

Dimensions," 3rd edition, p. 79. It may be noted that, when two of the director curves are right lines, the above equation is immediately interpretable.

29. It may be worth notice that the coordinate system of representation is applicable to the intersection of two surfaces. Suppose these

$$\text{are} \quad (ax + by + cz + dw)^m = 0,$$

$$\text{and} \quad (a'x + b'y + c'z + d'w)^n = 0.$$

Adopting Prof. Cayley's method, we form the equation to the cone passing through the intersection of these surfaces and having its vertex at  $\xi, \eta, \zeta, w$ . This equation is, when the same notation as before is utilized,

$$(Af + Fa + Bg + Gb + Ch + Hc)^{mn} = 0.$$

In this case the difficulties of interpretation begin with the fundamental equation itself.

In Salmon's "Geometry of Three Dimensions," 3rd edition, Art. 217, a particular case ( $m = n = 2$ ) is given, and the result given above is consistent therewith.

### *On Polygons circumscribed about a Cuspidal Cubic.*

By Mr. R. A. ROBERTS, M.A.

[Read April 6th, 1882.]

I propose to consider some cases in which an infinite number of closed polygons can be circumscribed about a cuspidal cubic and inscribed in another curve. In all the cases which I shall consider, the curve circumscribing the polygon is unicursal.

The equation of a cuspidal cubic being reduced to the form  $y^2 = x^2z$ , we may take  $1, y, y^2$  as the coordinates of a point on the curve, and the equation of the tangent at the point  $y$  is then  $2y^2x - 3y^2y + z = 0$ . Taking two tangents, we have, for the point  $x, y, z$  of their intersection,

$$2x = 1, \quad 3y = t + u, \quad z = t^2u \quad \dots\dots\dots(1),$$

where  $u = y_1 + y_2$ , and  $t = -\frac{y_1 y_2}{y_1 + y_2}$  is the parameter of the third tangent drawn from  $x, y, z$ .

Suppose we have  $u = \frac{f_m}{f_n}$ , where  $f_m, f_n$  are rational functions of  $t$  of the  $m^{\text{th}}$  and  $n^{\text{th}}$  degrees respectively; then the locus of  $x, y, z$  is a unicursal curve whose degree is equal to the greater of the numbers  $n+1$ ,

$m+2$ . This locus will touch the cubic in as many points as the equation  $2u-t=0$  has roots; for, substituting the values (1) in the equation of the curve, we obtain  $(2u-t)^2(u+4t)=0$ . The locus will, evidently, also touch the stationary tangent  $z$  in the point corresponding to  $t=0$ .

Let us take the equation

$$\phi(\mathfrak{J}_1, \mathfrak{J}_2) = a(\mathfrak{J}_1 + \mathfrak{J}_2)^2 + b\mathfrak{J}_1^2 \mathfrak{J}_2^2 + f\mathfrak{J}_1 \mathfrak{J}_2 + g(\mathfrak{J}_1 + \mathfrak{J}_2) + h\mathfrak{J}_1 \mathfrak{J}_2 (\mathfrak{J}_1 + \mathfrak{J}_2) = 0,$$

or

$$u = \frac{ft-g}{a-ht+bt^2};$$

then the locus of the intersection of the tangents is a unicursal cubic, touching the stationary tangent and having triple contact with the curve. This cubic is not limited in any further manner, as we have four constants at our disposal, and a unicursal cubic is determined by eight conditions.

Now, the equation  $\phi(\mathfrak{J}_1, \mathfrak{J}_2) = 0$  can, as is well known, be written in the form  $u_1 \pm u_2 = a$  a constant, where  $u_1$  is an elliptic function of the first kind, depending on  $\mathfrak{J}_1$ , and  $u_2$  a similar function depending on  $\mathfrak{J}_2$ . For a polygon of  $n$  sides, then, we have, taking the negative sign,  $u_1 - u_2 = a$ ,  $u_2 - u_3 = a$ , &c.,  $u_{n-1} - u_n = a$ ; from which we see that the polygon will close of itself, if  $a = \frac{2K}{n}$ , which determines a relation between the constants  $a$ ,  $b$ , &c. (see Schloemilch's "Théorie des Intégrales et des Fonctions Elliptiques," translated by Graindorge, p. 126). It is, in fact, by means of these equations, that the problem of inscribing polygons in one conic, which shall be circumscribed to another, is solved.

Thus, when the conditions stated above are satisfied, and also a further relation between the constants, we can have an infinite number of polygons circumscribed about a cuspidal cubic and inscribed in a nodal cubic. I proceed to find the two parameters corresponding to the nodes of this cubic. We could find them by the equations, in Salmon's "Curves," Art. 216 (c); but they can be more readily obtained by the following considerations. A node of the locus will evidently arise from the coexistence of the equations  $\phi(t_1, \mathfrak{J}) = 0$ ,  $\phi(t_2, \mathfrak{J}) = 0$ , where  $t_1, t_2, \mathfrak{J}$  are the parameters of three tangents passing through a point. Considering, then,  $\phi(t, \mathfrak{J}) = 0$  as a quadratic in  $t$  whose roots are  $t_1, t_2$ , we have

$$t_1 + t_2 = - \frac{\{h\mathfrak{J}^2 + (2a+f)\mathfrak{J} + g\}}{a + h\mathfrak{J} + b\mathfrak{J}^2},$$

$$t_1 t_2 = \frac{a\mathfrak{J}^2 + b\mathfrak{J}}{a + h\mathfrak{J} + b\mathfrak{J}^2},$$

which, combined with  $t_1 t_2 + \mathfrak{J}(t_1 + t_2) = 0$ , the condition that the tan-

gents should pass through a point, gives  $a+f+h\mathfrak{J}=0$ . For a cusp  $t_1=t_2$ , and we have, then, either

$$a^3+af-gh=0 \dots\dots\dots(2),$$

$$\text{or} \quad h^3(a^3+af-gh)-4b(a+f)^3+4fh^3(a+f)=0\dots\dots\dots(3),$$

the condition (2) being satisfied when the cusp is on  $z$ , the stationary tangent, and (3) when the cusp is on the given curve. Now, the equation (2) is the condition for the coexistence of the equations  $\phi(\mathfrak{J}_1, \mathfrak{J}_2) = \phi(\mathfrak{J}_2, \mathfrak{J}_3) = \phi(\mathfrak{J}_3, \mathfrak{J}_1) = 0$ , for an indefinite number of values of  $\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3$ , in which case an infinite number of triangles can be circumscribed about the given curve and inscribed in the locus. Hence we infer that, if a cuspidal cubic  $V$  have triple contact with a cuspidal cubic  $U$ , and the cusp of  $V$  lie on the stationary tangent of  $U$ , then an infinite number of triangles can be circumscribed about  $U$  and inscribed in  $V$ . When the relation (3) is satisfied, and also a further relation between the constants, we can have an infinite number of polygons circumscribing a cuspidal cubic  $U$  and inscribed in a cuspidal cubic  $V$ ; the two curves  $U$  and  $V$  having double contact with each other, and being such that the cusp of  $V$  lies on  $U$ , and the stationary tangent of  $U$  is touched by  $V$ .

There is another case in which an infinite number of triangles can be circumscribed about a cuspidal cubic and inscribed in another cuspidal cubic. Let us take three points, whose parameters are  $\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3$ , on the cubic  $U \equiv y^3 - x^2z = 0$ , such that

$$\mathfrak{J}_1 + \mathfrak{J}_2 + \mathfrak{J}_3 = 0,$$

$$4(\mathfrak{J}_1\mathfrak{J}_2 + \mathfrak{J}_2\mathfrak{J}_3 + \mathfrak{J}_3\mathfrak{J}_1)^3 + 27h(\mathfrak{J}_1\mathfrak{J}_2\mathfrak{J}_3)^3 = 0,$$

then it will be found that the tangents at these points form a triangle inscribed in the cubic  $V \equiv y^3 - kx^2z = 0$ . It may be remarked that the points of contact of the sides of this triangle lie on a line which touches  $ky^3 - x^2z = 0$ , and that the tangents to  $V$  at its vertices pass through a point which lies on  $y^3 - k^2x^2z = 0$ .

If we take the more general relation

$$\begin{aligned} \phi(\mathfrak{J}_1, \mathfrak{J}_2) &= a(\mathfrak{J}_1 + \mathfrak{J}_2)^2 + b\mathfrak{J}_1^2\mathfrak{J}_2^2 + c \\ &\quad + h\mathfrak{J}_1\mathfrak{J}_2(\mathfrak{J}_1 + \mathfrak{J}_2) + g(\mathfrak{J}_1 + \mathfrak{J}_2) + f\mathfrak{J}_1\mathfrak{J}_2 = 0, \end{aligned}$$

$$\text{or} \quad u^2(a - ht + bt^2) + u(g - ft) + c = 0,$$

the locus of the intersection of the tangents is a unicursal quartic, as may be seen thus:—If  $\alpha$  and  $\beta$  are the roots of the equation

$$(g - ft)^2 - 4c(a - ht + bt^2) = 0;$$

then, assuming  $t = \frac{\alpha + \beta\lambda}{1 + \lambda^2}$ , we get for  $u$  an expression of the form

$\frac{1}{p+q\lambda+r\lambda^2}$ , and, substituting these values in (1), we have

$$2x = (1+\lambda^2)(p+q\lambda+r\lambda^2),$$

$$3y = (\alpha+\beta\lambda^2)(p+q\lambda+r\lambda^2) + (1+\lambda^2)^2,$$

$$z = (\alpha+\beta\lambda^2)^2;$$

from which we see that the locus is a unicursal quartic touching the curve four times and the stationary tangent twice.

Thus, exactly as before, when a relation between the constants is satisfied, we can have an infinite number of polygons circumscribed about a cuspidal cubic and inscribed in a unicursal quartic which touches the cubic four times and its stationary tangent twice.

To determine the parameters of the nodes of this quartic, we have

$$\phi(t, \mathcal{J}) = 0, \quad \phi(t_1, \mathcal{J}) = 0, \quad t_1 t_2 + \mathcal{J}(t_1 + t_2) = 0,$$

whence 
$$t_1 + t_2 = -\frac{\{h\mathcal{J}^2 + (2a+f)\mathcal{J} + c\}}{b\mathcal{J}^2 + h\mathcal{J} + a},$$

$$t_1 t_2 = \frac{a\mathcal{J}^2 + g\mathcal{J} + c}{b\mathcal{J}^2 + h\mathcal{J} + a},$$

$\mathcal{J}$  being determined by the equation  $h\mathcal{J}^2 + (a+f)\mathcal{J} - c = 0$ . If we eliminate  $\mathcal{J}$  between the latter equation and  $t^2 - (t_1 + t_2)t + t_1 t_2 = 0$ , we shall obtain a sextic in  $t$  whose roots are the parameters of the three nodes which the quartic has in general.

If 
$$a^2 + af + bc - gh = 0 \dots\dots\dots(4),$$

the equations  $\phi(\mathcal{J}_1, \mathcal{J}_2) = \phi(\mathcal{J}_2, \mathcal{J}_3) = \phi(\mathcal{J}_3, \mathcal{J}_1) = 0$  coexist for an indefinite number of values of  $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ , and it is evident then that the three nodes of the quartic are replaced by a triple point. Hence we infer that, if a quartic  $V$  with a triple point touch a cuspidal cubic  $U$  four times, and the stationary tangent of the cubic twice, then an infinite number of triangles can be circumscribed about  $U$  and inscribed in  $V$ . The quartic  $U$  is not limited in any further manner, for such a curve is determined by ten conditions, and we have four constants at our disposal.

If 
$$\phi(\mathcal{J}_1) = \phi(\mathcal{J}_2) = \phi(\mathcal{J}_3) = \phi(\mathcal{J}_4),$$

where 
$$\phi(\mathcal{J}) = \frac{1}{\mathcal{J}^4}(\mathcal{J}^4 - p\mathcal{J}^3 + q\mathcal{J}^2 - r\mathcal{J} + s),$$

the four corresponding tangents will evidently form a quadrilateral whose six intersections of sides will lie on the same locus. Dividing  $\phi(\mathcal{J}_1) - \phi(\mathcal{J}_2) = 0$  by  $\mathcal{J}_1 - \mathcal{J}_2$ , this relation becomes

$$p\mathcal{J}_1^3\mathcal{J}_2^3 - q\mathcal{J}_1^2\mathcal{J}_2^2(\mathcal{J}_1 + \mathcal{J}_2) + r\mathcal{J}_1\mathcal{J}_2(\mathcal{J}_1 + \mathcal{J}_2)^2 - s(\mathcal{J}_1 + \mathcal{J}_2)^3 - r\mathcal{J}_1^2\mathcal{J}_2^2 \\ + 2s\mathcal{J}_1\mathcal{J}_2(\mathcal{J}_1 + \mathcal{J}_2) = 0,$$

or 
$$u = \frac{-t(rt+2s)}{pt^2+qt^2+rt+s};$$

from which we see that the locus of the intersection of the tangents is the unicursal quartic determined by the equations

$$\begin{aligned} 2x &= pt^2 + qt^2 + rt + s, \\ 3y &= t(pt^2 + qt^2 - s), \\ z &= -t^3(rt + 2s). \end{aligned}$$

Since, from these equations, we have

$$3sy + qz + t(2sx + 3ry + pz) = 0,$$

it follows that this quartic must have a triple point determined by  $3sy + qz = 0$ ,  $2sx + 3ry + pz = 0$ . This quartic has an inflexion and corresponding tangent in common with the given cubic, and also touches it in three points elsewhere.

If we consider the case when two sides of the quadrilateral coincide, it is evident that the two other sides will become tangents to the quartic. Thus we see that the six common tangents of the cubic and quartic form three pairs of lines, whose intersection lies on the quartic, and whose chords of contact touch the cubic, the points of contact of the latter lines lying on the quartic.

Let 
$$\phi(\mathcal{J}) = \frac{1}{9^3}(\mathcal{J}^3 - p\mathcal{J}^2 + q\mathcal{J} - r),$$

then the equations  $\phi(\mathcal{J}_1) + \phi(\mathcal{J}_2) = 0,$

$$\phi(\mathcal{J}_2) + \phi(\mathcal{J}_3) = 0, \quad \phi(\mathcal{J}_3) + \phi(\mathcal{J}_4) = 0, \quad \phi(\mathcal{J}_4) + \phi(\mathcal{J}_1) = 0,$$

being only equivalent to three conditions, will give rise to a series of quadrilaterals, of which the intersections (12), (23), (34), (41) will lie on the same locus. Writing  $\phi(\mathcal{J}_1) + \phi(\mathcal{J}_2) = 0$  in the form

$$\begin{aligned} 2\mathcal{J}_1^2\mathcal{J}_2^2 - p\mathcal{J}_1^2\mathcal{J}_2^2(\mathcal{J}_1 + \mathcal{J}_2) + q\mathcal{J}_1\mathcal{J}_2(\mathcal{J}_1 + \mathcal{J}_2)^2 - r(\mathcal{J}_1 + \mathcal{J}_2)^3 \\ - 2q\mathcal{J}_1^2\mathcal{J}_2^2 + 3r\mathcal{J}_1\mathcal{J}_2(\mathcal{J}_1 + \mathcal{J}_2) = 0, \end{aligned}$$

we get

$$u = \frac{-t(2qt + 3r)}{2t^2 + pt^2 + qt + r},$$

from which we see that the locus is a unicursal quartic determined by the equations

$$\begin{aligned} 2x &= 2t^3 + pt^2 + qt + r, \\ 3y &= t(2t^3 + pt^2 - qt - 2r), \\ z &= -t^3(2qt + 3r). \end{aligned}$$

This quartic has an inflexion and corresponding tangent in common with the given cubic, and also touches it in three points elsewhere. To find the parameters of the nodes, we have

$$\phi(\mathcal{J}_1) + \phi(\mathcal{J}) = 0, \quad \phi(\mathcal{J}_2) + \phi(\mathcal{J}) = 0, \quad \mathcal{J}_1\mathcal{J}_2 + \mathcal{J}(\mathcal{J}_1 + \mathcal{J}_2) = 0;$$

hence  $\mathfrak{J}_1, \mathfrak{J}_2$  are roots of the equation

$$\{1 + \phi(\mathfrak{J})\} t^2 - pt^2 + qt - r = 0;$$

and, since  $\frac{1}{\mathfrak{J}_1} + \frac{1}{\mathfrak{J}_2} + \frac{1}{\mathfrak{J}} = 0$ , if  $t$  is the third root of this equation,  $\frac{1}{\mathfrak{J}_1} + \frac{1}{\mathfrak{J}_2} + \frac{1}{t} = \frac{q}{r} = \frac{1}{t} - \frac{1}{\mathfrak{J}}$ , we have  $t = \frac{r\mathfrak{J}}{r+q\mathfrak{J}}$ , and  $\mathfrak{J}$  is determined by the cubic

$$(pq - 2r)\mathfrak{J}^3 + (q^2 + 2pr)\mathfrak{J}^2 + qr\mathfrak{J} + 2r^2 = 0.$$

If  $t_1, t_2, t_3$  are the roots of this equation, the parameters of the node corresponding to  $t_1$  will be the roots of  $t^2 - u_1 t - u_1 t_1 = 0$ ; this quartic will, therefore, be in general trinodal.

$$\text{Since } \phi(\mathfrak{J}_1) - \phi(\mathfrak{J}_2) = 0, \quad \phi(\mathfrak{J}_2) - \phi(\mathfrak{J}_3) = 0,$$

$$\text{or } p\mathfrak{J}_1^2\mathfrak{J}_2^2 - q\mathfrak{J}_1\mathfrak{J}_2(\mathfrak{J}_1 + \mathfrak{J}_2) + r(\mathfrak{J}_1^3 + \mathfrak{J}_2^3 + \mathfrak{J}_1\mathfrak{J}_2) = 0,$$

the locus of the intersections of the opposite sides of the quadrilateral (13), (24) will be a cuspidal cubic. If we consider the tangents corresponding to the roots of the equation  $\mathfrak{J}^2\phi(\mathfrak{J}) = 0$ , we see that the points of contact of these tangents and the vertices of the triangle formed by them lie on the quartic.

Of course, corresponding to all the theorems concerning polygons circumscribed about a cuspidal cubic, we shall have, by reciprocation, similar theorems concerning polygons inscribed in a cuspidal cubic.

*The Algebraic Solution of the Modular Equation for the Septic Transformation.* By G. S. ELY, Fellow in Johns Hopkins University, U.S.

[Read April 6th, 1882.]

$$\text{To transform } \frac{dx}{\sqrt{1-x^2} \cdot 1-\kappa^2 x^2} = \frac{M dy}{\sqrt{1-y^2} \cdot 1-\lambda^2 y^2},$$

Prof. Cayley writes for the septic transformation

$$\frac{1-y}{1+y} = \frac{1-x}{1+x} \left( \frac{1-ax+\beta x^3-\gamma x^5}{1+ax+\beta x^3+\gamma x^5} \right)^2,$$

and shows that the conditions necessary for the change of  $x$  into