On Curves obtained by an extension of Maclaurin's method of constructing Conics. By Samuel Roberts.

[*Read Feb. 8th*, 1883.]

In Maclaurin's construction, the points of a conic are marked by the vertex of a variable triangle, whose sides pass respectively through fixed points, and whose remaining vertices move on given straight lines. The generalisation above referred to is briefly noticed in Salmon's *Higher Plane Curves* (1st edition, p. 247), where the degree of the locus is given in several comprehensive cases. In what follows, I treat the subject in rather more detail by determining the degree and class of the loci under slightly more general conditions. There are special cases which seem to be of some interest; but many of these are of a simple character, and have been elsewhere given in a more or less isolated way. I confine myself generally to conditions which do not lead to very complicated singularities. Analogous results could probably be obtained in more intricate cases, if for any special purpose it became desirable. Thus, instead of ordinary double points or multiple points on a curve directrix, we might assume singularities equivalent to a certain number of double points and cusps, and so forth. It will often happen that, under such circumstances, the degree and class will still be given by the general formula.

For shortness I use the symbol C_m^n to denote a curve of the m^{th} degree and n^{th} class.

I.—If the sides BC, CA, AB of a triangle ABC pass respectively through the fixed points P, Q, R, while the vertices B, A move respectively on the curves $C_{m_1}^{n_1}$, $C_{m_2}^{n_2}$, and if P is multiple on the curve $C_{m_1}^{n_1}$ to the order p, Q is multiple on the curve $C_{m_2}^{n_2}$ to the order q, and R is multiple on the curve $C_{m_1}^{n_1}$ to the order r, and on the curve $C_{m_2}^{n_2}$ to the order r'; then the degree of the locus of the vertex C is

$$(m_1-p)(m_2-r')+(m_2-q)(m_1-r),$$

and the class is

$$2(m_1-p-r)(m_2-q-r')+n_1(m_2-r')+n_2(m_1-r)-2pq.$$

The points P, Q, R are supposed to be simply multiple on the directrices.

The circumstance to be specially noticed here and in subsequent cases, is that the degree of the locus is not affected by point singularities on the curve directrices, except when they exist in special situations, as for example at the points P, Q, R, or on the line joining

P, Q. This appears by the actual process of the generation, for point singularities on the directrices in general produce point singularities on the locus.

Let us take then, for $C^{n_1}_{m_1}$, m_1 straight lines, of which p pass through P and r through R; for $C^{n_2}_{m_2}$, m_2 straight lines, of which q pass through Q and r' through R.

If $m_1 = m_2 = 1$, $n_1 = n_2 = 0$, we have Maclaurin's construction for conics. If, however, P or R is on $C^{n_1}_{m_1}$, Q or R on $C^{n_2}_{m_2}$, the locus is a straight line through the point. If both P and R or Q and R are on the directrix of B in the first instance, or of A in the second, the locus is still a straight line. If P is on the directrix of B, and Q on that of A, or if the directrices meet in R, the degree is zero. The fact of P being on the directrix of A, or Q on that of B, does not affect the degree.

There results then, for the general degree,

$$\begin{aligned}&2(m_1-p-r)(m_2-q-r')+(m_1-p)\,q+(m_2-q)\,p\\&\quad+(m_1-p-r)\,r'+(m_2-q-r')\,r\\&=(m_1-p)(m_2-r')+(m_2-q)(m_1-r).\end{aligned}$$

We may now consider the general curves $C^{n_1}_{m_1}$, $C^{n_2}_{m_2}$. If any transversal is drawn through P, there are on it only $(m_1-p)(m_2-r')$ points of the locus distinct from P. Hence P is a multiple point of the order $(m_2-q)(m_1-r)$, and by symmetry Q is a multiple point of the order $(m_1-p)(m_2-r')$. Except when further special conditions are imposed, the tangents at these multiple points will be distinct; that is, the points are ordinary multiple points.

Consider now the tangents which can be drawn to the locus from and at the point P, counting those at P twice.

From P we can draw n_1-2p tangents to $C^{n_1}_{m_1}$ (unless a branch through P is rectilinear, which case is exceptional), and these are multiple tangents to the locus, having in general m_2-r' distinct points of contact. If we draw tangents to $C^{n_2}_{m_2}$ from R, they also are in general n_2-2r' in number (the case of a rectilinear branch through R being exceptional) and determine m_1-r tangents to the locus from P.

This includes all the ways in which tangents to the locus from P arise.

The necessity for excluding the case of rectilinear branches follows from the circumstance that n_1-2p does not express the number of tangents which can be drawn from a point ($p = 1$) on a straight line, or from a multiple point of the order p forming the intersection of lines, one or more of which are straight. Sometimes the general expression for the class will in such a case become negative, thus indicating the discontinuity, but it may happen otherwise.

Taking the number of tangents at and from P to the locus, we get the class

$$2\,(m_2-q)\,(m_1-r)+(n_1-2p)\,(m_2-r')+(n_2-2r')\,(m_1-r)$$
$$=2\,(m_1-p-r)\,(m_2-q-r')+n_1\,(m_2-r')+n_2\,(m_1-r)-2pq.$$

Certain results of the generation can be immediately discerned. The intersections of the curve directrices are points on the locus. The straight line joining Q, R meets $C^{n_1}_{m_1}$ in m_1-r points distinct from P, and these are multiple points on the locus of the order m_2-q-r'. Similarly the straight line joining P, R meets $C^{n_2}_{m_2}$ in m_2-r' points distinct from Q, and these are multiple points on the locus of the order m_1-p-r.

The degree of the locus suffers a reduction, if intersections of the directrices lie on the straight line joining P, Q; for the sum of the orders of the multiple points at P and Q is the general degree.

The degree may also suffer a reduction in consequence of transversals through P being common tangents to the directrices. These create additional double points. If there are t such common tangents, the class is reduced by $2t$.

A double point or cusp on $C^{n_1}_{m_1}$ occasions m_2-r' double points or cusps on the locus. *Mutatis mutandis*, the same holds for $C^{n_2}_{m_2}$.

The following particular cases occur :—

If $\quad p=q=r=r'=1,\;$ and $\;m_1=m_2=2,$
or if $\quad p=r=m_2=1,\;\; q=r'=0,\;\; m_1=2,$
or if $\quad q=r'=m_1=1,\;\; p=r=0,\;\; m_2=2,$
the locus is a conic.

If $\quad p=q=1,\;\; r=r'=0,\;\; t=2,\;\; m_1=m_2=2,$
the locus is of the fourth degree and fourth class, breaking up into two conics (Salmon's *Conics*, 5th edition, Ex. 7, p. 288).

If $p=r=r'=1,\; q=0,\; m_1=m_2=2,$ or if $p=q=r'=1,\; r=0,$
$m_1=m_2=2,$ or if $q=r=r'=1,\; p=0,\; m_1=m_2=2,$
the locus is a unicursal cubic.

If $\quad p=q=r=r'=0,\;\; m_1=m_2=2,\;\; t=0,$
the locus is of the eighth degree and sixteenth class. But, if $t=2$, the locus is of the twelfth class, breaking up into two unicursal quartics. The points P, Q are double on both; the other two double points correspond to the common tangents to the directrices through R.

If $p=q=1,\; m_1=m_2=2,\; r=r'=0$, and one intersection of the conic directrix lies on the straight line joining P, Q, the locus is of the third degree and sixth class.

The three intersections of the conics not on PQ are on the locus—call them 1, 2, 3. Then P is the coresidual of 1, 2, 3, Q, and Q is the coresidual of 1, 2, 3, P.

II. If in the construction of I. the points P, Q, R are collinear, the degree of the locus of C becomes

$$(m_1-p)(m_2-r')+p(m_2-q-r')+r'(m_1-r) = m_1m_2-pq-rr',$$

and the class is

$$2p(m_2-q-r')+2r'(m_1-r)+(n_1-2p)(m_2-r')+(n_2-2r')(m_1-r)$$
$$= m_2n_1+m_1n_2-rn_2-r'n_1-2pq.$$

When we take two straight lines for directrices, the locus is a straight line, unless P is on the directrix of B, and Q on that of A. If this is the case, the degree of the locus is zero.

Taking then, for $C_{m_1}^{n_1}$, m_1 straight lines of which p pass through P, and for $C_{m_2}^{n_2}$, m_2 straight lines of which q pass through Q, we get for the degree

$$(m_1-p)(m_2-q)+(m_1-p)q+(m_2-q)p,$$

which is the general degree for the case $r = r' = 0$.

Now, suppose, instead of straight lines, we take for $C_{m_1}^{n_1}$ M conics, p of which pass through P and r through R; and for $C_{m_2}^{n_2}$ M' conics, of which q pass through Q and r' through R.

For two conic directrices, both passing through R, the locus is a cubic, and so also if $p = q = 1$.

The degree for C_{2M}^{2M} and $C_{2M'}^{2M'}$ as directrices is then

$$4\{(M-p-r)(M'-q-r')+(M-p)q+(M'-p)p$$
$$+(M-p-r)r'+(M'-q-r')r\}+3(pq+rr')$$
$$= 4MM'-pq-rr'.$$

To meet the case of m_1 or m_2 odd, or both, we may add a straight line to C_{2M}^{2M} or $C_{2M'}^{2M'}$, or both; the addition is $2M$ or $2M'$ or $2(M+M'+1)$ in the respective cases. Hence the general expression is

$$m_1m_2-pq-rr'.$$

As to the class, the number of points on the locus lying on a transversal through P and distinct from P is $(m_1-p)(m_2-r')$. Hence there is a multiple point of the order $p(m_2-q-r')+r'(m-r)$ at P. In general, this will be an ordinary multiple point. Taking the tangents to the locus through P, and counting those at P twice, we have for the class

$$2p(m_2-q-r')+2r'(m-r)+(n_1-2p)(m_2-r')+(n_2-2r')(m-r)$$
$$= m_1n_2+m_2n_1-r'n_1-rn_2-2pq.$$

For a particular case, we have $p = q = 1$, $r = r' = 0$, $m_1 = m_2 = 2$; the locus is a cubic of the sixth class.

In fact, a general cubic can thus be drawn. For the tangents to the conics at P, Q are tangents to the locus at the same points, and their intersection is the common tangential of P, Q. If then we have given two tangents OP, OQ of a cubic through O, P, Q (P, Q being the points of contact), we can draw a conic having OP for a tangent at P, and passing through three arbitrary points. We can also draw another conic having OQ for a tangent at Q, and passing through the same three points. If we draw two straight lines through P and Q respectively, meeting in a fourth arbitrary point, we determine R. Thus we can describe by the present method a cubic through four arbitrary points, and having at P, Q the tangents OP, OQ, whose common tangential is O. The points O, P, Q are also optional.

The fourth intersection of the conics is common to the cubics of the system passing through the three arbitrary points first taken.

If a transversal through P touches the directrices in points distinct from R, there arises a double point on the locus. The class reduction is $2t$ for t such transversals.

Thus, if $m_1 = m_2 = 2$, $p = q = r = r' = 0$, $t = 2$, the locus breaks up into two conics. If we take for directrices two circles and a centre of similitude R, with two points P, Q collinear with R, one part of the locus is a circle, and the other a conic through the finite intersections of the circle directrices. Otherwise, the two factors of the locus are analogously related.

The result is equivalent (as to the circle factor of the locus) to an elementary property of homologous points on three circles.

The centres of similitude lie collinearly in threes. If we take three homologous points, one on each circle, and join them in pairs by straight lines, these will respectively pass through one of three collinear centres of similitude. Thus, if p, q, r are the homologous points, and P, Q, R the centres of similitude through which the straight lines qr, rp, rq respectively pass, pqr forms a triangle conditioned in the manner supposed in the general construction. Hence, if q, r move on their circles, p describes the third circle. But if we take anti-homologous points for q, r, the locus of p is a conic analogously related to the figure.

I have shown elsewhere* that, if $m_2 = 1$, $p = q = r = r' = 0$, the degree of the locus is m_1 and the class n_1; and that the locus is really a homographic transformation of $C_{m_1}^{n_1}$.

III. We will now suppose that the directrices $C_{m_1}^{n_1}$, $C_{m_2}^{n_2}$, of I., coincide with C_m^n. The degree of the locus is then

$$(m-p)(m-r-1)+(m-q)(m-r-1) = (2m-p-q)(m-r-1),$$

* Reprint of Mathematics from the *Educational Times*, Vol. xxix., p. 96.

and the class is

$$2(m-q)(m-r-1)+(n-2p)(m-r-1)+(n-2r)(m-r-2).$$

The curve C_m^n is assumed to have a multiple point of the order p at P, one of the order q at Q, and one of the order r at R.

Take for C_m^n M conics, of which p pass through P, q through Q, and r through R; these conics may be taken as distinct. Then, remembering that, in the case of a pair of conic directrices, either may be a directrix of B and also of A, we find for the degree

$$(8+8)\,\frac{(M-p-q-r)(M-p-q-r-1)}{2}$$
$$+\,(8+6)(M-p-q-r)(p+q)+4(M-p-q-r)$$
$$+\,(8+4)\{(M-p-q-r)r+pq\}$$
$$+\,(6+6)\left\{\frac{p(p-1)}{2}+\frac{q(q-1)}{2}\right\}+(4+4)\left(\frac{r\,.\,r-1}{2}\right)$$
$$+\,(5+6)\{p+q\}r+3(p+q+r)$$
$$=8(M-p-q-r)(M-p-q-r-1)$$
$$+(M-p-q-r)(14p+14q+12r+4)$$
$$+4r(r-1)+6\{p(p-1)+q(q-1)\}+11(p+q)r$$
$$+3(p+q+r)+12pq$$
$$=8M^2-4M-2M(p+q+2r)+p+q+pr+qr$$
$$=2m(m-1)-m(p+q+2r)+p+q+pr+qr,\ \text{ if } 2M=m.$$

To meet the case of m odd, we may add a straight line to C_m^n; the addition to the degree is

$$8(M-p-q-r)+(4+3)p+(4+3)q+(3+3)r=8M-p-q-2r,$$

giving the same result.

The points of the locus distinct from P on a transversal through that point, are in number $(m-p)(m-r-1)$. There is, therefore, a multiple point of the order $(m-q)(m-r-1)$ at P; and similarly one of the order $(m-p)(m-r-1)$ at Q.

Counting tangents from and at P as in previous cases, we find for the class

$$2(m-q)(m-r-1)+(n-2p)(m-r-1)+(n-2r)(m-r-2).$$

If $m=2$, $p=q=1$, $r=0$, the locus is a conic. This is an instructive example, for the locus remains the same for any conic directrix passing through P, Q, and any two points on the original directrix collinear with R. It remains the same, therefore, for an infinite number of straight directrices in pairs. In fact, we may take as directrices the two sides of any generating triangle, which pass through P and Q respectively. The locus is that of the points of contact of tangents from R to the system of conic directrices. This is an instance in which the class formula fails for the straight directrices, $n-2p$ being negative and illusory.*

IV. If, in III., P, Q, R are collinear, the degree is

$$(m-p)(m-r-1)+p(m-q-r-1)+r(m-r-1)$$
$$= m(m-1)-pq-r(r+1),$$

and the class is

$$2p(m-q-r-1)+2r(m-r-1)+(n-2p)(m-r-1)+(n-2r)(m-r-2)$$
$$= n(2m-3)-2(n-1)r-2pq.$$

As in the preceding case, we take for C_m^n M conics. Then the degree is

$$(4+4)\left\{\frac{(M-p-q-r)(M-p-q-r-1)}{2} + (M-p-q-r)(p+q+r)+qr+pr\right\}+$$
$$(4+3)pq+2(M-p-q-r)+3r(r-1)+4p(p-1)+4q(q-1)+2p+2q$$
$$= 2M(2M-1)-pq-r(r+1).$$

We may again meet the case of m odd by adding a straight line. The consequent addition is

$$4(M-p-q-r)+(2+2)(p+q+r)=4M.$$

The general degree then is

$$m(m-1)-pq-r(r+1).$$

On a transversal through P, there are $(m-p)(m-r-1)$ points of the locus distinct from P. There is then at P a multiple point of the order $p(m-q-r-1)+r(m-r-1)$. As in previous cases, we get the class above given.

If $m=n=2$, $p=q=1$, $r=0$, the locus is a straight line, evidently the polar of R with respect to the conic.

If $m=3$, $n=6$, $p=q=r=1$, the locus is also a cubic of the sixth class.

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* The lines RP, RQ are tangents to the locus at P, Q. Two points on the conic directrix collinear with R determine, therefore, a fifth point of the conic locus. Analytically, the result comes out as follows:—

Let RP, PQ, QR be the sides β, γ, α of the triangle of reference, and $C\gamma^2+2D\beta\gamma+2E\alpha\gamma+2F\alpha\beta=0$ the equation of the conic directrix.

Then, if a tranversal through R (α, β) meets the conic in the points (α', β'), (α'', β''), we have, putting $\alpha-k\beta=0$ for the equation of the transversal,

$$\frac{\beta'\beta''}{\gamma'\gamma''}=\frac{C}{2Fk}=\frac{\beta'\alpha''}{\gamma'\gamma''k};$$

and X, Y, Z being the current coordinates, so that

$$Z\beta'-Y\gamma'=0,\quad Z\alpha''-X\gamma''=0,$$

we have $CZ^2-2FXY=0$ independently of D, E.

V. If the p sides of a polygon pass through fixed points respectively and all the vertices but one move on curves $C_{m_1}^{n_1}, C_{m_2}^{n_2} \dots C_{m_{p-1}}^{n_{p-1}}$ respectively, the degree of the locus of the remaining vertex is $2m_1m_2 \dots m_{p-1}$, and the class is

$$2m_1m_2 \dots m_{p-1}+n_1m_2m_3 \dots m_{p-1}+n_2m_1m_3 \dots m_{p-1}+\dots+n_{p-1}m_1 \dots m_{p-2}.$$

It may be shown (Salmon's *Higher Plane Curves*, 1st ed., p. 247) that on a transversal through (say) the fixed point on a side of the describing vertex, there are $m_1m_2 \dots m_{p-1}$ points of the locus distinct from that point, and that the point itself is multiple in the locus to the order $m_1m_2 \dots m_{p-1}$. Or we can separate the polygon into triangles. Thus, if the polygon is a quadrilateral $M_1M_2M_3C$, C being the describing vertex, let P_1, P_2, P_3, P_4 be the fixed points through which pass respectively CM_1, M_1M_2, M_2M_3, M_3C. Produce M_1C and M_2M_3 to meet in K. Then the locus of K is of the degree $2M_1M_2$, with P_1, P_3 for multiple points of the order m_1m_2.

Again, the locus of C is, by the formula of I.,

$$m_3(2m_1m_2-m_1m_2)+m_3(2m_1m_2-m_1m_2) = 2m_1m_2m_3.$$

We can use the same process for the general polygon. As to the class, for the quadrilateral, the class of the locus of K is

$$2m_1m_2+m_1n_2+m_2n_1,$$

and the class of the locus of C is, by III.,

$$2(2m_1m_2-m_1m_2-m_1m_2)m_3+(2m_1m_2+n_1m_2+n_2m_1)m_3+n_3(2m_1m_2-m_1m_2)$$
$$= 2m_1m_2m_3+n_1m_2m_3+n_2m_1m_3+n_3m_1m_2.$$

We can now assume the formula for a polygon of $p-1$ sides (taking notice also of the multiple points at two of the fixed points), and thence, as above, show that the same formula holds for a polygon of p sides.

We may, indeed, similarly treat the case in which the fixed points are multiple points on the respective directrices with which they are associated.

It will be sufficient to take the case of a quadrilateral, $M_1M_2M_3C$. Suppose on $C_{m_1}^{n_1}$ that P_1 is a multiple point of the order p_1, and P_2 a multiple point of the order p_2. Let the corresponding orders for P_2, P_3 on $C_{m_2}^{n_2}$, and P_3, P_4 on $C_{m_3}^{n_3}$, be r_2, r_3 and s_3, s_4.

Then the degree of the locus of K is

$$(m_1-p_1)(m_2-r_2)+(m_1-p_2)(m_2-r_3),$$

and the class is

$$2(m_1-p_1-p_2)(m_2-r_2-r_3)+n_1(m_2-r_2)+n_2(m_1-p_1)-2p_1r_3, \text{ by I.}$$

The locus of C is then of the degree

$$(m_3-s_4)(m_1-p_2)(m_2-r_3)+(m_3-s_3)(m_1-p_1)(m_2-r_2),$$

and the class is

$$n_3(m_1-p_2)(m_2-r_3)+\{2(m_1-p_1-p_2)(m_2-r_2-r_3)+n_1(m_2-r_2)$$
$$+n_2(m_1-p_2)-2p_1r_3\}(m_3-s_3)-2s_4(m_1-p_2)(m_2-r_3)$$
$$=2(m_3-s_3-s_4)(m_2-r_3-r_2)(m_1-p_1-p_2)$$
$$+n_3(m_1-p_2)(m_2-r_3)+n_2(m_1-p_2)(m_3-s_3)+n_1(m_2-r_2)(m_3-s_3)$$
$$-2m_1s_4r_2-2m_2s_3p_1-2m_3r_3p_1+2s_4p_1r_3+2s_4r_2p_2+2s_4p_1r_2+2p_1r_3s_3.$$

VI. For the fixed points of I., we may substitute curves $C^{N_1}_{M_1}$, $C^{N_2}_{M_2}$, $C^{N_3}_{M_3}$, which are touched respectively by the sides BC, CA, AB of the generating triangle. These curves are supposed not to have special relations to the other directrices.

The degree of the locus is then $2N_1N_2N_3m_1m_2$, and the class is

$$N_1N_2N_3(2m_1m_2+n_1m_2+n_2m_1)+m_1m_2(N_1N_2M_3+N_1N_3M_2+N_2N_3M_1).$$

In fact, if we take for $C^{N_1}_{M_1}$ N_1 points, for $C^{N_2}_{M_2}$ N_2 points, and for $C^{N_3}_{M_3}$ N_3 points, the degree of the locus is $2N_1N_2N_3m_1m_2$ by what precedes, and this result remains unaltered by the double tangents arising from the points N_1 taken two and two, the points N_2 taken two and two, or the points N_3 taken two and two.

A double tangent of this kind does, however, affect the class, since it occasions, if existing on $C^{N_1}_{M_1}$ (now considered as a proper curve), $N_2N_3m_1m_2$ double points on the locus. The case is similar for the other directrices $C^{N_2}_{M_2}$ and $C^{N_3}_{M_3}$. Making allowance for this, and also for the existence of double tangents and inflexions, we must add to the first result $N_1N_2N_3(2m_1m_2+n_1m_2+n_2m_1)$ the following complement

$$m_1m_2(N_1N_2M_3+N_1N_3M_2+N_2N_3M_1).$$

If we generalise III. in a similar fashion, the degree obtained is $2N_1N_2m(m-1)$, and the class is

$$N_1N_2N_3\{n(2m-3)+2m(m-1)\}$$
$$+m(m-1)(N_1N_2M_3+N_1N_3M_2+N_2N_3M_1).$$

It will, without doubt, be remarked how far the cases which I have dealt with fall short of being exhaustive. Some additional ones could be added if we were content with the determination of the degree of the locus. The determination of the class is accompanied with special difficulties, and the few results I have obtained in this direction are imperfect. The problem, in its ultimate form, requires us to determine the degree and class of the locus when the directrices of the two vertices of the generating triangle, and those which are touched by the sides, coincide in one curve. Intermediate cases lead up to this.

On the use of certain Differential Operators in the Theory of Equations. By J. HAMMOND.

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1. If $D_1, D_2, D_3, \ldots$ denote operators defined by the equation

$$\phi\left(p_1-\frac{1}{y},\ p_2-\frac{p_1}{y},\ p_3-\frac{p_2}{y},\ \ldots\right)$$
$$=(1-y^{-1}D_1+y^{-2}D_2-y^{-3}D_3+\ldots)\,\phi\,(p_1, p_2, p_3, \ldots)\ldots\ldots\ldots(1),$$

it has been shown in a former paper (*Proceedings*, Vol. xiii., p. 79) that when

$$\phi\,(p_1, p_2, p_3, \ldots)=\Sigma a_1^{\lambda} a_2^{\lambda} \ldots a_l^{\lambda} \beta_1^{\mu} \beta_2^{\mu} \ldots \beta_m^{\mu} \gamma_1^{\nu} \gamma_2^{\nu} \ldots \gamma_n^{\nu} \ldots$$
$$=(\lambda^l . \mu^m . \nu^n \ldots) \text{ suppose,}$$

where $a_1, a_2, \ldots \beta_1, \beta_2, \ldots \gamma_1, \gamma_2, \ldots$ are roots of the equation

$$1-x^{-1}p_1+x^{-2}p_2-x^{-3}p_3+\ldots=0\ldots\ldots\ldots\ldots\ldots\ldots\ (2),$$

$$D_\kappa(\lambda^l . \mu^m . \nu^n \ldots)=0,\ \ D_\lambda(\lambda^l . \mu^m . \nu^n \ldots)=(\lambda^{l-1} . \mu^m . \nu^n \ldots),\ \ D_\lambda(\lambda)=1.$$

The operators of the present paper are the symmetric functions of the roots of the equation

$$1-y^{-1}D_1+y^{-2}D_2-y^{-3}D_3+\ldots=0\ \ \ldots\ldots\ldots\ldots\ (3),$$

considered simply as rational integral functions of $D_1, D_2, D_3, \ldots$, when operating on $(\lambda^l . \mu^m . \nu^n \ldots)$ and as differential operators on $\phi\,(p_1, p_2, p_3, \ldots)$. It is easy to see that these operators follow the ordinary laws of quantity in their combinations with constants and with each other; *e.g.*, if $\phi\,(p_1, p_2, p_3, \ldots)=(\lambda^l . \mu^m . \nu^n \ldots)$, we have

$$D_\lambda D_\mu \phi=(\lambda^{l-1} . \mu^{m-1} . \nu^n \ldots)=D_\mu D_\lambda \phi,$$

and D_λ, D_μ obey the commutative law. In the case of the remaining laws, a formal proof is hardly required.

The meaning of equation (3), or of its roots, does not at all concern us, as they are only used for purposes of concise definition; thus, if

$$1-y^{-1}D_1+y^{-2}D_2-y^{-3}D_3+\ldots=\left(1-\frac{a}{y}\right)\left(1-\frac{b}{y}\right)\left(1-\frac{c}{y}\right)\ldots,$$

and if $\Sigma a^\lambda b^\mu c^\nu \ldots=[\lambda . \mu . \nu \ldots]$, using [ ] instead of ( ) to show that the roots referred to are $a, b, \ldots$ instead of $a, \beta, \ldots$, we have

$$D_1=\Sigma a=[1],\ \ D_2=\Sigma ab=[1^2],$$
$$D_\kappa=[1^\kappa] \text{ and } D_1^2-2D_2=\Sigma a^2=[2].$$

This notation was suggested by Professor Cayley.

2. A line placed over any expression will be used to denote that all its symbols are to be combined by the ordinary laws of quantity; thus from (1) we obtain directly

$$D_\kappa = \frac{1}{\kappa\,!}\,\overline{\left(\frac{d}{dp_1}+p_1\frac{d}{dp_2}+p_2\frac{d}{dp_3}+\ldots\right)^\kappa} = \frac{1}{\kappa\,!}\,\overline{d_1^\kappa} \quad \ldots\ldots\ldots \ (4),$$

where, for shortness,

$$\left.\begin{aligned} d_1 &= \frac{d}{dp_1}+p_1\frac{d}{dp_2}+p_2\frac{d}{dp_3}+\&c. \\ d_2 &= \frac{d}{dp_2}+p_1\frac{d}{dp_3}+p_2\frac{d}{dp_4}+\&c. \\ d_3 &= \frac{d}{dp_3}+p_1\frac{d}{dp_4}+p_2\frac{d}{dp_5}+\&c. \\ &\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\ d_\lambda &= \frac{d}{dp_\lambda}+p_1\frac{d}{dp_{\lambda+1}}+p_2\frac{d}{dp_{\lambda+2}}+\&c. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \ (5).$$

Now $$d_\lambda D_\kappa = \frac{1}{\kappa\,!}\,d_\lambda\overline{d_1^\kappa} = \frac{1}{\kappa\,!}\overline{d_\lambda d_1^\kappa} + \frac{1}{(\kappa-1)\,!}\,\overline{d_1^{\kappa-1}d_{\lambda+1}},$$

the last term arising from the differentiation of D_κ considered as an explicit function of $p_1, p_2, p_3, \ldots$

This result may also be written

$$d_\lambda D_\kappa = \overline{d_{\lambda+1} D_{\kappa-1}} + \overline{d_\lambda D_\kappa} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (6).$$

Reducing by means of (6), we have,

$$\begin{aligned} &d_\lambda - d_{\lambda-1}D_1 + d_{\lambda-2}D_2 - d_{\lambda-3}D_3 + \ldots + (-)^{\lambda-1}d_1 D_{\lambda-1} \\ &= d_\lambda - (d_\lambda + \overline{d_{\lambda-1}D_1}) + (\overline{d_{\lambda-1}D_1} + \overline{d_{\lambda-2}D_2}) - (\overline{d_{\lambda-2}D_2} + \overline{d_{\lambda-3}D_3}) + \ldots \\ &\qquad \ldots + (-)^{\lambda-1}(\overline{d_2 D_{\lambda-2}} + \overline{d_1 D_{\lambda-1}}) \\ &= (-)^{\lambda-1}\,\overline{d_1 D_{\lambda-1}} = \frac{(-)^{\lambda-1}}{(\lambda-1)\,!}\,\overline{d^\lambda} = (-)^{\lambda-1}\lambda D_\lambda. \end{aligned}$$

Hence $$d_\lambda - d_{\lambda-1}D_1 + d_{\lambda-2}D_2 - d_{\lambda-3}D_3 + \ldots + (-)^\lambda\lambda D_\lambda = 0 \quad \ldots\ldots\ldots (7),$$

which, with reference to (3), is Newton's rule for the sums of the powers of the roots, and gives

$$\left.\begin{aligned} d_1 &= D_1 \\ d_2 &= D_1^2 - 2D_2 \\ d_3 &= D_1 - 3D_1 D_2 + 3D_3 \\ d_4 &= D_1^4 - 4D_1^2 D_2 + 2D_2^2 + 4D_1 D_3 - 4D_4 \\ &\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \ (8),$$

and in the general case $d_\lambda = [\lambda]$.

3. The differential operators corresponding to the other symmetric functions are found in terms of $d_1, d_2, d_3, \ldots$ by a process parallel to the expression of the symmetric functions in terms of the sums of the powers in the ordinary Algebraic Theory. For

$$d_\lambda d_\mu = \left(\frac{d}{dp_\lambda} + p_1 \frac{d}{dp_{\lambda+1}} + p_2 \frac{d}{dp_{\lambda+2}} + \ldots\right)\left(\frac{d}{dp_\mu} + p_1 \frac{d}{dp_{\mu+1}} + p_2 \frac{d}{dp_{\mu+2}} + \ldots\right)$$

$$= \overline{\left(\frac{d}{dp_\lambda} + p_1 \frac{d}{dp_{\lambda+1}} + \ldots\right)\left(\frac{d}{dp_\mu} + p_1 \frac{d}{dp_{\mu+1}} + \ldots\right)} + \frac{d}{dp_{\lambda+\mu}} + p_1 \frac{d}{dp_{\lambda+\mu+1}} + \ldots$$

Or, after abbreviation and transposition,

$$\overline{d_\lambda d_\mu} = d_\lambda d_\mu - d_{\lambda+\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9).$$

Comparing this with the well-known Algebraic formula

$$\Sigma \alpha^\lambda \beta^\mu = S_\lambda S_\mu - S_{\lambda+\mu},$$

we see that $\overline{d_\lambda d_\mu} = [\lambda . \mu]$.

All other formulæ of this class are deducible from (9); for example,

$$d_\lambda d_\mu d_\nu = d_\lambda (\overline{d_\mu d_\nu} + d_{\mu+\nu})$$

$$= \overline{d_\lambda d_\mu d_\nu} + \overline{d_{\lambda+\mu} d_\nu} + \overline{d_\mu d_{\nu+\lambda}} + \overline{d_\lambda d_{\mu+\nu}} + d_{\lambda+\mu+\nu},$$

which, after further reduction of the three middle terms on the right by means of (9), becomes

$$\overline{d_\lambda d_\mu d_\nu} = d_\lambda d_\mu d_\nu - d_\lambda d_{\mu+\nu} - d_\mu d_{\nu+\lambda} - d_\nu d_{\lambda+\mu} + 2d_{\lambda+\mu+\nu} \ldots\ldots \quad (10),$$

showing that $\overline{d_\lambda d_\mu d_\nu} = [\lambda . \mu . \nu]$.

But the law $\overline{d_\lambda d_\mu d_\nu \ldots} = [\lambda . \mu . \nu \ldots]$,

which is perfectly general when $\lambda, \mu, \nu, \ldots$ are all different, requires modification to meet the case where some of them are equal. This is a necessary consequence of the modification of the Algebraic formula which gives $\Sigma \alpha^\lambda \beta^\mu \gamma^\nu \ldots$ in terms of the sums of the powers, and makes no difference in formulæ such as (9) and (10), which are absolutely correct, even when all the suffixes $\lambda, \mu, \nu, \ldots$ are equal.

Thus, as particular cases of (9) and 10), we have

$$\left.\begin{aligned} \overline{d_\lambda^2} &= d_\lambda^2 - d_{2\lambda} \\ \overline{d_\lambda^3} &= d_\lambda^3 - 3d_\lambda d_{2\lambda} + 2d_{3\lambda} \\ \overline{d_\lambda d_\mu^2} &= d_\lambda d_\mu^2 - d_\lambda d_{2\mu} - 2d_\mu d_{\lambda+\mu} + 2d_{\lambda+2\mu} \end{aligned}\right\} \ldots\ldots\ldots\ldots \quad (11),$$

which, when compared with the corresponding Algebraic formulæ

$$2!\, \Sigma \alpha^\lambda \beta^\lambda = S_\lambda^2 - S_{2\lambda},$$

$$3!\, \Sigma \alpha^\lambda \beta^\lambda \gamma^\lambda = S_\lambda^3 - 3S_\lambda S_{2\lambda} + 2S_{3\lambda},$$

$$2!\, \Sigma \alpha^\lambda \beta^\mu \gamma^\mu = S_\lambda S_\mu^2 - S_\lambda S_{2\mu} - 2S_\mu S_{\lambda+\mu} + 2S_{\lambda+2\mu},$$

give the identities

$$\overline{d_\lambda^2} \div 2\,! = [\lambda^2], \quad \overline{d_\lambda^3} \div 3\,! = [\lambda^3], \quad \overline{d_\lambda\, d_\mu^2} \div 2\,! = [\lambda \,.\, \mu^2].$$

And, from these considerations, it follows in general that

$$\overline{d_\lambda^l\, d_\mu^m\, d_\nu^n \ldots} \div l\,!\; m\,!\; n\,! \ldots = [\lambda^l \,.\, \mu^m \,.\, \nu^n \ldots] \;\ldots\ldots\ldots\; (12).$$

4. Every known symmetric function formula now gives a relation between the operators, and *vice versâ*. Thus formulæ (6) and (7) (*Proceedings*, Vol. xiii., p. 81) give immediately

$$\overline{d_2^m} \div m\,! = D_m^2 - 2D_{m-1}D_{m+1} + 2D_{m-2}D_{m+2} - \ldots + (-)^m\, 2D_{2m},$$

$$\overline{d_2^m\, d_1} \div m\,! = D_m D_{m+1} - 3D_{m-1}D_{m+2} + 5D_{m-2}D_{m+3} - \ldots$$
$$\ldots + (-)^m (2m+1)\, D_{2m+1},$$

and if, in (6) of the present paper, we put $\lambda = 1$, $\kappa = n-1$, we have

$$d_1 D_{n-1} = \overline{d_2 D_{n-2}} + \overline{d_1 D_{n-1}}$$

the last term of which is nD_n, since $D_\kappa = \overline{d_1^\kappa} \div \kappa\,!$.

Hence $$\overline{d_2\, d_1^{n-2}} \div (n-2)\,! = D_1 D_{n-1} - nD_n,$$

which gives the Algebraic formula

$$(2 \,.\, 1^{n-2}) = p_1 p_{n-1} - np_n.$$

The general formula (6) gives, when interpreted,

$$p_\kappa \Sigma a^\lambda = (\lambda+1 \,.\, 1^{\kappa-1}) + (\lambda \,.\, 1^\kappa) \;\ldots\ldots\ldots\ldots\ldots\; (13).$$

If now in (13) we make $\kappa + \lambda = n = \text{const.}$ and $(\lambda \,.\, 1^{n-\lambda}) = u_\lambda$, we have $$u_{\lambda+1} + u_\lambda = p_{n-\lambda}\, \Sigma a^\lambda.$$

Whence, the value of u_2 being known,

$$(2 \,.\, 1^{n-2}) = u_2 = p_1 p_{n-1} - np_n,$$
$$(3 \,.\, 1^{n-3}) = u_3 = p_{n-2}(p_1^2 - 2p_2) - p_{n-1}p_1 + np_n,$$
$$(4 \,.\, 1^{n-4}) = u_4 = p_{n-3}(p_1^3 - 3p_1p_2 + 3p_3) - p_{n-2}(p_1^2 - 2p_2) + p_{n-1}p_1 - np_n,$$

... ...

and generally

$$u_\lambda = p_{n-\lambda+1} \Sigma a^{\lambda-1} - p_{n-\lambda+2} \Sigma a^{\lambda-2} + \ldots + (-)^{\lambda+1} np_n \;\ldots\ldots\ldots\; (14).$$

This includes Newton's series, giving

$$\Sigma a^n = p_1 \Sigma a^{n-1} - p_2 \Sigma a^{n-2} + p_3 \Sigma a^{n-3} - \text{\&c.},$$
$$\Sigma a^{n-1}\beta = p_2 \Sigma a^{n-2} - p_3 \Sigma a^{n-3} + p_4 \Sigma a^{n-4} - \text{\&c.},$$
$$\Sigma a^{n-2}\beta\gamma = p_3 \Sigma a^{n-3} - p_4 \Sigma a^{n-4} + p_5 \Sigma a^{n-5} - \text{\&c.},$$

... ...

the last term in every case being $\pm\, np_n$.

5. If in (8) we consider $d_1, d_2, d_3 \ldots$ as operating on $\phi\,(p_1, p_2, p_3, \ldots)$, and their equivalents

$$D_1,\; D_1^2-2D_2,\; D_1^3-3D_1D_2+3D_3 \ldots \text{ on } (\lambda^l.\mu^m.\nu^n\ldots),$$

we obtain a set of linear differential equations of the first order, all of them satisfied by ϕ.

Now, recalling the law of operation of D on ( ), viz.,

$$D_\kappa(\lambda^l.\mu^m.\nu^n\ldots)=0,\; D_\lambda(\lambda^l.\mu^m.\nu^n\ldots)=(\lambda^{l-1}.\mu^m.\nu^n\ldots),\; D_\lambda(\lambda)=1,$$

it is clear that any number less than the weight of ϕ, say κ, none of whose partitions are contained in $(\lambda^l.\mu^m.\nu^n\ldots)$, corresponds to a differential equation of the form $d_\kappa\phi=0$; and, whenever a sufficient number of such equations can be found, we are able to calculate the value of ϕ without reference to symmetric functions of inferior weight. A case in point is $\Sigma\alpha^n$, where the differential equations are

$$d_1, d_2, d_3 \ldots d_{n-1}\,\phi=0,$$

which, since they are of the first order, are more convenient to use than $D_1, D_2, D_3 \ldots D_{n-1}\,\phi=0$.

If $\phi=(3^4.1)$, it is easily seen that $d_2\phi=0$, $d_5\phi=0$, $d_8\phi=0$, $d_{11}\phi=0$, and these equations are, in this case, more than sufficient to completely determine ϕ; in fact, without using $d_8\phi=0$, we find

$$\begin{aligned}(3^4.1)=p_4^2p_5&-2p_3p_5^2-p_3p_4p_6+5p_2p_5p_6-5p_1p_6^2+4p_3^2p_7-7p_2p_4p_7\\&+2p_1p_5p_7+8p_6p_7-4p_2p_3p_8+11p_1p_4p_8-13p_5p_8+7p_2^2p_9\\&-10p_1p_3p_9-p_4p_9-7p_1p_2p_{10}+17p_3p_{10}+10p_1^2p_{11}-13p_2p_{11}\\&-10p_1p_{12}+13p_{13}.\end{aligned}$$

Whence

$$\begin{aligned}d_1\phi=(3^4)=p_4^3&-3p_3p_4p_5+3p_2p_5^2+3p_3^2p_6-3p_2p_4p_6-3p_1p_5p_6+3p_6^2\\&-3p_2p_3p_7+6p_1p_4p_7-3p_5p_7+3p_2^2p_8-3p_1p_3p_8-3p_4p_8\\&-3p_1p_2p_9+6p_3p_9+3p_1^2p_{10}-3p_2p_{10}-3p_1p_{11}+3p_{12}.\end{aligned}$$

Any of the other differential equations, $d_3\phi=3\,(3^3.1)$, $d_4\phi=4\,(3^3)$, &c., may be used to verify the value of ϕ with the help of a table of symmetric functions.

It may be noticed that, if by any method a table of symmetric functions of weight n has been calculated, the complete table of weight $n-1$ may be deduced from it by means of the operator d_1.

If $\phi=(5.4.3.2.1)$, the differential equations are found thus:

$$\begin{aligned}d_1\phi&=D_1(5.4.3.2.1)=(5.4.3.2),\\d_2\phi&=(D_1^2-2D_2)(5.4.3.2.1)\\&=D_1^2(5.4.3.2.1)-2(5.4.3.1)=-2(5.4.3.1),\\d_3\phi&=(D_1^3-3D_1D_2+3D_3)(5.4.3.2.1)\\&=(-3D_1D_2+3D_3)(5.4.3.2.1)\\&=-3(5.4.3)+3(5.4.2.1),\\d_4\phi&=(4D_1D_3-4D_4)(5.4.3.2.1)=4(5.4.2)-4(5.3.2.1),\end{aligned}$$

... ...

where, since partitions of all numbers up to 15 are contained in $5.4.3.2.1$, there is no equation of the form $d_\kappa\phi = 0$. The differential equations of the second order, $d_1 d_1 \phi = 0$, $d_3 d_3 \phi = 0$, $d_3 d_{11} \phi = 0$, $d_1 d_{13} \phi = 0$, may in this case be utilized; and a method will be given by which any symmetric function whatever may be calculated independently, assuming only the value of the coefficient of the general term of $\Sigma\alpha^n$ and the laws of combination of the operators.

6. The weight of an operator is the quantity by which it reduces the weight of its subject. Thus the weight of either d_λ or D_λ is λ, and so for every operator of the present paper; $\dfrac{d^m}{dp_\lambda dp_\mu dp_\nu \ldots}$ is of order m and weight $\lambda+\mu+\nu+\ldots$, and any operator of the form $\overline{d_\lambda d_\mu d_\nu \ldots}$ is of order m and weight $\lambda+\mu+\nu+\ldots$, where m is the number of the suffixes.

In the expanded value of any operator those differential coefficients may be rejected as useless which are either of higher order than the degree of the subject, or of higher weight than the weight of the subject.

Thus $$d_\lambda d_\mu = \overline{d_\lambda d_\mu} + d_{\lambda+\mu} = \overline{\left(\frac{d}{dp_\lambda} + p_1 \frac{d}{dp_{\lambda+1}} + \ldots\right)\left(\frac{d}{dp_\mu} + p_1 \frac{d}{dp_{\mu+1}} + \ldots\right)} + \frac{d}{dp_{\lambda+\mu}} + p_1 \frac{d}{dp_{\lambda+\mu+1}} + \ldots,$$

and if the subject be of weight $\lambda+\mu+1$, suppose

$$\phi = Ap_{\lambda+\mu+1} + Bp_1 p_{\lambda+\mu} + Cp_\lambda p_{\mu+1} + Dp_{\lambda+1} p_\mu + Ep_1 p_\lambda p_\mu + \text{other terms},$$

then

$$d_\lambda d_\mu \phi = \left(\frac{d^2}{dp_\lambda dp_\mu} + p_1 \frac{d^2}{dp_{\lambda+1} dp_\mu} + p_1 \frac{d^2}{dp_\lambda dp_{\mu+1}} + \frac{d}{dp_{\lambda+\mu}} + p_1 \frac{d}{dp_{\lambda+\mu+1}}\right)\phi$$
$$= p_1 (E+D+C+B+A).$$

The operator d_n, when performed on a subject of weight n, reduces to $\dfrac{d}{dp_n}$, and any operator of the form $\overline{d_\lambda d_\mu d_\nu \ldots}$ performed on a subject whose weight is equal to its own, reduces to $\dfrac{d}{dp_\lambda}\cdot\dfrac{d}{dp_\mu}\cdot\dfrac{d}{dp_\nu}\ldots$, since none but the over-weighted differential coefficients are affected with multipliers containing $p_1, p_2, p_3, \ldots$

If $\phi = \Sigma\alpha^n$, and if in (9) and (10) respectively we put $\lambda+\mu = n$ and $\lambda+\mu+\nu = n$, we have, since in this case $d_\lambda = 0$, $d_\mu = 0$, $d_\nu = 0$,

$$\frac{d^2\phi}{dp_\lambda dp_{n-\lambda}} = -\frac{d\phi}{dp_n}$$
$$\frac{d^3\phi}{dp_\lambda dp_\mu dp_{n-\lambda-\mu}} = 2\frac{d\phi}{dp_n}.$$

And, by an easy extension,

$$\frac{d^m\phi}{dp_\lambda dp_\mu dp_\nu \dots} = (-)^{m-1}(m-1)!\,\frac{d\phi}{dp_n} \quad\dots\dots\dots\dots\dots(15),$$

where $\phi = \Sigma\alpha^n$, and $\lambda+\mu+\nu+\dots = n$, the m suffixes $\lambda, \mu, \nu\dots$ being not necessarily all unequal.

For, by the nature of the operation d_κ, we have universally

$$d_\kappa.\overline{d_\lambda d_\mu d_\nu\dots} = \overline{d_\kappa d_\lambda d_\mu d_\nu\dots} + \overline{d_{\lambda+\kappa} d_\mu d_\nu\dots} + \overline{d_\lambda d_{\mu+\kappa} d_\nu\dots} + \overline{d_\lambda d_\mu d_{\nu+\kappa}\dots} + \&c. \quad\dots\dots\dots\dots(16),$$

where, if $d_\kappa = 0$, which happens when the subject is $\phi = \Sigma\alpha^n$, or if $\overline{d_\lambda d_\mu d_\nu\dots} = 0$, or in some other cases, the left-hand side vanishes; and if, further, $\kappa+\lambda+\mu+\nu+\dots = n$, the weight of the subject, (16) reduces to

$$\frac{d^{m+1}}{dp_\kappa dp_\lambda dp_\mu dp_\nu\dots} + \frac{d^m}{dp_{\lambda+\kappa} dp_\mu dp_\nu\dots} + \frac{d^m}{dp_\lambda dp_{\mu+\kappa} dp_\nu\dots} + \frac{d^m}{dp_\lambda dp_\mu dp_{\nu+\kappa}\dots} + \dots = 0 \quad\dots\dots\dots\dots(17),$$

where it is not necessary that the suffixes should be all unequal.

Now, if (15) holds for differentials of the m^{th} order, each term of the m^{th} order in (17) is equal to $(-)^{m-1}(m-1)!\,\frac{d}{dp_n}$, and, since there are m of them, the term of the $(m+1)^{\text{th}}$ order is equal to $(-)^m m!\,\frac{d}{dp_n}$; hence (15), which has been seen to hold in the cases $m=1$ and $m=2$, is true for all positive integral values of m.

From (15) and the known value of the coefficient of p_n in $\Sigma\alpha^n$, viz., $(-)^{n+1}n$, it follows at once that the general term of $\Sigma\alpha^n$ is

$$\frac{(-)^{r+n}(r-1)!\,n}{a!\,b!\,c!\dots l!}\,p_1^a p_2^b p_3^c \dots p_n^l \quad\dots\dots\dots\dots\dots\dots(18),$$

where $a+b+c+\dots+l = r$, and the indices $a, b, c\dots l$ are the positive integral, including zero, solutions of the equation

$$a+2b+3c+\dots+nl = n.$$

Hence

$$\begin{aligned}\Sigma\alpha^n = {} & p_1^n - np_1^{n-2}p_2 + np_1^{n-3}p_3 - np_1^{n-4}\left\{p_4 - (n-3)\frac{p_2^2}{2!}\right\} \\ & + np_1^{n-5}\{p_5 - (n-4)p_2p_3\} \\ & - np_1^{n-6}\left\{p_6 - (n-5)\left(p_2p_4 + \frac{p_3^2}{2!}\right) + (n-4)(n-5)\frac{p_2^3}{3!}\right\} \\ & + np_1^{n-7}\left\{p_7 - (n-6)(p_2p_5 + p_3p_4) + (n-5)(n-6)\frac{p_2^2p_3}{2!}\right\} \\ & - np_1^{n-8}\left\{p_8 - (n-7)\left(p_2p_6 + p_3p_5 + \frac{p_4^2}{2!}\right)\right. \\ & \left. + (n-6)(n-7)\left(\frac{p_2^2p_4}{2!} + \frac{p_2p_3^2}{2!}\right) - (n-5)(n-6)(n-7)\frac{p_2^4}{4!}\right\}\end{aligned}$$

$$+np_1^{n-9}\Big\{p_9-(n-8)(p_2p_7+p_3p_6+p_4p_5)$$

$$+(n-7)(n-8)\left(\frac{p_2^2p_5}{2!}+p_2p_3p_4+\frac{p_3^3}{3!}\right)-(n-6)(n-7)(n-8)\frac{p_2^3p_3}{3!}\Big\}$$

$$-np_1^{n-10}\Big\{p_{10}-(n-9)\left(p_2p_8+p_3p_7+p_4p_6+\frac{p_5^2}{2!}\right)$$

$$+(n-8)(n-9)\left(\frac{p_2^2p_6}{2!}+p_2p_3p_5+\frac{p_2p_4^2}{2!}+\frac{p_3^2p_4}{2!}\right)$$

$$-(n-7)(n-8)(n-9)\left(\frac{p_2^3p_4}{3!}+\frac{p_2^2p_3^2}{2!\,2!}\right)$$

$$+(n-6)(n-7)(n-8)(n-9)\frac{p_2^5}{5!}\Big\}+\&c. \qquad\qquad (19).$$

7. Formula (12), combined with the principle of rejecting overweighted differential coefficients, furnishes a simple proof of a law of symmetry, discovered by Prof. Cayley in 1856, but given without proof at the end of the tables in Salmon's *Higher Algebra*.

For, if $(\lambda^l.\mu^m.\nu^n\,...)=...+Ap_\lambda^{l'}p_\mu^{m'}p_\nu^{n'}...+...$

and $(\lambda^{l'}.\mu^{m'}.\nu^{n'}\,...)=...+A'p_\lambda^{l}p_\mu^{m}p_\nu^{n}...+...$ $\Big\}$ .... (a),

the first equation of (a) combined with (12) gives

$$\overline{d_\lambda^l d_\mu^m d_\nu^n\,...}\div l!\,m!\,n!\,...=...+AD_{\lambda'}^{l'}D_{\mu'}^{m'}D_{\nu'}^{n'}...+... \quad ...\ (a'),$$

since each side of (a') is equal to

$$[\lambda^l.\mu^m.\nu^n\,...].$$

Now, using each side of (a') as an operator on the opposite side of the second equation of (a), since

$$(\overline{d_\lambda^l d_\mu^m d_\nu^n\,...}\div l!\,m!\,n!\,...)\,p_\lambda^l p_\mu^m p_\nu^n\,...$$

$$=\left(\frac{d}{dp_\lambda}\right)^l\left(\frac{d}{dp_\mu}\right)^m\left(\frac{d}{dp_\nu}\right)^n...\frac{p_\lambda^l p_\mu^m p_\nu^n\,...}{l!\,m!\,n!...}=1,$$

and $$D_{\lambda'}^{l'}D_{\mu'}^{m'}D_{\nu'}^{n'}\,...\,(\lambda^{l'}.\mu^{m'}.\nu^{n'}\,...)=1,$$

and no other terms survive the operation, we have $A=A'$, which is the first part of the law of symmetry.

If, moreover, $p_\lambda^l p_\mu^m p_\nu^n\,...=...+B\,(\lambda^{l'}.\mu^{m'}.\nu^{n'}\,...)+...$

and $p_{\lambda'}^{l'}p_{\mu'}^{m'}p_{\nu'}^{n'}\,...=...+B'(\lambda^l.\mu^m.\nu^n\,...)+...$ $\Big\}$ .... (b),

the second equation of (b), combined with (12), gives, as before,

$$D_{\lambda'}^{l'}D_{\mu'}^{m'}D_{\nu'}^{n'}\,...=...+B'\,\overline{d_\lambda^l d_\mu^m d_\nu^n\,...}\div l!\,m!\,n!\,...+... \quad ...(b');$$

and, using each side of equation (b') as an operator on the opposite side of the first equation of (b), precisely the same reasoning as before gives us $B'=B$, which is the second part of the law of symmetry.

8. In this concluding article a method, of universal application, for calculating symmetric functions, is illustrated by the calculation of $(3^2.2^3.1)$.

If $F(D_1, D_2, D_3, \ldots)$ is a rational integral function, it is manifest, from the nature of the operation D_κ, that all terms in F may be rejected except such as are factors of $D_\lambda D_\mu D_\nu \ldots$, where the subject is $(\lambda . \mu . \nu \ldots)$, of the same type; *i.e.*, the $\lambda . \mu . \nu \ldots$ in ( ) corresponding exactly with the suffixes of the D's. Thus, if the subject is $(3^2 . 2^3 . 1)$, all terms are to be rejected except factors of $D_3^2 D_2^3 D_1$.

When $F(D_1, D_2, D_3, \ldots) = d_n$, the known value (18) of the general term of $\Sigma\alpha^n$ enables us to write down at once the terms to be retained. Thus, retaining only factors of $D_3^2 D_2^3 D_1$ in the expressions (8) for $d_1, d_2, d_3, \ldots$, we have

$$d_1 = D_1$$
$$d_2 = -2D_2$$
$$d_3 = -3D_1 D_2 + 3D_3$$
$$d_4 = 2D_2^2 + 4D_1 D_3$$
$$d_5 = 5D_1 D_2^2 - 5D_2 D_3$$
$$d_6 = -2D_2^3 - 12D_1 D_2 D_3 + 3D_3^2$$
$$d_7 = -7D_1 D_2^3 + 7D_2^2 D_3 + 7D_1 D_3^2$$
$$d_8 = 24D_1 D_2^2 D_3 - 8D_2 D_3^2$$
$$d_9 = -9D_2^3 D_3 - 27D_1 D_2 D_3^2$$
$$d_{10} = -40D_1 D_2^3 D_3 + 15D_2^2 D_3^2$$
$$d_{11} = 66D_1 D_2^2 D_3^2$$
$$d_{12} = -24D_2^3 D_3^2$$
$$d_{13} = -130D_1 D_2^3 D_3^2.$$

With the values of $d_1, d_2, d_3, \ldots$ thus obtained we form products of weight w, and of $1, 2, 3, \ldots m$ factors, where w is the weight of the function to be calculated and m its degree; rejecting in the process all terms that would formerly have been rejected. When this has been done, there remains, of each product, but a single term of the same type as the function to be calculated, and of these terms only the numerical coefficient is retained. These coefficients and their corresponding products are placed opposite each other in contiguous columns,—in the illustrative example thus:

d_{13}	-130	-130	p_{13}
$d_1 d_{12}$	-24	$+106$	$p_1 p_{12}$
$d_2 d_{11}$	-132	-2	$p_2 p_{11}$
$d_3 d_{10}$	-165	-35	$p_3 p_{10}$
$d_4 d_9$	-90	$+40$	$p_4 p_9$
$d_5 d_8$	-160	-30	$p_5 p_8$
$d_6 d_7$	-119	$+11$	$p_6 p_7$
$d_1 d_1 d_{11}$	0	-80	$p_1^2 p_{11} \div 2\,!$
$d_1 d_2 d_{10}$	-30	$+31$	$p_1 p_2 p_{10}$
$d_1 d_3 d_9$	-27	-8	$p_1 p_3 p_9$
$d_1 d_4 d_8$	-16	-2	$p_1 p_4 p_8$
$d_1 d_5 d_7$	-35	$+8$	$p_1 p_5 p_7$
$d_1 d_6 d_6$	-12	-10	$p_1 p_6^2 \div 2\,!$
$d_2 d_2 d_9$	-108	-14	$p_2^2 p_9 \div 2\,!$
$d_2 d_3 d_8$	-192	$+5$	$p_2 p_3 p_8$
$d_2 d_4 d_7$	-84	-3	$p_2 p_4 p_7$
$d_2 d_5 d_6$	-150	$+1$	$p_2 p_5 p_6$
$d_3 d_3 d_7$	-189	0	
$d_3 d_4 d_6$	-114	0	
$d_3 d_5 d_5$	-225	0	
$d_4 d_4 d_5$	-80	0	

The numbers in the third column are the coefficients of terms whose literal part is given in the fourth, and when these numbers have been found, the calculation is completed. This may be effected in two distinct ways.

First, considering the operative character of the symbols, if $\phi = (3^2 . 2^3 . 1)$, we have

$$D_1 D_2^3 D_3^2 (3^2 . 2^3 . 1) = 1,$$

and $\quad d_{13} = -130 D_1 D_2^3 D_3^2, \quad d_1 d_{12} = -24 D_1 D_2^3 D_3^2,$ &c.,

giving $\quad d_{13}\phi = -130, \quad d_1 d_{12}\phi = -24, \quad d_2 d_{11}\phi = -132,$ &c.

Whence

$$\frac{d\phi}{dp_{13}} = -130, \quad \frac{d^2\phi}{dp_1\, dp_{12}} + \frac{d\phi}{dp_{13}} = -24, \quad \frac{d^2\phi}{dp_2\, dp_{11}} + \frac{d\phi}{dp_{13}} = -132, \text{ \&c.,}$$

and therefore $\quad \dfrac{d^2\phi}{dp_1\, dp_{12}} = 106, \quad \dfrac{d^2\phi}{dp_2\, dp_{11}} = -2,$ &c.

So also $\quad d_1 d_1 d_{11}\phi = 0$

$$= \left(\frac{d}{dp_1} + p_1 \frac{d}{dp_2} + \dots\right)\left(\frac{d}{dp_1} + p_1 \frac{d}{dp_2} + \dots\right)\left(\frac{d}{dp_{11}} + p_1 \frac{d}{dp_{12}} + p_2 \frac{d}{dp_{13}}\right)\phi,$$

which gives, after expanding and rejecting the over-weighted differential coefficients, as in Art. 6,

$$\frac{d^3\phi}{dp_1^2\, dp_{11}} + 2\frac{d^2\phi}{dp_1\, dp_{12}} + \frac{d^2\phi}{dp_2\, dp_{11}} + \frac{d\phi}{dp_{13}} = 0,$$

in which, substituting the values of the three last terms found above, we have finally $\quad \dfrac{d^3\phi}{dp_1^2\, dp_{11}} = -80.$

In this way any of the numbers in the third column may be found, but the coefficients of terms of a lower degree must be found before those of terms of the next higher degree.

Second, considering the symbols as symmetric functions of (3), viz., $d_\lambda = \Sigma a^\lambda = [\lambda]$, &c., using the Algebraic formulæ corresponding to (9) and (10), and others of the same kind, we have

$$\text{co. } p_1 p_2^3 p_3^2 \text{ in } \Sigma a^\lambda \beta^\mu = \text{co. } p_1 p_2^3 p_3^2 \text{ in } (S_\lambda S_\mu - S_{\lambda+\mu})$$
$$= \text{co. } D_1 D_2^3 D_3^2 \text{ in } (d_\lambda d_\mu - d_{\lambda+\mu}).$$

Thus, if

$$\lambda = 10, \quad \mu = 3, \quad \text{co. } p_1 p_2^3 p_3^2 \text{ in } \Sigma a^{10}\beta^3 = -165 + 130 = -35,$$

and therefore, by the law of symmetry,

$$\text{co. } p_3 p_{10} \text{ in } (3^2 . 2^3 . 1) = -35.$$

So also

$$\text{co. } p_1 p_2^3 p_3^2 \text{ in } \Sigma a^\lambda \beta^\mu \gamma^\nu$$
$$= \text{co. } p_1 p_2^3 p_3^2 \text{ in } (S_\lambda S_\mu S_\nu - S_\lambda S_{\mu+\nu} - S_\mu S_{\nu+\lambda} - S_\nu S_{\lambda+\mu} + 2S_{\lambda+\mu+\nu})$$
$$= \text{co. } D_1 D_2^3 D_3^2 \text{ in } (d_\lambda d_\mu d_\nu - d_\lambda d_{\mu+\nu} - d_\mu d_{\nu+\lambda} - d_\nu d_{\lambda+\mu} + 2d_{\lambda+\mu+\nu}),$$

and if $\lambda = 2,\ \mu = 4,\ \nu = 7$, we have

$$\text{co. } p_1 p_2^3 p_3^2 \text{ in } \Sigma\alpha^7\beta^4\gamma^2 = -84+132+90+119-260 = -3,$$

and therefore, by the law of symmetry,

$$\text{co. } p_2 p_4 p_7 \text{ in } (3^2.2^3.1) = -3.$$

In this way any of the numbers in the third column may be found, when those in the second are known. The method is in its essence that given in all the text-books for finding symmetric functions from the sums of the powers, but it is simplified by the rejection of superfluous terms and by the application of the law of symmetry.

On a Generalization of the Nine-Points Properties of a Triangle. *By* Captain P. A. MacMahon, R.A.

[*Read Feb. 8th,* 1883.]

In the triangle ABC (Fig. p. 130), let O be the centre of the circle ABC, T the orthocentre; through O and T draw the lines OL, TN making angles α and $\pi-\alpha$ respectively with the side BC, meeting that side in the points L and N; again, draw the lines OI, TM making angles $\pi-\alpha$ and α with the same side, meeting it in the points I and M; obtain in a similar manner eight other points, four on each of the other sides: these twelve points lie six and six upon two equal circles of radius $\frac{1}{2}R$ cosec α, R being the radius of the circle ABC.

These two circles also pass each through six other points, corresponding to the points bisecting TA, TB, TC which lie upon the nine-points circle of the triangle.

When $\alpha = \dfrac{\pi}{2}$, the two circles considered here coalesce into the nine-points circle. Also, as will be seen, the twelve other points mentioned coalesce in this case into three.

1. Let S be the nine-points centre, and draw SP at right angles to OT, and OP making an angle α with SP. Then P is the centre of the circle passing through the points L and N and the corresponding points on the other sides of the triangle.

Draw OA', TC' perpendiculars to BC, meeting it in the points A and C'.

Join PL, PN, PT, SA', SC'.

Since, in the two triangles LOP, $A'OS$,

$$\text{angle } LOP = \text{angle } A'OS,$$