ON THE ARITHMETIC CONTINUUM

By E. W. HOBSON.

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THE present communication is concerned with an important point relating to the definition of the irrational numbers of the arithmetic continuum which has been recently* raised by Prof. König. A distinction is introduced by König between those elements of the continuum which are capable of being "finitely defined" (*endlich definiert*) and those which are not capable of being defined in finite terms,† and he argues that the former elements form an enumerable aggregate E within the continuum, *i.e.*, an aggregate of cardinal number $\aleph_0$. The existence of those elements of the continuum which do not belong to the aggregate E being, in accordance with König's view, incapable of establishment by means of definitions in finite terms, such existence must be established by the method of postulation; each such element and the continuum itself being regarded as "possible conceptions" (*mögliche Begriffe*), that is, such that the postulation of them does not lead to contradiction. König refers for an analysis of this idea of the continuum to a treatment of the subject which has been given by Hilbert. A refusal to go beyond what can be defined by finite laws can only, in König's view, lead to the denial of the existence of the continuum and of the continuum problem.

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* "Ueber die Grundlagen der Mengenlehre und das Kontinuumproblem," *Math. Annalen*, Vol. LXI., September, 1905.

† König's explanation of what he means by finite definition is as follows:—"Ein Element des Kontinuums soll 'endlich definiert' heissen, wenn wir mit Hilfe einer zur Fixierung unseres wissenschaftlichen Denkens geeigneten Sprache in endlicher Zeit ein Verfahren (Gesetz) angeben können, das jenes Element des Kontinuums von jedem anderen begrifflich sondert, oder—anders ausgedrückt—für ein beliebig gewähltes k die Existenz einer und nur einer zugehörigen Zahl a_k ergibt. Dabei muss aber ausdrücklich betont werden, dass die hierin geforderte 'endliche' begriffliche Sonderung nicht mit der Forderung eines wohldefinierten oder gar endlichen Verfahrens zur Bestimmung der a_k verwechselt werden darf." The last clause refers to the definition of the continuum employed by König, that, if $a_1, a_2, \ldots, a_k, \ldots$ is an enumerable sequence of positive integers (of type ω), the continuum is the aggregate of objects such as $(a_1, a_2, \ldots, a_k, \ldots)$. The idea that those numbers which are capable of finite definition form an enumerable set, with deductions similar to those made by König, occurred independently to Prof. A. C. Dixon, who communicated his views to me before the publication of König's paper.

König's theory, if well founded, is obviously of great importance in relation to our views as to the fundamental nature of the arithmetic continuum. I propose therefore to examine the distinction which König introduces between those elements of the continuum which belong to the aggregate E, and to the remainder, with a view to determine whether the distinction in relation to definition is well founded or not.

It will here be shewn that the distinction referred to is not a valid one, although it may be true that there exists in the continuum an enumerable set of irrational numbers each of which is capable of a formal definition of a character which is, in a certain aspect, simpler than definitions applicable to irrational numbers in general. The enumerable set referred to contains those irrational numbers, such as e, π, $\sqrt{2}$, ..., which are in common use in arithmetical analysis. It will, however, be shewn, by means of a discussion of the possible modes of formal definition of irrational numbers, that the distinction between the definitions applicable to the special class, and to other irrational numbers, is not of such a character as to justify our speaking of some irrational numbers as capable of finite definition, and of others as not so.

König argues that the irrational numbers capable of being defined in a form which in each case involves only a finite number of letters and symbols (*Buchstaben und Interpunktionszeichen*) must form an enumerable aggregate. Such irrational numbers, together with the rational numbers, he regards as finitely defined, and the other numbers of the continuum he characterizes as, in some special sense, ideal elements, these latter being incapable of finite definition. It must, however, in the first place be remarked that, if it be regarded as essential to a finite definition that it be expressible by means of a finite number of words and symbols, each of which has a definite and *unique* meaning, then no irrational number whatever is capable of such a definition. The simplest possible definition of any of the ordinary irrational numbers, such as π, e, ..., involves the use of a symbol n (or of some form of words equivalent to the use of such symbol), to which no unique meaning is attached, but which is capable of denoting the numbers of the integer sequence 1, 2, 3, ..., of type ω. Thus, for example, a definition of the number e may be given, in which the expression $1+\sum_1^n \frac{1}{n!}$ occurs, and no arithmetical definition of the number e can be given which dispenses with the use in some form of the "variable" n, of which the meaning is not unique.

König's argument may be applied to the aggregate of definitions of those irrational numbers, each of which can be defined in a form involving only a finite number of words and signs, and including the use of this one "variable" n (1, 2, 3, ...). Assuming that it is possible to arrange such definitions in order, on the basis of the number and arrangement of the letters and signs, including n, which are employed in them, these numbers form an enumerable aggregate of type ω. However difficult it may be to imagine how this ordering of the definitions could be carried out, I shall assume, for the purposes of the present discussion, that this can be done, and therefore that the aggregate E_ι of all such numbers is enumerable. If to E_ι there be added the aggregate E_r of all the rational numbers, we have an aggregate E which I presume to be identical with König's aggregate E, and which I accordingly denote by the same letter.

An irrational number in general is an object which has a definite ordinal relation with the rational numbers in their so-called natural order. This conception of the nature of an irrational number is perhaps most immediately expressed in Dedekind's form of definition, in which an irrational number is regarded as being defined by a certain kind of section (*Schnitt*) of the aggregate of rational numbers; but it is also essential in Cantor's theory of irrational numbers. A particular irrational number is defined when we are supplied with the means of deciding, in respect of any arbitrarily assigned rational number, whether the irrational number ordinally precedes or ordinally succeeds the assigned rational number. Any definition of an irrational number, no matter how such definition be expressed, which satisfies this requisite, I shall speak of as an *adequate* definition of the irrational number in question. It is difficult to understand how any irrational number of which it is impossible to give in some form an *adequate* definition can be said to be defined at all, or to have a determinate ordinal relation with the rational numbers. To have recourse to the method of postulation, in order to provide elements in the continuum, would appear to involve the postulation of the existence of entities which are not clearly distinguishable from one another; for, if determinate ordinal relationship with the rational numbers is supposed to appertain to them, we are unable, in default of adequate definitions, to ascertain what that relationship in any particular case may be.

Confining our attention to the interval (0, 1), as we may do without real loss of generality, it is clear that a definition of a particular irrational number in this interval, which is adequate in the sense explained above, must supply us with directions for calculating, for any prescribed integer n, by an arithmetic process, the first n figures of the decimal representation of the irrational number. The number of steps in the process must be definite for any assigned value of n, but has no upper limit for all values of n. It is also clear that any form of definition which supplies us with the directions mentioned above is an adequate definition, in that it enables us to assign the order of the number relatively to any arbitrarily assigned number. In particular, the numbers which belong to the enumerable set E are all capable of adequate definition.

Now let us assume that we have an enumerable set of numbers $x_1, x_2, \ldots, x_n, \ldots$ all in the interval (0, 1), and that each one of these numbers has an adequate definition. The set being enumerable, it can be placed in correspondence with the numbers 1, 2, 3, ... of the integer series, so that x_n is identifiable for each value of n.

Let the decimal representation of $x_1, x_2, \ldots$ be expressed as follows :—

$$\begin{aligned} x_1 &= \cdot p_{11} p_{12} p_{13} \ldots p_{1r} \ldots, \\ x_2 &= \cdot p_{21} p_{22} p_{23} \ldots p_{2r} \ldots, \\ \ldots & \quad \ldots \quad \ldots \quad \ldots \\ x_n &= \cdot p_{n1} p_{n2} p_{n3} \ldots p_{nr} \ldots, \\ \ldots & \quad \ldots \quad \ldots \quad \ldots . \end{aligned}$$

In case any of the numbers are rational numbers not expressed by recurring decimals, we may suppose all the figures to be 0 after some fixed one. On the above hypothesis that x_n is, for any particular value of n, an identifiable number possessing an adequate definition, we are in possession of the means of calculating the digit p_{nr}, for any assigned values of n and r, by a finite process dependent upon those assigned values. In particular, we have the means of calculating p_{nn}, for any assigned value of n. It can now be shewn that an adequate definition can be given of a number which is not contained in the set $x_1, x_2, \ldots$. For example, let us consider the number N represented by $\cdot a_1 a_2 \ldots a_n \ldots$, where $a_n = p_{nn} + (-1)^{p_{nn}}$, and is thus essentially different from p_{nn}: if $p_{nn} = 9$, then $a_n = 8$; and, if $p_{nn} = 0$, then $a_n = 1$; and so on. This number N is adequately defined; for, in accordance with the hypothesis, we have the means of calculating p_{nn}, and thence a_n, and thus $\cdot a_1 a_2 \ldots a_n$, can be calculated in a finite number of steps. Moreover, N cannot be identical with any of the numbers $x_1, x_2, \ldots$; for it differs from each of them in at least one figure.

Let us now assume that the set of numbers $x_1, x_2, x_3, \ldots$ contains all the numbers of the set E which are in the interval (0, 1).

We have seen that the new number N is also capable of adequate definition; it thus appears that there exist numbers capable of adequate definition which do not belong to the set E. It does not, however, follow that the number N is capable of being defined in a form of words and signs which involves only the use of the one "variable" n, and which would be such that the definitions of $x_1, x_2, \ldots,$ together with their law of order, were all collected together and merged in one definition. In fact, this cannot be the case; otherwise N would itself belong to E, contrary to hypothesis. It thus appears that any formal definition of the number N must involve a reference to the numbers $x_1, x_2, \ldots$ explicitly, or to some other such sequence of numbers not identical with the integer sequence $1, 2, 3, \ldots$.

We are thus led to the consideration of a type of definition of irrational numbers. A definition of this type contains, besides the "variable" n, a reference to an aggregate $a_1, a_2, a_3, \ldots$ of numbers* which must be regarded as having been already defined and arranged in the order type ω. If we use one general symbol a_n to denote any of the already defined numbers $a_1, a_2, a_3, \ldots,$ in the same sense in which n denotes any of the numbers $1, 2, 3, \ldots,$ then the particular definition is expressible by a finite number of words and unique signs together with the symbols n and a_n; and such a definition is an adequate definition of a particular irrational number. We may, for convenience, speak of the numbers $a_1, a_2, a_3, \ldots$ as the parameters of the definition. All the numbers capable of being defined in forms which involve the use of one and the same set of parameters would, by a repetition of König's argument, form an enumerable set. It does not, however, follow that all numbers capable of a definition of this kind, when various sets of parameters are taken into account, form an enumerable set. In fact, reasoning similar to that employed above may be applied to shew that this cannot be the case. For, let us suppose, if possible, that all the numbers so definable form an enumerable set which can be denoted by $x_1, x_2, x_3, \ldots x_n, \ldots$; then the same reasoning as was applicable when $\{x_n\}$ was taken to be the set E suffices to shew that a number $\overline{N}$, not belonging to the $\{x_n\}$, admits of a definition of the type considered; and thus that there is a contradiction in the assumption that all the numbers definable in this manner are contained in the enumerable set $\{x_n\}$. The proof is, in fact, a modification of one of Cantor's proofs that the arithmetic continuum is not enumerable; and it completes that proof, by shewing that a number can be adequately *defined* which does not belong to the assumed enumerable set.

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* It is easy to construct directly definitions of irrational numbers in which the parameters cannot be taken to be the integers $1, 2, 3, \ldots$. For example, an irrational number M may be adequately defined as follows:—Let the decimal representation of M be such that the n-th figure is identical with the n-th figure in the decimal representation of $2^{1/P_n}$, where P_n denotes the n-th of the prime numbers $2, 3, 5, 7, 11, \ldots$. Since P_n is not expressible in finite terms as a function of n, the parameters of the definition may be taken to be the set of prime numbers.

Any particular number which admits of a definition of the type considered is capable of definition in a variety of ways, involving the use of various sets of parameters. For example, we may, in defining the number N, make use of the rational parameters $\cdot p_{11}$, $\cdot p_{21}p_{22}$, $\cdot p_{31}p_{32}p_{33}$, ..., instead of the parameters represented by endless decimals.

An irrational number belonging to the set E has the peculiarity that it is capable of definition in a form which involves the use of the numbers 1, 2, 3, ..., denoted by n, without the employment of any other sequence of the same type ω. As has been shewn, other definitions of irrational numbers can be given which involve the use of sets of parameters $a_1, a_2, \ldots$, which can be denoted by a_n; these parameters being numbers which have already been defined, and must be taken as data in the definitions in question. The definitions of the numbers of E are only that particular case of the more general type of definition which arises when $a_1, a_2, \ldots$ can be identified with 1, 2, 3, ..., which are therefore the parameters used in a definition of one of the numbers of E. The possession of this peculiarity does not, however, justify the use of the term "finite definition" as in any peculiar sense applicable to the numbers of E. These numbers, like the others, are only capable of a definition involving the use of an ordered infinity of numbers (parameters) regarded as data in virtue of previous definition. The term "finitely defined" is, in fact, an expression not free from ambiguity. In one sense every irrational number capable of a definition involving the use of a set of parameters, and which is therefore adequately defined, is finitely defined, since a single variable may be used to denote the parameters. But, as each such definition, whether the irrational number belong to E or not, implies the existence of an infinity of separate entities taken as data, and contains directions for carrying on a process which is essentially endless, it follows that such definition cannot, in a more fundamental sense of the term, be said to be finite. Nevetheless, the process of making an ordinal comparison of the defined number with any assigned rational number is a finite one in which only a finite number of parameters is employed.

It appears therefore that, on the assumption of the possibility of ordering those formal definitions in which a finite number of words, signs, and symbols are employed, in the order type ω, the distinction drawn by König between finitely defined numbers and others not finitely defined is not a valid one, and that the numbers which are capable of formal definition involving only a finite number of words, signs, and symbols do not form an enumerable set. König asserts that it is necessary to admit the existence of elements in the continuum which go beyond "finite laws" in his sense of the term, and that there exist elements "die wir nicht 'zu Ende' denken können," and which are yet free from contradiction. Of no irrational number can the expression "zu Ende denken" be rightly used if we regard the number as in process of formation, from the point of view of the process itself. The warrant of the uniqueness of the object defined is contained in the definition itself, the determinancy of the process being the test of the adequacy of the definition to supply us with the conception of a distinct object. It is unnecessary that any part of the process have been actually carried out, and all questions as regards the mere practicability of the process are irrelevant. Any definition which is adequate in the sense defined above, no matter how the definition may be expressed, or what implications are contained in it, is sufficient to supply us with the conception of an object which has a definite ordinal relation with the rational numbers.

If we regard the continuum as containing every number capable of adequate definition, in whatever form, and as containing no elements which are to be regarded as in any special sense ideal, it appears that the continuum so conceived has the properties which are essential to its fitness to be regarded as the domain of the real variable. For it is connex, *i.e.*, having given two numbers in it, other numbers ordinally between the two can be defined: this follows from the connexity of the aggregate of rational numbers. Again it is perfect; for, if $x_1, x_2, x_3, \ldots$ be any defined convergent sequence of numbers, each of which is adequately defined, the limit x of the sequence is definable adequately in a form involving the use of $x_1, x_2, x_3, \ldots$ as parameters; and, conversely, any adequately defined number can be exhibited as the limit of a convergent sequence of other numbers, in particular of rational numbers. The continuum so conceived thus possesses the two properties of being connex, and of being perfect, and these are sufficient for the purposes of arithmetic analysis.

Some mathematicians appear to have the impression that there must in some sense exist in the continuum numbers incapable of adequate definition, and only capable of representation as endless decimals in which each figure is to be regarded as quite arbitrarily assigned. In the first place, as has just been shewn, the arithmetic continuum is complete for the purposes for which it exists, without taking into account such nebulous entities as lawless decimals, even if any precise meaning can be assigned to the assertion of their existence. Moreover, in respect of such a lawless infinite decimal, it cannot be rightly asserted that the object exists as a single whole; only so much of it exists at any one time as is represented by the figures which have been actually chosen at that time, and the number of such figures must be finite. In the case of an adequately defined irrational number, on the other hand, its existence is quite independent of the number of figures in the decimal representation of it which may at any one time have been calculated, or indeed of whether any of them are ever calculated. The process of arbitrarily choosing figures one after the other, without cessation, involves the idea of endlessness only, and this is quite distinct from the truly infinite process which can be regarded as defining a definite object. In the latter case the process regarded from outside is a completed one embodied in the law which dominates it; in the former case it is impossible to regard the process from the outside.

For reasons which I have elsewhere* explained, there appears to be no adequate reason for thinking that any unenumerable aggregate is capable of being normally ordered (*wohlgeordnet*), and this of course includes the case of the continuum. The proof which König has given that the continuum cannot be normally ordered depends, however, on the distinction which he has drawn between those elements which are finitely defined and those which are only ideal, and stands or falls with the validity of this distinction.

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* "On the General Theory of Transfinite Numbers and Order Types," *Proc. London Math. Soc.*, Ser. 2, Vol. 3.