that, when it is possible to inscribe in U a polygon of a given number of sides which shall be circumscribed about V, only one further condition is satisfied. The double tangents of V are evidently determined by the equations $\phi(\vartheta_1, \vartheta_2) = 0$, $\phi(\vartheta_1, \vartheta_3) = 0$, combined with the condition that the three points $\vartheta_1, \vartheta_2, \vartheta_3$ should lie on a line, and also by the equations $\phi(\vartheta_1, \vartheta_2) = 0$, $\phi(\vartheta_3, \vartheta_4) = 0$, combined with the two conditions that the four points $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ should lie on a line. The first condition gives $3(n-2)$ double tangents, and the second the remainder. In the case of the triangle, these $3(n-2)$ double tangents are replaced by $n-2$ triple tangents, and then no further condition is satisfied.

If the parameters ϑ_1, ϑ_2 of a node of U satisfy the equation $\phi(\vartheta_1, \vartheta_2) = 0$, V is evidently divisible by the equation of that node. Thus, for every node of U whose parameters satisfy $\phi(\vartheta_1, \vartheta_2) = 0$, we must subtract one from the class of V, and, it can be shown, two from the points of contact of U and V. We can show, also, that every double tangent of V which becomes an inflexional tangent, touches U. The above result will be found to include several that have been already obtained.

On certain Relations between Volumes of Loci of Connected Points.

By Mr. E. B. Elliott.

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1. It is known (*Messenger of Mathematics*, Vol. vii., p. 151) that when a straight line, either extensible or not, occupies a closed infinite series of positions and corresponding states of extension in a plane, the area enclosed by the path of any point of it may be expressed linearly in terms of those enclosed by the paths of three chosen points of reference on it, or, more usefully, in terms of two such areas and a relative area easily interpreted generally, but of especial simplicity when the moving straight line is inextensible. We know also that, if a lamina, subject or not to homogeneous strain, move in a plane, in like manner the area enclosed by the path of any point in it can be expressed linearly in terms of those given by three chosen ones of its points and certain relative areas. This theorem, though practically arrived at by Mr. Leudesdorf (*Messenger*, Vol. vii., p. 125; Vol. viii., p. 11), has not, I believe, been definitely stated by him or by others, except for the case of the kinematics of a *rigid* lamina. Its general form is at once obtained by interpreting generally, instead of as he does specially, the analytical result obtained in the second of his two papers referred to, and may be stated,—"If a lamina subject to homogeneous strain

occupy a closed cycle of positions in a plane, a single definite state of strain corresponding to each position, then, if (A), (B), (C) be the areas enclosed by the paths of three points of reference A, B, C in it, (a), (b), (c) the areas that would be traced out by radii from fixed points always equal and parallel to CB, AC, BA respectively, and (P) the area enclosed by the path of P whose areal coordinates with regard to the triangle A, B, C are always x, y, z, this last is given by the equation $(P) = x(A)+y(B)+z(C)-yz(a)-zx(b)-xy(c)$" ...(1).

Considering this equation for a moment before passing to the main subject of our paper, we see, putting (P) constant, that the locus of points P of the lamina which surround areas equal to any given one, is one of a system of concentric similar and similarly situated conics, which in particular (Kempe's *Theorem*), in the case where the lamina is rigid, become the concentric circles, or straight lines if n be zero,

$$k = x(A)+y(B)+z(C)-n\pi(a^2yz+b^2zx+c^2xy).$$

Reference to special triangles in the lamina naturally simplifies the result (1), and materially helps the geometrical interpretation of conclusions. For instance, let (A_1), (B_1), (C_1) be the areas corresponding to the vertices of a triangle self-conjugate with regard to the real or imaginary conic which is the locus of points surrounding zero areas, then the area given by the point whose coordinates with regard to this triangle are x, y, z, is found to be

$$(P_1) = x^2(A_1)+y^2(B_1)+z^2(C_1) \text{ ....(2)},$$

the vanishing of the yz, zx, and xy terms giving the area relations

$$(a_1) = (B_1)+(C_1),\ (b_1) = (C_1)+(A_1),\ (c_1) = (A_1)+(B_1).$$

The form this takes for the case of rigid kinematics is noteworthy. In virtue of the known values $n\pi a_1^2$, $n\pi b_1^2$, $n\pi c_1^2$ for (a_1), (b_1), (c_1), the values of (A_1), &c. are at once $n\pi b_1c_1\cos A_1$, &c., and the relation becomes

$$(P_1) = 2n\pi\Delta_1(x^2\cot A_1+y^2\cot B_1+z^2\cot C_1) \text{ ....(3)},$$

where Δ_1 is the area of the triangle.

Another convenient special triangle of reference is one inscribed in one of the locus-conics. The form taken by the relation is too immediate to need statement.

2. Again, it is familiar (*Messenger*, Vol. vii., p. 153) that, if a straight line occupy a complete doubly infinite series of positions and corresponding states of extension in space, so that the various points of it traverse entire closed surfaces, the volume of any one of such surfaces can be expressed linearly in terms of those of four chosen ones, or of three chosen ones and a relative volume. The present paper is an endeavour to complete a series of space theorems by re-

placing the straight line here, first by a plane and then by a space, just as in two dimensions the area-theorem on the kinematics of a lamina completes the series introduced by that on the kinematics of a rod.

3. Let then, first, A, B, C be three points whose positions have a (1, 1, 1) correspondence, and cover respectively three closed surfaces of volumes (A), (B), (C); and let it be required to express as simply as possible the volume surrounded by the positions of the point P in the plane of ABC, whose areal coordinates with regard to the triangle ABC are always x, y, z. Considering the plane of ABC as a lamina, the alterations of size and shape in whose parts are due to constantly varying homogeneous strain, such a point P is of course always the same one of the lamina.

Referred to rectangular axes fixed in space, let (p_1, q_1, r_1), (p_2, q_2, r_2), (p_3, q_3, r_3) be the coordinates in corresponding positions of A, B, C; and suppose points filling the interiors of the volumes under consideration to be connected just as points covering their bounding surfaces are; then the volume element of the locus of P is

$$d(P) = (x\,dp_1 + y\,dp_2 + z\,dp_3)(x\,dq_1 + y\,dq_2 + z\,dq_3)(x\,dr_1 + y\,dr_2 + z\,dr_3) \quad ..(4).$$

Expanded, this is a sum of multiples of 27 volume elements mostly hard to interpret. These 27 belong, however, to only ten classes, corresponding to the ten multiples x^3, y^3, z^3, x^2y, x^2z, y^2z, xy^2, xz^2, yz^2, xyz. It occurs, then, that the sum may be expressed as one of ten independent elements only, and that these ten may be so chosen as to be interpreted without difficulty. It is possible to take the ten suitably in a large variety of ways. They may be the volume elements of the loci of ten points in the lamina; or some of them may be relative volume elements. The ten here chosen, that the form of result may have as much symmetry as possible, are the volume elements of (A), (B), (C), those of (A'), (B'), (C') the loci of the middle points of sides of the triangle ABC, that of (G) the locus of the centroid of the triangle, and those of (B, C), (C, A), (A, B) the relative loci of the vertices B, C, A with regard to C, A, B respectively. If the coefficients of these elements in the expansion of $d(P)$ be a, b, c, $8a'$, $8b'$, $8c'$, $27g$, a'', b'', c'' respectively, we have, for determining them, the equations which express the identity of the right-hand side of (4) with the following:—

$$\begin{aligned}
&a\,dp_1\,dq_1\,dr_1 + b\,dp_2\,dq_2\,dr_2 + c\,dp_3\,dq_3\,dr_3 \\
&+ a'd(p_2+p_3)\,d(q_2+q_3)\,d(r_2+r_3) + b'd(p_3+p_1)\,d(q_3+q_1)\,d(r_3+r_1) \\
&+ c'd(p_1+p_2)\,d(q_1+q_2)\,d(r_1+r_2) \\
&+ g\,d(p_1+p_2+p_3)\,d(q_1+q_2+q_3)\,d(r_1+r_2+r_3) \\
&+ a''d(p_2-p_3)\,d(q_2-q_3)\,d(r_2-r_3) + b''d(p_3-p_1)\,d(q_3-q_1)\,d(r_3-r_1) \\
&+ c''d(p_1-p_2)\,d(q_1-q_2)\,d(r_1-r_2),
\end{aligned}$$

that is to say, (1) three equations of which the type is

$$x^3 = a+b'+c'+g-b''+c'',$$

(2) three like $\quad x^2y = c'+g-c'',$

(3) three like $\quad xy^2 = c'+g+c'',$

(4) the one $\quad xyz = g:$

and these suffice to determine uniquely the coefficients desired; viz.,

$$g = xyz,$$

$$2a'' = yz^2-y^2z,\quad 2b'' = zx^2-z^2x,\quad 2c'' = xy^2-x^2y,$$

$$2a' = yz^2+y^2z-2xyz,\ 2b' = zx^2+z^2x-2xyz,\ 2c' = xy^2+x^2y-2xyz,$$

$$a = x^3+xyz-xy^2-xz^2,\ b = y^3+xyz-yz^2-yx^2,\ c = z^3+xyz-zx^2-zy^2.$$

Inserting and integrating throughout, we have then, finally,

$$\begin{aligned}(P) = x\,(x^2+yz-y^2-z^2)\,(A)+y\,(y^2+zx-z^2-x^2)\,(B)&\\ +z\,(z^2+xy-x^2-y^2)\,(C)&\\ +4yz\,(y+z-2x)\,(A')+4zx\,(z+x-2y)\,(B')+4xy\,(x+y-2z)\,(C')&\\ +27xyz\,(G)&\\ -\tfrac{1}{2}yz\,(y-z)\,(B,C)-\tfrac{1}{2}zx\,(z-x)\,(C,A)-\tfrac{1}{2}xy\,(x-y)\,(A,B)\ \ldots\ldots(5).&\end{aligned}$$

It is to be observed that $(B, C) = -(C, B)$, and so for the other relative volumes.

4. Putting the expression (5) for (P) constant, we obtain the equation in areal coordinates x, y, z of the locus in the moving plane of points of it which surround volumes equal to a given volume. This locus is, then, a curve of the third order. This is true whatever the value of (P) be—positive, zero, or negative. The loci are, therefore, real curves for all values of (P). The only case of exception to these statements is that in which the range of positions is such that

$$(A) = (B) = (C) = (A') = (B') = (C') = (G),$$

and

$$(B, C) = 0 = (C, A) = (A, B),$$

i.e. the case of no rotation and no strain for which

$$(P) = (x+y+z)^3\,(A) = (A),$$

and is necessarily constant for all points of the lamina.

Moreover, the form of equation (5) shows that the locus cubics have all common asymptotes, real or imaginary, and indeed have all three-point contact at their common points at infinity.

Our present theorem as to volumes does not seem to admit, as does the corresponding theorem of area description, of a general simplification of statement applicable to all cases where the kinematics under consideration is that of an *indeformable* plane. In some instances the relative volumes (B, C), (C, A), (A, B) will be zeroes, as in many also where there is deformation; in others they will be spherical volumes $\pm\frac{4}{3}\pi a^3$, $\pm\frac{4}{3}\pi b^3$, $\pm\frac{4}{3}\pi c^3$. Among other difficulties in the way of a general estimation of the relative volumes, that of interpretation of sign is not the least considerable.

5. It is useful to consider how the cubic equation of the (P)-volume locus can be simplified by reference to particular triangles in the lamina. Now, in a cubic curve it is possible to inscribe a triangle such that, as well as its vertices, the middle points of its sides lie upon the curve; for, to express that the three vertices and three mid-points lie on the cubic, just six conditions are needed in six independent coordinates of the vertices, which can therefore be determined so as to satisfy them. Let, then, such a triangle $A_1B_1C_1$ inscribed in the cubic corresponding to volume zero, be taken as that of reference. (5) will become, if x, y, z be the coordinates of P with regard to it,

$$(P) = 27xyz\,(G_1) - \tfrac{1}{2}yz\,(y-z)\,(B_1, C_1) - \tfrac{1}{2}zx\,(z-x)\,(C_1, A_1) - \tfrac{1}{2}xy\,(x-y)\,(A_1, B_1) \quad\ldots\ldots(6),$$

The value of this result is lessened by the probability that the triangle $A_1B_1C_1$ need not be real. To the following special reference no such objection can apply. It is known that some one of the four triangles with regard to which the equation of a given cubic reduces to the canonical form $\alpha x^3+\beta y^3+\gamma z^3+\delta xyz = 0$, must necessarily be real, viz., the real "Wendepunktsdreieck" each of whose sides connects one real with two conjugate imaginary points of inflexion of the cubic. Let us now refer to $A_2B_2C_2$, this "Wendepunktsdreieck" for the cubic whose points surround zero volume. We obtain immediately, by expressing the identity of the above canonical form with the result of putting $(P) = 0$ in (5), and replacing (A), &c. by (A_2), &c., the reference volumes corresponding to the present triangle, that

$$\alpha = (A_2),\quad \beta = (B_2),\quad \gamma = (C_2),$$

and by a little manipulation that

$$(A_2') = \tfrac{1}{8}\{(B_2)+(C_2)\},\ (B_2') = \tfrac{1}{8}\{(C_2)+(A_2)\},\ (C_2') = \tfrac{1}{8}\{(A_2)+(B_2)\},$$
$$(B_2, C_2) = (B_2)-(C_2),\ (C_2, A_2) = (C_2)-(A_2),\ (A_2, B_2) = (A_2)-(B_2);$$

and thus get finally, for the volume of the locus in space of P whose coordinates with regard to this triangle are x, y, z,

$$(P) = x^3(A_2)+y^3(B_2)+z^3(C_2)+\{27(G_2)-(A_2)-(B_2)-(C_2)\}\,xyz \ldots(7).$$

Once more, let the triangle formed by the common asymptotes of the locus cubics be taken as that of reference, so that the equation of any of them [and in particular of the one for which the value of (P) is zero] takes the form $(\alpha x+\beta y+\gamma z)(x+y+z)^2+\delta xyz = 0$. A like method to that last used, it being noticed that $x+y+z=1$, produces for this case the result

$$(P)=x(A_3)+y(B_3)+z(C_3)+9\{3(G_3)-(A_3)-(B_3)-(C_3)\}xyz \ldots (8),$$

with the connecting relations between the reference volumes

$$(B_3, C_3)=0=(C_3, A_3)=(A_3, B_3),$$

$$(A_3')=\tfrac{1}{2}\{(B_3)+(C_3)\},\ (B_3')=\tfrac{1}{2}\{(C_3)+(A_3)\},\ (C_3')=\tfrac{1}{2}\{(A_3)+(B_3)\}.$$

This triangle formed by the asymptotes may of course be real, or may have only a single side real, according to circumstances. Even in the latter case, the result (8), though in itself unintelligible, may lead to sound geometrical conclusions, as in the like case of the following article.

Yet one more special triangle of reference may be mentioned, that which gives the zero-locus cubic the form of equation

$$(\alpha x^2+\beta y^2+\gamma z^2)(x+y+z)+\delta xyz=0.$$

Referred to it, the equivalent of (5) is

$$(P)=x^2(A_4)+y^2(B_4)+z^2(C_4)+3\{9(G_4)-(A_4)-(B_4)-(C_4)\}xyz \ldots (9),$$

and the connecting relations between the reference volumes are

$$(B_4, C_4)=0=(C_4, A_4)=(A_4, B_4),$$

$$(A_4')=\tfrac{1}{4}\{(B_4)+(C_4)\},\ (B_4')=\tfrac{1}{4}\{(C_4)+(A_4)\},\ (C_4')=\tfrac{1}{4}\{(A_4)+(B_4)\}.$$

This triangle of reference has its sides clearly parallel to those of the last.

6. In just the same way as (5) and its special forms (6), (7), (8), (9) have been obtained, we may find like formulæ determining any relative volume (P, Q), *i.e.*, the volume which would be generated by radii vectores from a fixed point always equal to and in the direction of the relative radius vector QP, P and Q being two points of the moving plane. The general value for (P, Q) thus found is, x, y, z and x', y', z' being the constant areal coordinates of P and Q respectively, merely the result of substituting in the right-hand member of (5), $x-x'$, $y-y'$, $z-z'$ instead of x, y, z. We thus obtain the following theorem like in form to that on absolute volumes:—*The locus of points in the moving plane which surround any the same relative volume with regard to any point of it is a cubic curve.*

The expressions for (P, Q) in terms of the constants of the special triangles of reference $A_1B_1C_1$, $A_2B_2C_2$ are in like manner obtained from the right-hand side of (6) and (7) by insertion of $x-x'$, $y-y'$, $z-z'$ for x, y, z. The cases of (8) and (9) where the values of (P) contain terms of the first or second degree, as well as cubic terms in x, y, z, are however different; for, instead of having unity or a power of $x+y+z$ multiplying such terms, the expressions for (P, Q) will have the like power of $x-x'+y-y'+z-z'$ or zero. Thus the corresponding result to (8), the triangle of reference being that formed by the asymptotes of the set of cubics giving different values of (P), is

$$(P, Q) = 9\{3(G_3)-(A_3)-(B_3)-(C_3)\}(x-x')(y-y')(z-z').$$

A glance at this assures us of the following:—*The cubics which are the loci of points in the moving plane that give equal relative volumes with regard to a fixed point in that plane have common asymptotes, viz., parallels through that point to the common asymptotes of the cubics in the plane which are the loci of points giving equal absolute volumes.* Moreover, the result tells us that, if either $x=x'$, $y=y'$, or $z=z'$, the relative volume (P, Q) vanishes. Hence,—*There are in the moving plane three directions, of which all or only one may be real, such that any radius QP parallel to either of them generates a vanishing relative volume; viz., the directions of the above asymptotes.*

The theorem of area description in plane kinematics, which corresponds to that of the present article, is that the locus of points of the varying lamina which surround equal relative values k with regard to a fixed one of it, is the conic with that point as centre, to use the same notation as in (1),

$$(a)(y-y')(z-z')+(b)(z-z')(x-x')+(c)(x-x')(y-y') = k;$$

and, in particular, if the lamina be indeformable, is a circle.

7. Let us now proceed to the general case of one space of three dimensions, supposed to change its position with two degrees of freedom in a fixed space so as to occupy an entirely closed doubly infinite cycle of positions, and so that the loci of its points include closed volumes in that fixed space. We suppose that the form of any part of it is not necessarily unalterable, but that there is a single definite state of homogeneous strain corresponding to each position. Let a tetrahedron of reference be taken in the varying, and three rectangular axes of p, q, r in the fixed space.

Considering points filling the interiors of the various loci to be connected by possible positions of the varying space just as those upon the surfaces themselves are, the element of internal volume of the locus in the fixed space of a point P of the moving one whose coordinates (tetrahedral volume ratio) with regard to the chosen tetrahedron are, for all its states of homogeneous strain, x, y, z, w, is

$$d(P) = (x\,dp_1+y\,dp_2+z\,dp_3+w\,dp_4)(x\,dq_1+y\,dq_2+z\,dq_3+w\,dq_4)$$
$$(x\,dr_1+y\,dr_2+z\,dr_3+w\,dr_4) \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

where (p_1, q_1, r_1), (p_2, q_2, r_2), (p_3, q_3, r_3), (p_4, q_4, r_4) are the coordinates with regard to the fixed axes of the four vertices A, B, C, D.

Multiplied out, this expression for $d(P)$ contains 64 elements mostly uninterpretable, but belonging to only 20 classes, each with one of the 20 homogeneous products of three dimensions of x, y, z, w for common coefficient. It can then, doubtless, be expressed linearly in terms of 20 intelligible elements. Those chosen here for trial are the elements of (A), (B), (C), (D) the volumes of the loci of the vertices, of (AB), (AC), (AD), (BC), (BD), (CD) those of the loci of middle points of sides, of (G_a), (G_b), (G_c), (G_d) those of the loci of centroids of faces, and of (A, B), (A, C), (A, D), (B, C), (B, D), (C, D) the volumes of the relative loci of the first vertex mentioned in each case with regard to the second; and it is seen as follows, that coefficients a_1, a_2, a_3, a_4, $8b_{12}$, $8b_{13}$, $8b_{14}$, $8b_{23}$, $8b_{24}$, $8b_{34}$, $27c_1$, $27c_2$, $27c_3$, $27c_4$, d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34} of these elements may be uniquely determined to make the linear expression in terms of them practicable. We have, in fact, to make the right-hand members of (10) identical with

$$\sum_{i=1}^{i=4} a_i\, dp_i\, dq_i\, dr_i + \sum_{i=1,\, j=i+1}^{i=j-1,\, j=4} b_{ij}\,(dp_i+dp_j)\,(dq_i+dq_j)\,(dr_i+dr_j)$$
$$+\sum_{i=1}^{i=4} c_i\,(\Sigma\, dp-dp_i)\,(\Sigma\, dq-dq_i)\,(\Sigma\, dr-dr_i)$$
$$+\sum_{i=1,\, j=i+1}^{i=j-1,\, j=4} d_{ij}\,(dp_i-dp_j)\,(dq_i-dq_j)\,(dr_i-dr_j)\,;$$

that is to say, we have merely to solve the 20 linear equations in 20 unknowns,—

$$\begin{aligned} a_1+b_{12}+b_{13}+b_{14}+c_2+c_3+c_4+d_{12}+d_{13}+d_{14} &= x^3,\\ a_2+b_{12}+b_{23}+b_{24}+c_1+c_3+c_4-d_{12}+d_{23}+d_{24} &= y^3,\\ a_3+b_{13}+b_{23}+b_{34}+c_1+c_2+c_4-d_{13}-d_{23}+d_{34} &= z^3,\\ a_4+b_{14}+b_{24}+b_{34}+c_1+c_2+c_3-d_{14}-d_{24}-d_{34} &= w^3,\\ b_{12}+c_3+c_4-d_{12} &= x^2y, \end{aligned}$$

and five exactly similar expressions equated to x^2z, x^2w, y^2z, y^2w, z^2w respectively,
$$b_{12}+c_3+c_4+d_{12} = xy^2,$$
with five like equations with xz^2, xw^2, yz^2, yw^2, zw^2 for right-hand sides, and the four

$$c_1 = yzw, \quad c_2 = xzw, \quad c_3 = xyw, \quad c_4 = xyz.$$

Hence at once, the c's being already given by the last four equations, the type of the d's is $\quad d_{12} = \tfrac{1}{2}xy\,(y-x),$

that of the b's is $\quad b_{12} = \tfrac{1}{2}xy\,(x+y-2z-2w),$

and the a's are
$$\begin{aligned} a_1 &= x\,(x^2-y^2-z^2-w^2+yz+yw+zw),\\ a_2 &= y\,(y^2-x^2-z^2-w^2+xz+xw+zw),\\ a_3 &= z\,(z^2-x^2-y^2-w^2+xy+xw+yw),\\ a_4 &= w\,(w^2-x^2-y^2-z^2+xy+xz+yz). \end{aligned}$$

We have, then, by integrating over all positions,

$$(P) = a_1(A) + a_2(B) + a_3(C) + a_4(D)$$
$$+8\{b_{12}(AB) + b_{13}(AC) + b_{14}(AD) + b_{23}(BC) + b_{24}(BD) + b_{34}(CD)\}$$
$$+27\{c_1(G_a) + c_2(G_b) + c_3(G_c) + c_4(G_d)\}$$
$$+d_{12}(A, B) + d_{13}(A, C) + d_{14}(A, D) + d_{23}(B, C) + d_{24}(B, D) + d_{34}(C, D) \quad \ldots\ldots\ldots\ldots(11),$$

with the above found cubic expressions in x, y, z substituted for the coefficients, a result which upon putting $w = 0$ is seen to include, as it should, the previously obtained one (5).

Putting the value of (P) constant, we obtain that the locus in the moving space of points of it whose loci in the fixed one contain volumes equal to a given one, is a cubic surface. For all values of (P), positive, zero, and negative, there is then such a real locus surface. Moreover, the form of equation shows that all the cubic surfaces thus obtained as loci have contact of the closest possible kind at an infinite distance.

8. The most convenient special tetrahedron to take for that of reference in the moving space, is that real one with regard to which the cubic $(P) = 0$ reduces to the form

$$\alpha x^3 + \beta y^3 + \gamma z^3 + \delta w^3 + \alpha' yzw + \beta' xzw + \gamma' xyw + \delta' xyz = 0.$$

The reference volumes being now those corresponding to this tetrahedron, the vanishing of the coefficients of x^2y, xy^2, &c. gives us twelve conditions in these, viz.,

six like $\qquad 4(AB) - \tfrac{1}{2}(A, B) - (B) = 0,$

and six like $\qquad 4(AB) + \tfrac{1}{2}(A, B) - (A) = 0;$

whence $8(AB) = (A) + (B)$, &c., and $(A, B) = (A) - (B)$, &c.,

and the value of (P), x, y, z, w being now coordinates referred to this tetrahedron, reduces to

$$(P) = (A)x^3 + (B)y^3 + (C)z^3 + (D)w^3$$
$$+27\{(G'_a)yzw + (G'_b)xzw + (G'_c)xyw + (G'_d)xyz\} \ldots\ldots(12),$$

where $\qquad 27(G'_a) = 27(G_a) - (B) - (C) - (D)$, &c.

Two other special tetrahedra of reference may be mentioned, viz., (1) one of those whose vertices and the mid-points of whose edges lie upon the cubic surface $(P) = 0$ (determined by the satisfaction of ten conditions among twelve quantities), referred to which (2) is replaced by

$$(P) = 27\{(G_a)yzw + (G_b)xzw + (G_c)xyw + (G_d)xyz\}$$
$$-\tfrac{1}{2}\Sigma\{(A, B)xy(x-y)\} \ldots\ldots\ldots\ldots(13),$$

and (2) that referred to which the cubic surface $(P) = 0$ takes an equation of the form

$$(\alpha x+\beta y+\gamma z+\delta w)\,(x+y+z+w)^2+\alpha' yzw+\beta' xzw+\gamma' xyw+\delta' xyz = 0,$$

which enables us to reduce (11) to

$$(P) = x\,(A)+y\,(B)+z\,(C)+w\,(D)$$
$$+27\,(G_a'')\,yzw+(G_b'')\,xzw+(G_c'')\,xyw+(G_d'')\,xyz\}\ldots\ldots(14),$$

in which $27\,(G_a'') = 27\,(G_a)-9\,\{(B)+(C)+(D)\}$, &c.

9. The relative volume (P, Q) of the locus in the fixed space of one point $P\ (x, y, z, w)$ of the moving one with regard to another $Q\,(x', y', z', w')$ of it, is derived from (P) by the substitution for x, y, z, w of $x-x'$, $y-y'$, $z-z'$, $w-w'$ in the general expression for it (11). Thus the locus of points giving, with regard to a specified one, a constant relative volume, is a cubic surface. Of the expressions (12), (13), (14), and others simplified by special reference, the terms below the third degree in x, y, z, w will have corresponding to them no terms in the corresponding expressions for (P, Q) for a like reason to that adduced in §6. The simplified form corresponding to (14) for relative volumes will assure us immediately of the fact, that there are six directions, not necessarily all real, of lines QP through any point Q of the moving space which sweep out zero relative volumes.

On certain Quartic Curves, which have a Cusp at Infinity, whereat the Line at Infinity is a Tangent. By Henry M. Jeffery.

[*Read Dec. 14th*, 1882.]

1. In a recent memoir, published in these *Proceedings* (Vol. xiii., No. 185), quartics were classified, which were met and touched by the line at infinity in four coincident points,

$$\kappa\alpha^4 = u_3.$$

I now proceed to the next division of quartics (Salmon's "Higher Plane Curves," p. 213), which are met and touched by the same line in three coincident and met in one other real point (see § 31), and may be thus denoted $\kappa\alpha^3\beta = u_3$.

Inasmuch as the three points may coincide in a cusp, triple point, or stapete-point, the title of the memoir is imperfect.

The quartics are usually unicusped ovals, pirum-shaped, but occasionally cardioidal, and are either unipartite or bipartite.