$$

It is shown in mathematical treatises on the subject that the coefficients  $a_r$ ,  $b_r$  may be obtained by the following formulæ :

$$(2) \quad \pi a_r = \int_0^{2\pi} y \cos rx dx, \quad \pi b_r = \int_0^{2\pi} y \sin rx dx.$$

The use of these formulæ, however, presupposes a knowledge of  $y = f(x)$ , while in the experimental investigations under consideration only a few values of  $y$  are known, and the object is to find  $y$  in order to get the component periods or frequencies.

For the present purpose it is necessary therefore to use an approximation which can be carried to the degree of accuracy required by the problem in hand.

In either formula (2) the integral may be regarded as the limit of the sum of finite quantities :  $y_1 \cos rx_1 \Delta x$ ,  $y_2 \cos rx_2 \Delta x$ , etc.; so that denoting these approximate values of  $a_r$ ,  $b_r$  by  $R$ ,  $r$  respectively, we have according to the principles of the calculus :

$$(3) \quad R = a_r (\text{appr.}) = \frac{1}{\pi} (y_0 \cos rx_0 \Delta x + y_1 \cos rx_1 \Delta x + y_2 \cos rx_2 \Delta x + \text{etc.}), \\ r = b_r (\text{appr.}) = \frac{1}{\pi} (y_1 \sin rx_1 \Delta x + y_2 \sin rx_2 \Delta x + \text{etc.}),$$

where  $y_0$ ,  $y_1$ ,  $y_2 \dots$  are ordinates corresponding to abscissas  $x_0$ ,  $x_1$ ,  $x_2 \dots$ ,  $\Delta x$  apart; these coördinates being given or obtained as the result of measurements made on the curve to be analyzed.

Suppose the period to be divided into  $2n$  equal parts, so that we have  $x_0 = 0$ ,  $x_1 = \Delta x$ ,  $x_2 = 2x_1 = 2\Delta x$ ,  $x_3 = 3x_1 = 3\Delta x$ , etc.

$$\Delta x = \frac{2\pi}{2n} = \frac{\pi}{n}.$$



$$\begin{aligned} 18b &= y_1 \sin 20^\circ + y_2 \sin 40^\circ + \cdots y_{35} \sin 700^\circ \\ &\vdots \\ 18q &= y_1 \sin 170^\circ + y_2 \sin 340^\circ + \cdots y_{35} \sin 5950^\circ \end{aligned}$$

The sines and cosines of the multiple angles in the above formulæ may all be expressed in terms of the cosines of angles in the first quadrant by means of the well-known trigonometrical relations.

When similar terms are combined  $A, B, C \dots Q$ , the coefficients of the cosine terms, and  $a, b, c \dots q$ , the coefficients of the sine terms in the Fourier series will appear as the sum of a series of products of partial coefficients ( $A_0, A_1, A_2 \dots A_8; B_0, B_2 \dots B_8; C_0, C_3, C_6$ ; etc., and similarly for small letters) and cosines of the multiples of  $10^\circ$  in the first quadrant.

To avoid suffices so far as possible denote  $y_0, y_1, y_2 \dots y_{35}$  by (o), (1), (2)  $\dots$  (35) respectively.

$A, B, C \dots Q$ , then denote the coefficients respectively in the 1st, 2d, 3d  $\dots$  17th frequency terms of the cosines and  $a, b, c \dots q$  those of the sines. The capitals and small letters affected with suffices denote the corresponding partial coefficients, the suffix itself denoting the multiple of  $10^\circ$  occurring in the term.

The expressions for the coefficients may then be presented as follows:

18A			18a		
$A_0$	(0) — (18)	$\cos 0^\circ$	$a_0$	(9) — (27)	
$A_1$	(1) — (17) — (19) + (35)	$\cos 10^\circ$	$a_1$	(8) + (10) — (26) — (28)	
$A_2$	(2) — (16) — (20) + (34)	$\cos 20^\circ$	$a_2$	(7) + (11) — (25) — (29)	
$A_3$	(3) — (15) — (21) + (33)	$\cos 30^\circ$	$a_3$	(6) + (12) — (24) — (30)	
$A_4$	(4) — (14) — (22) + (32)	$\cos 40^\circ$	$a_4$	(5) + (13) — (23) — (31)	
$A_5$	(5) — (13) — (23) + (31)	$\cos 50^\circ$	$a_5$	(4) + (14) — (22) — (32)	
$A_6$	(6) — (12) — (24) + (30)	$\cos 60^\circ$	$a_6$	(3) + (15) — (21) — (33)	
$A_7$	(7) — (11) — (25) + (29)	$\cos 70^\circ$	$a_7$	(2) + (16) — (20) — (34)	
$A_8$	(8) — (10) — (26) + (28)	$\cos 80^\circ$	$a_8$	(1) + (17) — (19) — (35)	

where it is to be understood:

$$18A = A_0 \cos 0^\circ + A_1 \cos 10^\circ + A_2 \cos 20^\circ + \cdots A_8 \cos 80^\circ$$

$$18a = a_0 \cos 0^\circ + a_1 \cos 10^\circ + a_2 \cos 20^\circ + \cdots a_8 \cos 80^\circ$$

also that, *e. g.*,

$$A_4 = (4) - (14) - (22) + (32) = y_4 - y_{14} - y_{22} + y_{32}$$

and

$$a_6 = (3) + (15) - (21) - (23) = y_3 + y_{15} - y_{21} - y_{23}$$

18B

$$\begin{aligned}
B_0 &= \begin{cases} (0) + (18) = B_0' \\ -(9) - (27) = -B_0'' \end{cases} \\
B_2 &= \begin{cases} (1) + (35) + (17) + (19) = B_2' \\ -(8) - (28) - (10) - (26) = -B_2'' \end{cases} \\
B_4 &= \begin{cases} (2) + (34) + (16) + (20) = B_4' \\ -(7) - (29) - (11) - (25) = -B_4'' \end{cases} \\
B_6 &= \begin{cases} (3) + (33) + (15) + (21) = B_6' \\ -(6) - (30) - (12) - (24) = -B_6'' \end{cases} \\
B_8 &= \begin{cases} (4) + (32) + (14) + (22) = B_8' \\ -(5) - (31) - (13) - (23) = -B_8'' \end{cases}
\end{aligned}$$

18b

$$\begin{aligned}
b_1 &= \begin{cases} (4) - (32) - (14) + (22) = b_1' \\ + (5) - (31) - (13) + (23) = b_1'' \end{cases} \\
b_3 &= \begin{cases} (3) - (33) - (15) + (21) = b_3' \\ + (6) - (30) - (12) + (24) = b_3'' \end{cases} \\
b_5 &= \begin{cases} (2) - (34) - (16) + (20) = b_5' \\ + (7) - (29) - (11) + (25) = b_5'' \end{cases} \\
b_7 &= \begin{cases} (1) - (35) - (17) + (19) = b_7' \\ + (8) - (28) - (10) + (26) = b_7'' \end{cases}
\end{aligned}$$

where it is to be understood, *e. g.*, that

$$B_0 = B_0' - B_0'', \text{ etc.,}$$

and

$$b_3 = b_3' + b_3'', \quad b_5 = b_5' + b_5'', \quad \text{etc.}$$

18D

$$\begin{aligned}
D_0 &= B_0' + B_0'' \\
D_2 &= -B_2' - B_2'' \\
D_4 &= B_4' + B_4'' \\
D_6 &= -B_6' - B_6'' \\
D_8 &= B_8' + B_8''
\end{aligned}$$

18d

$$\begin{aligned}
d_1 &= b_5' - b_5'' \\
d_3 &= b_3' - b_3'' \\
d_5 &= b_7' - b_7'' \\
d_7 &= b_1' - b_1''
\end{aligned}$$

The partial coefficients of the other components may all now be expressed in terms of those just obtained for  $A$ ,  $B$ ,  $D$ ,  $a$ ,  $b$ ,  $d$ .

For convenience they are arranged in tabular form and the partial coefficients  $A_0, A_1, M_0, M_1$ , etc., are indicated by suffices only under the letter to which they belong.

$A$	$Q$	$E$	$M$	$G$	$K$	$a$	$q$	$e$	$m$	$g$	$k$
0	0	0	0	0	0	0	0	0	0	-0	-0
1	-1	5	-5	7	-7	1	-1	5	-5	-7	7
2	2	-8	-8	-4	-4	2	2	-8	-8	4	4
3	-3	-3	3	-3	3	3	-3	-3	3	3	-3
4	4	-2	-2	8	8	4	4	-2	-2	-8	-8
5	-5	-7	7	1	-1	5	-5	-7	7	-1	1
6	6	6	6	6	6	6	6	6	6	-6	-6
7	-7	1	-1	-5	5	7	-7	1	-1	5	-5
8	8	4	4	-2	-2	8	8	4	4	2	2

Thus the partial coefficients of  $Q, E, M, G$ , and  $K$  are the same as those of  $A$  but with different signs and in a different order, *e. g.* :

$$A_2 = Q_2 = -E_8 = -M_8 = -G_4 = -K_4,$$

$$a_5 = -q_5 = -e_7 = m_7 = -g_1 = k_1.$$

In like manner those of  $B, J$  and  $N$  are the same ; also  $P, H$  and  $D$ .

$B$	$J$	$N$	$P$	$H$	$D$	$b$	$j$	$n$	$p$	$h$	$d$
0	0	0	0	0	0						
2	-8	-4	2	-8	-4	1	5	-7	-1	-5	7
4	-2	8	4	-2	8	3	-3	3	3	-3	3
6	6	6	6	6	6	5	-7	-1	-5	7	1
8	4	-2	8	4	-2	7	1	5	7	1	5

$$C_0 = A_0 - A_6 = O_0, \quad c_0 = -a_0 + a_6 = o_0,$$

$$C_3 = A_1 - A_5 - A_7 = -O_3, \quad c_3 = -a_1 + a_5 + a_7 = -o_3,$$

$$C_6 = A_2 - A_4 - A_8 = O_6, \quad c_6 = -a_2 + a_4 + a_8 = o_6,$$

$$F_0 = B_0 - B_6, \quad L_0 = D_0 - D_6, \quad I_0 = C_0 - C_6,$$

$$F_6 = B_2 - B_4 - B_8, \quad L_6 = -D_2 + D_4 + D_8, \quad i_0 = -c_0 + c_6,$$

$$f_3 = -b_1 + b_5 + b_7, \quad l_3 = -d_1 + d_5 + d_7,$$

The results thus obtained for the partial coefficients may now be easily embodied in a simple scheme, which may be checked at every point by the values given above. The working of this scheme is exhibited in the case of a particular example (given in full, with values of measured ordinates, on pages 268, 269) as follows :

## SCHEME.

Computation of Partial Coefficients (cosine terms.) (Left-hand side of computation sheet.)

		(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
(1) <sup>1</sup>	$y's$	35	62	57	44	28	17	15	20	25	28
(2) <sup>1</sup>	$y's$		17	8	9	23	39	44	44	43	43
(3)		35	79	65	53	51	56	59	64	68	71
(4) <sup>2</sup>		29	64	70	75	81	80	74	71	72	—15 <sup>3</sup>
(5)	$A's$	6	15	—5	—22	—30	—24	—15	—7	—4	—15 <sup>3</sup>
<hr/>											
		(0)	(2)	(4)	(6)	(8)					
(6)	$B'$	64	143	135	128	132					
(7)	$B''$	71	140	135	133	136					
(8)	$B's$	—7	3	0	—5	—4	6	15	—5		
(9)		5	0					15	24	30	
(10)			4						7	4	
<hr/>											
(11)	$F's$	—2	7				$C's$	(0) 21	(3) 46	(6) 29	
								—29			
(12)	$D's$	135	283	270	261	268	$I_0$	— 8(÷18 = coeff.)			
(13)		261	270								
(14)			268								
(15)	$L's$	396	821	$L_0+L_6=1217=36(\bar{y})$ .							

<sup>1</sup> Line (1) contains the ordinates  $y_0, y_1, \dots, y_9$ . Line (2) contains the ordinates  $y_{35} = 17, y_{34} = 8, y_{33} = 9$ , etc., etc.

<sup>2</sup> Line (4) contains sums  $y_{18}, y_{17} + y_{19}, y_{16} + y_{20}$ , etc., from right-hand side of computation sheet.

<sup>3</sup> Difference belonging to the right-hand side of computation sheet.

## SCHEME (continued).

Computation of partial coefficients (sine terms.) (Right-hand side of computation sheet.)

(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	
28 <sup>1</sup>	27	23	25	37	48	56	53	44		(1)
43 <sup>1</sup>	45	48	49	43	33	19	17	20	29	(2)
71 <sup>1</sup>	—18	—25	—24	—6	15	37	36	24		(3)
—15	—18	—24	—29	—22	5	35	49	45		(4)
—15	—36	—49	—53	—28	20	72	85	69	<i>a's</i>	(5)
<hr/>										
					(1)	(3)	(5)	(7)		
					—10	—2	13	21	<i>b'</i>	(6)
					—16	—5	1	0	<i>b''</i>	(7)
(0)	(3)	(6)			—26	—7	14	21	<i>b's</i>	(8)
15	36	49						14	<i>b<sub>5</sub></i>	(9)
72	20	—28						26	— <i>b<sub>1</sub></i>	(10)
	85	69								
87	141	90	<i>c's</i>					61	<i>f<sub>3</sub></i>	(11)
		—87	— <i>c<sub>0</sub></i>							
(÷18=coeff.)	3	<i>i<sub>0</sub></i>			(7)	(3)	(1)	(5)		
					6	3	12	21	<i>d's</i>	(12)
								—12	— <i>d<sub>1</sub></i>	(13)
								6	+ <i>d<sub>7</sub></i>	(14)
								15	<i>l<sub>3</sub></i>	(15)

<sup>1</sup> Ordinates and sum from the left-hand side of computation sheet.

## EXPLANATION OF COMPUTATION OF PARTIAL COEFFICIENTS.

*Cosine Terms.*

Arrange the measured ordinates  $y_0, y_1, y_2 \dots y_{35}$  in two horizontal lines (1) and (2) across the computation sheet putting  $y_0 \dots y_{17}$  in line (1), and  $y_{18} \dots y_{35}$  in reversed order in line (2) so that  $y_{35}$  is under  $y_1, y_{34}$  under  $y_2$ , and so on.

*Left-hand Side :*

1. Add line (1) to line (2) putting sums under columns (0)  $\dots$  (9) in line (3) and sums (10)  $\dots$  (18) reversed in line (4) so that sum in column (10) is under sum of column (8), (11) under (7), etc., *i. e.*, on left-hand side of the sheet.
2. Subtract line (4) from (3), the differences in order in (5) are the partial coefficients,  $A_0, A_1, \dots A_8$ , which are subsequently to be multiplied into  $\cos 0^\circ, \cos 10^\circ \dots \cos 80^\circ$  respectively and the products added to get  $18A$ .
3. Add (3) to (4) putting sums (0)  $\dots$  (4) in line (6), and (9)  $\dots$  (5) respectively under them in line (7). Subtract line (7) from (6), the differences in order in (8) are  $B_0, B_2, B_4, B_6, B_8$ , subsequently to be multiplied into  $\cos 0^\circ, \cos 20^\circ, \dots \cos 80^\circ$  respectively.
4. Arrange the  $B$ 's in two columns as indicated in lines (9) and (10), reversing the signs of  $B_4, B_6, B_8$ . The algebraic sums will be  $F_0$  and  $F_6$  as shown in line (11).
5. Add (6) to (7), the sums in line (12) are in order,  $D_0, D_4, D_8, -D_6$  and  $-D_2$ .
6. Arrange the  $D$ 's in two columns as indicated in lines (13) and (14). The sums in (15) are the  $L$ 's.
7. Arrange the  $A$ 's in three columns as indicated in lines (8), (9) and (10), reversing the signs of  $A_4, A_5, A_6, A_7, A_8$ . The algebraic sums will be the  $C$ 's in line (11).
8. Form  $I_0 = C_0 - C_6$  under  $C_0$  in line (12).

*Sine Terms.**Right-hand Side :<sup>1</sup>*

1. Subtract line (2) from (1) placing differences in (3) in columns marked (10)  $\dots$  (17) and differences (9)  $\dots$  (1) under them in line (4) so that (9) is by itself and (8) is under (10), (7) under (11), etc.
2. Add line (3) to (4) putting sums in columns (9)  $\dots$  (17). The sums in order are the partial coefficients  $a_0, a_1 \dots a_8$ , which are

<sup>1</sup> It is interesting and instructive to compare these directions with those given above for the "Left-hand Side."



subsequently to be multiplied into  $\cos 0^\circ, \cos 10^\circ \dots \cos 80^\circ$  respectively and the products added to get  $18a$ .

3. Subtract (3) from (4) placing differences (14)  $\dots$  (17) in line (6), and differences (13)  $\dots$  (10) under them respectively in line (7). Add line (6) to (7), the differences in order in (8) are  $b_1, b_3, b_5, b_7$ , subsequently to be multiplied into  $\cos 10^\circ, \cos 30^\circ, \cos 50^\circ, \cos 70^\circ$  respectively.

4. Form  $f_3 = b_7 + b_5 - b_1$  to the right in lines (8), (9), (10) and (11).

5. Subtract line (7) from (6), the differences in line (12) are in order,  $d_7, d_3, d_1$  and  $d_5$ .

6. Form  $l_3 = d_5 - d_1 + d_7$  in lines (12), (13), (14) and (15).

7. Form  $c_0 = -a_0 + a_6 = 0$ ,  $c_3 = -a_1 + a_5 + a_7 = -0_3$  and  $c_6 = -a_2 + a_4 + a_8 = 0_6$  to the left in lines (8)  $\dots$  (11).

8. Form  $i_0 = -c_0 + c_6$  under  $c_6$  in line (12).

The partial coefficients for  $E, G, K, M$  and  $Q$  are the same as those obtained in line (5) for  $A$ , but with different signs and in a different order.  $A$  and  $Q$  are in the same order with the signs alternately the same and opposite; similarly for the pairs,  $E$  and  $M, G$  and  $K$ .

In a similar way the partial coefficients of  $B, J$  and  $N$ , and of  $P, H$  and  $D$  are the same differing only in sign and order; the partial coefficients of  $C$  and  $O$  differ only in sign. The foregoing remarks apply also to the partial coefficients of the sine terms, substituting small letters for the capitals.

These relations are given in compact form for reference in the tables on page 262 and suggest the following scheme for performing the necessary multiplications by the cosines of the multiples of  $10^\circ$  and the additions to obtain the final coefficients:

SCHEME (*continued*).

*Multiplication by the Cosines.*

	(1)	(2)	(3)		<i>Natural Cosines.</i>		
$A$	$A$	$Q$	$E$	$M$	$G$	$K$	
(0)	6	6.0	6.0		6.0		1.000 + 1.000 + 1.000
(1)	15	14.8		9.6		5.2	.985 + .643 + .342
(2)	5	-4.7	0.9		3.9		.940 - .174 - .766
(3)	-22	-19.1		19.1		19.1	.866 - .866 - .866
(4)	-30	-23.0	28.2		-5.2		.766 - .940 + .174
(5)	-24	-15.4		8.2		-23.7	.643 - .342 + .985
(6)	-15	-7.5	-7.5		-7.5		.500 + .500 + .500
(7)	-7	-2.4		-6.9		4.5	.342 + .985 - .643
(8)	-4	-0.7	-3.1		3.8		.174 + .766 - .940
Sums	-29.9	-22.1	+24.5	+30.0	+1.0	+5.1	
Coeff.	-52.0	-7.8	+54.5	-5.5	+6.1	-4.1	(to be divided by 18.)

	(1)	(2)	(3)		(1)	(2)	(3)	
<i>B</i>	<i>B</i>	<i>J</i>	<i>N</i>	Natural Cosines.	<i>P</i>	<i>H</i>	<i>D</i>	<i>D</i>
				(1) (2) (3)				
(0)	-7	-7.0	-7.0	-7.0	1.000+1.000+1.000	135	135.0+135.0	+135 (0)
(2)	+3	2.8	-0.5	-2.3	.940 -.174-.766	-266	49.2+216.8	-283-(4)
(4)	0	0.0	0.0	0.0	.766 -.940+.174	+206.8-253.8	+46.8	+270 (8)
(6)	-5	-2.5	-2.5	-2.5	.500 +.500+.500	-130.5-130.5-130.5		-261 (6)
(8)	-4	-0.7	-3.1	+3.7	.174 +.766-.940	+46.6+205.3-251.9	+268	-(2)
Coeff. -7.4-13.1-8.1				(÷18)	-8.1 +5.2 +16.2			
	<i>C</i>	<i>C</i>	<i>O</i>	cos.	<i>c</i>	<i>o</i>	<i>c</i>	<i>F</i> cos. <i>L</i> <i>f</i> <i>l</i>
(0)	21	21.0	21.0	1.000	87.0	87.0	87	-2.0 1.000 396.0
(3)	46	39.8	-39.8	+.866	-122.1	122.1	141	3.5 .500-410.5 61 .866 15
(6)	29	14.5	14.5	.500	45.0	45.0	90	
Coeff. 75.3 -4.3					254.1	9.9	+1.5	-14.5 52.8 13.0

## MULTIPLICATION BY THE COSINES.

*Left-hand Side and Middle:*

Since the coefficients are the same, in the same order and with alternately the same and opposite signs in the pairs  $A, Q; E, M; Q, R$ , but the order different in the several pairs, the natural cosines are arranged in three different orders in columns (1), (2) and (3) in such a way and with such signs that, being multiplied by the partial coefficients  $A$  in order, the proper products are obtained for  $Q, E, M, G$  and  $K$ .

Similar remarks, *mutatis mutandis*, apply to the calculation of the remaining coefficients for both cosine and sine terms. The processes will be readily understood by comparing carefully the tables on page 262 with the following scheme:<sup>1</sup>

1. Place the partial coefficients  $A_0 A_1 \dots A_8$  in the left-hand column. Multiply them by the corresponding cosines in the same line in columns (1), (2) and (3), putting the products (1) alternately in  $A$  and  $Q$  columns; (2) in a similar way in  $E$  and  $M$  columns; (3) in  $G$  and  $K$  columns.

2. Since each pair has the same coefficients in the same order but alternating in sign, the sum of the  $A$  and  $Q$  columns  $-29.9 - 22.1 = -52.0 = 18A$ , their difference,  $-29.9 + 22.1 = -7.8 = 18Q$ , similarly for the other pairs.

3. Place the partial coefficients  $B_0, B_2, B_4, B_6, B_8$ , in the left-hand column. Multiply them by the corresponding cosines in the same line in columns (1), (2) and (3), placing the products in columns  $B$ ,

<sup>1</sup> Explanations are given here only for  $A, B, C$ , etc. The processes on the right hand side for  $a, b, c$ , etc., are entirely similar.

$J$ ,  $N$  respectively. The sums will give  $18B$ ,  $18J$  and  $18N$ , respectively equal to  $-7.4$ ,  $-13.1$ ,  $-8.1$ .

4. For convenience the  $D$ 's must be arranged in the right-hand column in the order  $D_0, -D_4, D_8, D_6$  and  $-D_2$ . Proceeding as in (3)  $18P = -8.1$ ,  $18H = +5.2$  and  $18D = +16.2$  are obtained.

5.  $C$ ,  $O$ ,  $F$  and  $L$  are similarly obtained as shown.

6. Dividing the results thus obtained in the lines marked "Coeff." by 18, the coefficients  $A$ ,  $B$ ,  $C$ , etc., and  $a$ ,  $b$ ,  $c$ , etc., are obtained and may be substituted in the following (approximate) form of Fourier's Theorem :

$$y = (\bar{y})^1 + A \cos x + B \cos 2x + C \cos 3x + \dots Q \cos 17x^2 \\ + a \sin x + b \sin 2x + c \sin 3x + \dots q \sin 17x^2$$

The phase of each simple harmonic may then be computed from the formulæ:  $\tan \varphi_1 = A/a$ ,  $\tan \varphi_2 = B/b$ , etc., and the amplitudes from the formulæ,

$$a_1 = \sqrt{a^2 + A^2}, \quad a_2 = \sqrt{b^2 + B^2}, \quad \text{etc.},$$

in the usual way.<sup>3</sup>

#### CASE, $2n = 12$ . GIVING FREQUENCIES 1-5.†

When only a few simple harmonics are present the case in which  $2n = 12$ , requiring 12 given ordinates, may be applied.

As this case is of frequent occurrence it seems desirable to append the values of the coefficients and the scheme of computation. The notation is the same as that already used.

$$\begin{array}{rcll} & 6A & & 6a \\ A_0 = (0) - (6) & = E_0 \left| \begin{array}{c} \cos 0^\circ \\ \cos 30^\circ \end{array} \right| & a_0 = (3) - (9) & = e_0 \\ A_1 = (1) - (5) - (7) + (11) & = -E_1 \left| \begin{array}{c} \cos 0^\circ \\ \cos 30^\circ \end{array} \right| & & \\ & & a_1 = (2) + (4) - (8) - (10) & = -e_1 \end{array}$$

<sup>1</sup> The term  $(\bar{y})$  depends on the choice of the axis of  $x$  and may generally be omitted. Its value is  $(\bar{y}) = \bar{L}_0 + \bar{L}_6 / 36 = \Sigma y / 36$ , i. e., the mean value of the 36  $y$ 's.

<sup>2</sup> 36 divisions will give the first 17 frequencies or simple periods.  $2n$  divisions will give the first  $n-1$  coefficients. The formulæ give illusory results for the remaining  $n+1$  coefficients.

<sup>3</sup> Dr. Bevier prepared a printed blank for use in his investigations in the "Acoustic Analyses of the Vowels" in accordance with the scheme presented here. In it he printed all the signs of the products to be inserted as—. This lessens the labor, and promotes accuracy by calling special attention to every sign; the symbol — being readily changed to + when necessary. A copy of this blank with a special example worked out is given on pages 268, 269.



<i>A</i>	1	17	5	13	7	11	Natural Cosines			1	17	5	13	7	11	<i>a</i>						
+ 6	+ 6.0		+ 6.0		+ 6.0		1000	+ 1000	+ 1000		-15.0		-15.0			-15						
+15		+14.8		+ 9.6		+ 5.2	.985	+ .613	+ .342	-35.5		-23.1		-12.3		-36						
- 5	- 4.7		+ 0.9		+ 3.9		.910	- .174	- .766	-46.1			+ 8.5		+37.5	-49						
-22		-19.1		+19.1		+19.1	.866	- .866	- .866	-45.9		+ 45.9		+45.9		-53						
-30	-23.0		+28.2		-5.2		.766	- .940	- .174	-21.4			+ 26.3		- 4.9	-28						
-24		-15.4		+ 8.2		-23.7	.613	- .342	+ .985	+12.9		- 6.8		+19.7		+20						
-15	- 7.5		- 7.5		- 7.5		.500	+ .500	+ .500	+36.0			+ 36.0		+ 36.0	+72						
- 7		- 2.4		- 6.9		+ 4.5	.342	+ .985	- .643	+29.1		+ 83.7		-54.7		+ 85						
- 4	- 0.7		- 3.1		+ 3.8		.174	+ .766	- .940	+12.0			+ 52.9		-64.9	+ 69						
Coeff.							-29.9	-22.1	+24.5	30.0	1.0	5.1										
	-52.0	- 7.8		+54.5	- 5.5	6.1	- 4.1															
<i>B</i>	2	10	14	Natural Cosines			16	8	4	<i>D</i>	<i>b</i>	2	10	14	Natural Cosines			16	8	4	<i>d</i>	
- 7	- 7.0	- 7.0	- 7.0	1000	+ 1000	+ 1000	+ 135.0	+ 135.0	+ 135.0	+ 135		-26	-25.6	-16.8	+ 8.9	.985	+ .613	- .342	- 5.8	- 3.8	+ 2.1	- 6
+ 3	+ 2.8	0.0	- 2.3	.910	- .174	- .766	- 266.0	+ 49.2	+ 216.8	- 283		- 7	- 6.1	+ 6.1	- 6.1	.866	- .866	+ .866	+ 2.6	- 2.6	+ 2.6	+ 3
- 0	- 0.0	0.0	0.0	.766	- .940	+ .174	+ 206.8	- 253.8	+ 46.8	+ 270		+14	+ 9.0	- 4.8	-13.8	.643	- .342	- 9.5	- 7.7	+ 4.1	+11.8	-12
- 5	- 2.5	- 2.5	- 2.5	.500	+ .500	+ .500	-130.5	-130.5	-130.5	-261		+21	+ 7.2	+20.7	+13.5	.342	+ .985	+ .613	+ 7.0	+ 20.7	+ 13.4	+21
- 4	- 0.7	- 3.1	+ 3.7	.174	+ .766	- .940	+ 46.6	+ 205.3	- 251.9	+ 268		-15.5	+ 5.2	+ 2.5					-3.9	+18.4	+29.9	
	-7.4	-13.1	-8.1				- 8.1	+ 5.2	+ 16.2													
<i>C</i>	3	15	Natural Cosines			15	3	15	<i>c</i>	12	6	12	6	12	9	<i>I</i> <sub>0</sub>	<i>i</i> <sub>0</sub>					
+21	+21.0	+21.0		1000		+ 87.0	+ 87.0	+ 87								-8						
+46	+39.8	-39.8		+ .866		+122.1	+122.1	+141		+396.0	1000	-2.0		+15.0		.866	+61					
+29	+14.5	+14.5		.500		+ 45.0	+ 45.0	+ 90		-410.5	.500	+3.5										
	+75.3	- 4.3		254.1		+ 9.9				- 14.5		+1.5		+12.9			+52.8					
<i>R</i>	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17					
	-2.8	-0.4	+4.2	+0.9	+3.0	+0.1	+0.3	+0.3	-0.4	-0.7	-0.2	-0.8	-0.3	-0.5	-0.2	-0.5	-0.4					
<i>r</i>	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17					
	-4.1	-0.9	+14.2	+1.7	+11.6	+2.9	-0.7	-1.0	+0.2	+0.3	+0.5	+0.7	-0.5	+0.1	+0.6	-0.2	-0.3					

The numbers 1-17 in black faced type above indicate the orders of the simple harmonics present in the compound harmonic curve.

$$A_2 = (2) - (4) - (8) + (10) = E_2 |\cos 60^\circ| a_2 = (1) + (5) - (7) - (11) = e_2$$

$$B_0 = \begin{cases} (0) + (6) = B_0' \\ -(3) - (9) = -B_0'' \end{cases} \quad b_1 = \begin{cases} (1) - (5) + (7) - (11) = b_1' \\ + (2) - (4) + (8) - (10) = b_1'' \end{cases}$$

$$B_2 = \begin{cases} (1) + (5) + (7) + (11) = B_2' \\ -(2) - (4) - (8) - (10) = -B_2'' \end{cases}$$

$$C_0 = A_0 - A_2 \quad c_0 = -a_0 + a_2$$

$$D_0 = B_0' + B_0'' \quad d_1 = b_1' - b_1''$$

$$D_2 = -B_2' - B_2''$$

where  $B_0' = y_0 + y_6$ ,  $B_0'' = y_3 + y_9$ , etc.,  
and  $b_1' = y_1 - y_5 + y_7 - y_{11}$ ,  $b_1'' = y_2 - y_4 + y_8 - y_{10}$ , etc.

As a particular example we take every third ordinate of the series of 36 ordinates given in the case  $2n = 36$ .

(0)	(1)	(2)	(3)	(4)	(5)	(6)	
35	44	15	28	25	56		
	9	44	43	49	19	29	
35	53	59	71	-24	37		
29	75	74	-15	-29	35		
A's 6	-22	-15	-15	-53	72		a's
64	128			5	2		
71	133	C <sub>0</sub>	c <sub>0</sub>		5		
B's -7	-5	6	72		-7		b's
		15	-15				
D's 135	-261	(Coeff.) 21	87 (Coeff.)		3		d's
A	1	5	cos.	1	5	a	b
6	6.0	1.000		-15.0	-15	-7	-6.1 .866
-22		-19.1	.866	-45.9	-53	d	4
-15	-7.5		500	36.0	72	3	2.6 .866
	-1.5	-19.1		-45.9	21.0		
(Coeff.) <sup>1</sup>	-20.6	+17.6		-24.9	66.9		
B	2	cos.	4	D			
-7	-7.0	1.000	135.0	135			
-5	-2.5	.500	-130.5	-261		3	3
(Coeff.) <sup>1</sup>	-9.5		+4.5			21	87
R =	1	2	3	4	5		
	-3.4	-1.6	+3.5	+0.8	+2.9		
r =	-4.2	-1.0	+14.5	+0.4	+11.2		

RUTGERS COLLEGE, NEW BRUNSWICK, N. J.

<sup>1</sup> The results in line marked (Coeff.) must be divided by 6.