

$$\begin{aligned}
 & \int_0^r \int_0^\pi \sin \theta \, d\phi \, d\theta \\
 & \times \frac{r^3 - a^3 - \lambda^3 + 2a\lambda \cos \theta}{[r^3 + a^3 + \lambda^3 - 2a\lambda \cos \theta - 2r\lambda \cos a + 2ar (\cos a \cos \theta - \sin a \sin \theta \cos \phi)]^{\frac{1}{2}}} \\
 & = \frac{2\pi (r^3 - \lambda^3)}{(r^3 + \lambda^3 - 2r\lambda \cos a)^{\frac{1}{2}}}.
 \end{aligned}$$

Prof. Tait wished to know whether the solution could be easily effected by direct analytical processes.

Prof. H. J. S. Smith, F.R.S., Vice-President, then read two papers : Second Notice "On the Characteristics of the Modular Curves," and a Note relating to the "Theory of the Division of the Circle." Prof. Cayley spoke on the subject of both papers, asking, in the course of his remarks, if a solution had been effected for the inscription of a regular heptagon, assuming the trisection of an angle.

Mr. Tucker then read abstracts of papers by Prof. Minchin "On the Astatic Conditions of a Body acted on by given Forces," and by Mr. C. Leudesdorf "On certain Extensions of Frullani's Theorem."

The following presents were received :—

"Journal of Education," April, 1878.

"Elements of Dynamic," Part i., Kinematic, by Prof. W. K. Clifford, F.R.S.; Macmillan, 1878. From the Author.

"Atti della R. Accad. dei Lincei . . .," Serie terza; "Transunti," Vol. ii., Fasc. 4<sup>o</sup>, Marzo, 1878; Roma.

"Crelle's Journal," 85 Band, 1<sup>st</sup> Heft; Berlin, 1878.

"Jahrbuch über die Fortschritte der Mathematik . . .," achter Band, Jahrgang, 1876, Heft i.; Berlin, 1878.

"Anvendelse af en Sætning af Maxwell til at finde de billigste Bygningskonstruktioner af Dr. H. G. Zeuthen."

"Educational Times," April, 1878.

"Journal of Institute of Actuaries," No. cx., Jan. 1878.

"Proceedings of Royal Society," No. 186, Vol. xxvii.

"Monatsbericht," Januar, 1878; Berlin.

### *On Astatic Equilibrium.* By Prof. MINCHIN.

[Read April 11th, 1878.]

1. When a body is in equilibrium under the action of forces applied at given points in the body, with fixed magnitudes, and directions fixed in space, it will, under certain conditions, remain in equilibrium when

it is displaced in any manner. These are the *Astatic Conditions* of the given forces. They have been investigated at great length in Moigno's "Statique" (Dixième Leçon), and also in a memoir by M. Darboux (Bordeaux, 1877), which has been recently presented to this Society, but which I have not yet thoroughly examined. I believe, however, that several of the results in the following paper have not been noticed by either of these writers.

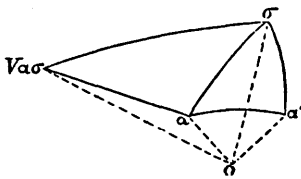
The method of treatment which I have adopted is one which involves only the very elements of quaternions—the most simple scalar and vector operations, in fact.

2. The equilibrium of a rigid body is preserved by two conditions—namely, the vanishing of the Resultant of Translation of the acting forces, and the vanishing of the Principal Moment calculated for any origin.

3. The general displacement of a rigid body can always be produced by a motion of translation, together with a motion of rotation about some axis, and such is the displacement which must be contemplated in discussing the Astatic Conditions. It is obvious, however, that a motion of translation can alter nothing in the equilibrium of the forces, so that we may confine our attention to a displacement produced by the rotation of the body about an axis.

4. Suppose that a body is rotated about an axis drawn through an origin  $O$  in the direction of a unit vector  $\sigma$ , what does any vector,  $a$ , drawn from  $O$  to a fixed point in the body, become?

Let a sphere, described with  $O$  as centre, meet the directions of  $\sigma$ ,  $a$ , and  $Va\sigma$  in points represented in the figure by these letters. Suppose that after rotation the vector  $a$  takes the direction denoted by  $a'$ ; let the angle between  $Oa$  and  $O\sigma$  be  $\theta$ , and let the body rotate through an angle  $\phi$ . Assume



$$a' = x\sigma + y\sigma + zVa\sigma;$$

$$\therefore Saa' = x\sigma^2 + yS\sigma\sigma,$$

$$S\sigma a' = xS\sigma\sigma + y\sigma^2,$$

$$Sa'Va\sigma = z(Va\sigma)^2 = -2\sigma^2 \sin^2 \theta.$$

$$\text{Now } S\sigma a' = -r^2 a (\cos^2 \theta + \sin^2 \theta \cos \phi), \text{ since } ra' = ra;$$

$$\text{also } S\sigma a' = -ra \cdot \cos \theta, \quad Sa'Va\sigma = r^2 a \cdot \sin^2 \theta \sin \phi.$$

Hence we have

$$x = \cos \phi; \quad y = (1 - \cos \phi) ra \cdot \cos \theta = -(1 - \cos \phi) S\sigma\sigma; \quad z = \sin \phi;$$

$$\therefore a' = a \cos \phi - (1 - \cos \phi) \sigma S\sigma\sigma + \sin \phi Va\sigma,$$

which determines the vector into which  $a$  is transformed by rotation.

5. Let  $a_1, a_2, a_3, \dots$  be the vectors from a fixed origin to the points in the body at which forces represented in magnitudes and directions by the vectors  $\omega_1, \omega_2, \omega_3, \dots$  are applied. Then the resultant of translation is  $\omega_1 + \omega_2 + \omega_3 + \dots$ , or  $\Sigma \omega$ ; and if  $G$  is the vector axis of the resultant couple for the assumed origin,

$$G = V(a_1 \omega_1 + a_2 \omega_2 + \dots).$$

The conditions of equilibrium are, then,

$$\Sigma \omega = 0, \quad \Sigma V a \omega = 0.$$

Now by the rotation of the body about any axis through  $O$ ,  $G$  becomes,

$$\text{by §4,} \quad \cos \phi \Sigma V a \omega + 2 \sin^2 \frac{\phi}{2} \Sigma (V \omega \sigma S a \sigma) - \sin \phi \Sigma (V \cdot \omega V a \sigma).$$

$\Sigma \omega$ , of course, remains unchanged. This expression must vanish independently of  $\phi$  and  $\sigma$ ; hence we must have

$$\Sigma (V \omega \sigma S a \sigma) \equiv 0 \dots \dots \dots (1),$$

$$\Sigma (V \cdot \omega V a \sigma) \equiv 0 \dots \dots \dots (2),$$

The second of these equations gives

$$\sigma (S a_1 \omega_1 + S a_2 \omega_2 + \dots) - (a_1 S \omega_1 \sigma + a_2 S \omega_2 \sigma + \dots) = 0 \dots \dots \dots (3),$$

which, since  $\sigma^2 = -1$ , involves the conditions

$$S a_1 \omega_1 + S a_2 \omega_2 + \dots \equiv 0,$$

$$a_1 S \omega_1 \sigma + a_2 S \omega_2 \sigma + \dots \equiv 0,$$

The first of these expresses that the Virial of the given forces vanishes.

That these conditions must hold may be otherwise seen thus:—Equation (3) may be written  $\phi \sigma = g \sigma$ , if  $\phi \sigma \equiv \Sigma a S \omega \sigma$ , and this equation is, in general, satisfied by only three definite values of  $\sigma$ .

It is not difficult to see that a combination of (1) and (2) gives the equation

$$\omega_1 S a_1 \sigma + \omega_2 S a_2 \sigma + \dots \equiv 0,$$

or  $\phi' \sigma \equiv 0$ , where  $\phi'$  is the conjugate of  $\phi$ . But if  $\phi \sigma \equiv 0$ ,  $\phi' \sigma$  will also vanish identically, so that the conditions expressed by (1) and (2) may both be represented by the equation  $\phi \sigma \equiv 0$ .

Again, it will be easily seen that the equations  $\Sigma S a \omega = 0$ ,  $\Sigma V a \omega = 0$  are both deducible from the equation  $\phi \sigma \equiv 0$ ; so that finally the Astatic conditions are

$$\Sigma \omega = 0 \dots \dots \dots (4),$$

$$\Sigma a S \omega \sigma \equiv 0.$$

6. The identical vanishing of the linear vector function  $\phi \sigma$  will be guaranteed if this function vanishes when any three non-coplanar

vectors  $i, j, k$  are substituted for  $\sigma$ . For simplicity, we shall suppose that  $i, j, k$  are a rectangular set of unit vectors.

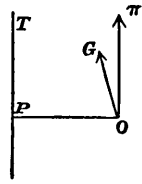
Hence the Astatic Conditions are

$$\Sigma \omega = 0 \dots \dots \dots (5),$$

$$\Sigma a S i \omega = 0, \quad \Sigma a S j \omega = 0, \quad \Sigma a S k \omega = 0 \dots \dots \dots (6).$$

7. To find the equation of Poinsot's Axis.

For any assumed origin,  $O$ , let  $G$  be the vector axis of the principal couple, and let  $\Pi$  (or  $\Sigma \omega$ ) be the Resultant of Translation. Then it is well known that Poinsot's Axis is constructed by taking a point  $P$  (in the proper direction) on the right line  $OP$  perpendicular to  $\Pi$  and  $G$ , at a distance from  $O$  equal to  $\frac{TG}{T\Pi} \sin \phi$ , where  $\phi$  is the angle between  $G$  and



$\Pi$ . Hence the vector  $OP$  is  $\frac{V\Pi G}{T^2\Pi}$ ; and the equation of Poinsot's Axis

$$\text{is} \quad \rho = x\Pi + \frac{V\Pi G}{T^2\Pi}.$$

8. To find the conditions that a system of forces should be astatically equivalent to a single force.

Suppose that a force  $\Pi_1$ , acting at the extremity of a vector  $A_1$ , drawn to a point fixed in the body, astatically equilibrates the given system.

Denote  $\Sigma \omega$  by  $\Pi$ , and  $\Sigma a S i \omega$ ,  $\Sigma a S j \omega$ ,  $\Sigma a S k \omega$  by  $I, J, K$  respectively. Then we have, by (5) and (6),

$$A_1 S i \Pi = I, \quad A_1 S j \Pi = J, \quad A_1 S k \Pi = K,$$

which give the conditions

$$\frac{I}{S i \Pi} = \frac{J}{S j \Pi} = \frac{K}{S k \Pi}.$$

These conditions are always satisfied when the given forces form a parallel system, since  $\frac{I}{S i \Pi}$  obviously becomes  $\frac{\Sigma a T \omega}{T \Pi}$ , which is also the common value of  $\frac{J}{S j \Pi}$  and  $\frac{K}{S k \Pi}$ , and the vector to the centre of parallel forces.

In general, the vectors  $\frac{I}{S i \Pi}$ ,  $\frac{J}{S j \Pi}$ ,  $\frac{K}{S k \Pi}$  are those drawn from the origin to the points of application,  $P_i, P_j, P_k$ , of three systems of parallel forces which are obtained by resolving each force into three rectangular components in the directions  $i, j, k$ .

In the present case, then, the points  $P_i, P_j, P_k$  must coincide.

9. To find the conditions that a system of forces should be astatically equivalent to two forces.

Let the two forces be  $\Pi_1$  and  $\Pi_2$  at points whose vectors are  $A_1$  and  $A_2$ .

Hence  $\Pi_1 + \Pi_2 + \Pi = 0$  ..... (7),

$$A_1 Si\Pi_1 + A_2 Si\Pi_2 = -I$$
 ..... (8),

$$A_1 Sj\Pi_1 + A_2 Sj\Pi_2 = -J$$
 ..... (9),

$$A_1 Sk\Pi_1 + A_2 Sk\Pi_2 = -K$$
 ..... (10).

The last three equations show that  $I, J, K$  must be coplanar with  $A_1$  and  $A_2$ . Hence  $SIJK = 0$  ..... (11).

But there is another condition to be satisfied; for if  $lI + mJ + nK \equiv 0$ , when  $l, m, n$  are given scalars, we have, by multiplying (8), (9), and (10) by  $l, m, n$ , and adding,

$$lSi\Pi_1 + mSj\Pi_1 + nSk\Pi_1 = 0$$
 ..... (12),

$$lSi\Pi_2 + mSj\Pi_2 + nSk\Pi_2 = 0$$
 ..... (13),

which, by (7), give the second condition,

$$lSi\Pi + mSj\Pi + nSk\Pi = 0$$
 ..... (14).

Now it is easy to see that (11) and (14) signify that *the three centres,  $P_1, P_2, P_3$  of parallel forces are in one right line*. We shall call this the *Line of Centres*. Moreover, *the points of application of  $\Pi_1$  and  $\Pi_2$  must also lie on this line*. For (8) can be written

$$A_1 Si\Pi_1 + A_2 Si\Pi_2 + Si\Pi \frac{I}{Si\Pi} = 0;$$

and since, by (7),  $Si\Pi_1 + Si\Pi_2 + Si\Pi = 0$ , this equation shows that the points at which  $\Pi_1$  and  $\Pi_2$  act lie on a right line through  $P_3$ ; similarly, they lie on a right line through  $P_2$ ; therefore, &c.

Again, the forces  $\Pi_1$  and  $\Pi_2$  are obviously known in magnitudes and directions from equations (7) ... (10) as soon as their points of application are assumed; and in all cases they are both parallel to a given plane; for (12) gives

$$S(li + mj + nk)\Pi_1 = 0,$$

and a similar equation in  $\Pi_2$ , which show that  $\Pi_1$  and  $\Pi_2$  are both perpendicular to the vector  $li + mj + nk$ , as is also the Resultant of Translation of the given system, by (14).

If we assume  $A_1$  and  $A_2$ , i.e., take

$$A_1 = -\frac{I}{a} + x\left(\frac{I}{a} - \frac{J}{b}\right), \quad A_2 = -\frac{I}{a} + y\left(\frac{I}{a} - \frac{J}{b}\right),$$

where, for shortness, we use  $a, b, c$  for  $-Si\Pi$ ,  $-Sj\Pi$ ,  $-Sk\Pi$ , equations (8), (9), (10) give

$$\begin{aligned} Si\Pi_1 &= -\frac{ay}{x-y}, & Sj\Pi_1 &= -\frac{b(y-1)}{x-y}, & Sk\Pi_1 &= -\frac{cny+bm}{n(x-y)}, \\ Si\Pi_2 &= \frac{ax}{x-y}, & Sj\Pi_2 &= \frac{b(x-1)}{x-y}, & Sk\Pi_2 &= \frac{cnx+bm}{n(x-y)}, \end{aligned}$$

which of course determine  $\Pi_1$  and  $\Pi_2$ .

If we take  $\Pi_1$  and  $\Pi_2$  at right angles to each other, we have

$$Si\Pi_1 Si\Pi_2 + Sj\Pi_1 Sj\Pi_2 + Sk\Pi_1 Sk\Pi_2 = 0,$$

which gives

$$n^2(a^2+b^2+c^2)xy+bn(cn-bn)(x+y)+b^2(m^2+n^2)=0.$$

Now it is obvious that the distance of the extremity of  $A_1$  from  $P_i$  is  $x \cdot P_i P_j$ , and the distance of the extremity of  $A_2$  from  $P_i$  is  $y \cdot P_i P_j$ ; and this last equation shows that these distances ( $\xi_1$  and  $\xi_2$ ) are connected by an equation of the form

$$\xi_1 \xi_2 + p(\xi_1 + \xi_2) + q = 0,$$

and that, therefore, the points at which  $\Pi_1$  and  $\Pi_2$  are applied are conjugate points of an involution system on the line of centres  $P_i P_j P_k$ .

The distance of the centre of this involution system from  $P_i$  is, of

course, 
$$\frac{b(bn-cn)}{n(a^2+b^2+c^2)} P_i P_j,$$

so that the vector to this centre is

$$-\frac{I}{a} + \frac{b(bn-cn)}{n(a^2+b^2+c^2)} \left( \frac{I}{a} - \frac{J}{b} \right),$$

or

$$-\frac{aI+bJ+cK}{a^2+b^2+c^2}.$$

Now it is easy to see that this is the vector to the centre of a system of parallel forces whose common direction is that of the Resultant of Translation of the given system.

For, the component of the force  $\varpi_1$  in this direction is  $-\frac{S\varpi_1\Pi}{T\Pi}$ ; therefore the vector of the centre of the parallel system is

$$\frac{a_1 S\varpi_1\Pi + a_2 S\varpi_2\Pi + \dots}{S\varpi_1\Pi + S\varpi_2\Pi + \dots},$$

or

$$-\frac{a_1 S(ai+bj+ck)\varpi_1 + a_2 S(ai+bj+ck)\varpi_2 + \dots}{T^2\Pi},$$

or

$$-\frac{aI+bJ+cK}{a^2+b^2+c^2}.$$

Hence, when a system of forces is astatically equilibrated by two rectangular forces, the points of application of these latter must lie on the line of centres, and be conjugate points of an involution system whose centre is the centre of a system parallel to the Resultant of Translation.

Call this point the centre of the line of centres.

It remains to be proved that the line of centres is unique. For this purpose we shall show that, if the given forces are each resolved in the direction of any vector, the centre of this system will lie on the line  $P_1 P_2 P_3$ .

If  $\vec{r}$  is the assumed vector, let  $\vec{r} = xi + yj + zk$ ; then the vector to the centre of forces parallel to  $\vec{r}$  is

$$\frac{a_1 S \vec{r} \omega_1 + a_2 S \vec{r} \omega_2 + \dots}{S \vec{r} \omega_1 + S \vec{r} \omega_2 + \dots}, \quad \text{or} \quad -\frac{xI + yJ + zK}{ax + by + cz}.$$

If  $I'$  is this vector, we have

$$(ax + by + cz) I' - ax \frac{I}{-a} - by \frac{J}{-b} - cz \frac{K}{-c} = 0;$$

and since the sum of the multipliers of  $I'$ ,  $-\frac{I}{a}$ ,  $-\frac{J}{b}$ , and  $-\frac{K}{c}$ , is zero, and the extremities of the latter three are collinear, the extremity of  $I'$  must lie on the line of centres.

The relations between the vectors  $A_1$  and  $A_2$  and the forces  $\Pi_1$  and  $\Pi_2$  will perhaps be better seen if we use  $\Omega$  for the vector to the centre of the line of centres, and  $\theta$  for the unit vector in the direction of the line of centres.

Clearly then we may put, when  $\Pi_1$  and  $\Pi_2$  are rectangular,

$$\left. \begin{aligned} A_1 &= \Omega + x\theta \\ A_2 &= \Omega - \frac{h^2}{x}\theta \end{aligned} \right\} \dots\dots\dots (15),$$

$$I = -a\Omega - s\theta, \quad J = -b\Omega - s'\theta, \quad K = -c\Omega - s''\theta,$$

where  $h^2$ ,  $s$ ,  $s'$ ,  $s''$  are all given constants, and  $x$  any variable scalar.

Equations (8), (9), (10) then determine  $\Pi_1$  and  $\Pi_2$ , and give

$$\Pi_1 = -\frac{h^2 \Pi + x\Theta}{h^2 + x^2}, \quad \Pi_2 = -\frac{x^2 \Pi - x\Theta}{h^2 + x^2} \dots\dots\dots (16),$$

where  $\Theta = si + s'j + s''k$ , so that  $\Theta$  is a given vector parallel to the plane to which  $\Pi$ ,  $\Pi_1$ , and  $\Pi_2$  are all parallel.

It is easy to prove in different ways that the vectors  $\Pi$  and  $\Theta$  are perpendicular to each other. One simple method consists in the fact that  $S\Pi_1 \Pi_2$  must be zero independently of the value of  $x$ . It may be seen otherwise thus:

$$aI + bJ + cK = -(a^2 + b^2 + c^2) \Omega - (as + bs' + cs'') \theta;$$

but

$$\Omega = -\frac{aI + bJ + cK}{a^2 + b^2 + c^2};$$

$\therefore as + bs' + cs'' = 0$ , which is the condition that  $\Theta$  and  $\Pi$  should be perpendicular.

The condition  $S\Pi_1\Pi_2 = 0$  further gives

$$h^2\Pi^2 - \Theta^2 = 0,$$

or, if  $R$  be  $T\Pi$ ,

$$T\Theta = hR \dots\dots\dots(17).$$

Corresponding to different values of  $x$  we can represent the magnitudes and directions of  $\Pi_1$  and  $\Pi_2$ . Since they equilibrate  $\Pi$ , they must, of course, be represented by the two sides of a right-angled triangle described, in a plane fixed in space, on  $\Pi$  as hypotenuse.

Draw from any origin,  $O$ , two right lines,  $OA$  and  $OB$ , in the directions of  $\Pi$  and  $\Theta$  respectively. Then, if the components of  $\Pi_1$  along  $OA$  and  $OB$  are  $X_1$  and  $Y_1$ , we have

$$X_1 = \frac{h^2 R}{h^2 + x^2}, \quad Y_1 = \frac{h R x}{h^2 + x^2};$$

therefore, if  $\Pi_1 = \overline{OP_1}$ , we have

$$\tan P_1OA = \frac{x}{h},$$

which determines the direction of  $\Pi_1$ .

Again, we can prove the following theorem:—

*As their points of application along the line of centres vary, the two rectangular forces which astatically equilibrate the system trace out a hyperbolic paraboloid.*

For if  $\rho$  is the vector to any point on the surface traced out by  $\Pi_1$ ,

$$\rho = \Omega + x\theta + y \frac{h^2\Pi + x\Theta}{h^2 + x^2},$$

where  $x$  and  $y$  are any variable scalars; or, if the centre of the line of centres is taken as origin of vectors,

$$\rho = x\theta + \frac{h^2 y}{h^2 + x^2} \Pi + \frac{xy}{h^2 + x^2} \Theta.$$

The Cartesian equation of this surface, referred to the line of centres as axis of  $x$ , and those of  $\Pi$  and  $\Theta$  as axes of  $y$  and  $z$ , respectively, is

$$xy = hz.$$

Since the two forces  $\Pi_1$  and  $\Pi_2$  are equivalent to the given system, the discussion of these forces may replace the discussion of the system. For example, to find Poinso's Axis, we have, from (15) and (16),

$$VA_1\Pi_1 + VA_2\Pi_2 = -V(\Omega\Pi + \theta\Theta).$$

Hence, from § 7, Poinso's Axis is

$$\rho = x\Pi + \Omega - \frac{1}{h^2} V\theta\Theta\Pi;$$



or, if the centre of the line of centres is taken as origin of vectors,

$$\rho = x\Pi - \frac{1}{R^2} V\theta\theta\Pi,$$

so that Poinso't's Axis will coincide with the Resultant of Translation at the centre of the line of centres if  $V\theta\theta\Pi = 0$ , or, in other words, if the line of centres is perpendicular to the plane (of  $\theta$  and  $\Pi$ ) to which the forces  $\Pi_1$ ,  $\Pi_2$ , and  $\Pi$  are parallel. The body may, of course, be turned round so that this perpendicularity occurs.

*A system of forces can be astatically equilibrated by two forces when all the forces of the system are parallel to one plane.*

For let the unit vector perpendicular to this plane be taken as  $k$ . Then  $Sk\varpi_1 = 0$ ,  $Sk\varpi_2 = 0$ , .....; therefore  $K = 0$ , and the requisite conditions (11) and (14) are satisfied.

*A system of forces may be astatic for displacements about a particular axis without being astatic for displacements about other axes.*

If  $\sigma$  is the unit vector in the direction of the axis of displacement, the conditions of continuous equilibrium are

$$\Sigma\varpi = 0,$$

$$\cos\phi \Sigma V_a\varpi + 2\sin^2\frac{\phi}{2} \Sigma (V\varpi\sigma S_a\sigma) - \sin\phi \Sigma V. \varpi V_a\sigma = 0;$$

the latter, holding for all values of  $\phi$ , gives

$$\Sigma V_a\varpi = 0, \quad \Sigma (V\varpi\sigma S_a\sigma) = 0, \quad \Sigma V. \varpi V_a\sigma = 0.$$

Consider the case in which all the forces lie in one plane, and let the axis of displacement be any axis perpendicular to this plane. Also take the origin of vectors in the plane.

Then these conditions become, since

$$S_{a_1}\sigma = S_{a_2}\sigma = \dots = 0, \quad S\varpi_1\sigma = S\varpi_2\sigma = \dots = 0, \\ \Sigma\varpi = 0, \quad \Sigma V_a\varpi = 0, \quad \Sigma S_a\varpi = 0.$$

I have shown ("Statics," pp. 101, 102) how these conditions are otherwise deduced, it being observed that  $\Sigma S_a\varpi$  is obviously the Virial of the forces.

Such a system of forces as this can always be equilibrated (for the displacements considered) by a single force. For let the force be  $\Pi_1$  at the extremity of a vector  $A_1$ . Then the conditions are

$$\Pi_1 + \Pi = 0, \quad A_1\Pi_1 + \Sigma a\varpi = 0; \\ \therefore \Pi_1 = -\Pi, \quad A_1 = \frac{\Sigma a\varpi}{\Pi},$$

the expression for  $A_1$  being evidently a vector, since  $a_1$ ,  $\varpi_1$ , ... and  $\Pi$  are all coplanar.

We thus arrive at the "Centre" of the system, and it is very easy

to prove that (see "Statics," p. 101) this point is characterised by the vanishing of the sum of the moments and of the Virial of the forces about it.

10. To investigate the astatic equivalence of a system of forces to three forces.

Let the forces be  $\Pi_1, \Pi_2, \Pi_3$  at the extremities of vectors  $A_1, A_2, A_3$ .

Then

$$\Pi_1 + \Pi_2 + \Pi_3 = -\Pi \dots\dots\dots (18),$$

$$A_1 Si \Pi_1 + A_2 Si \Pi_2 + A_3 Si \Pi_3 = -I \dots\dots\dots (19),$$

$$A_1 Sj \Pi_1 + A_2 Sj \Pi_2 + A_3 Sj \Pi_3 = -J \dots\dots\dots (20),$$

$$A_1 Sk \Pi_1 + A_2 Sk \Pi_2 + A_3 Sk \Pi_3 = -K \dots\dots\dots (21).$$

Now it is well known that the extremities of four vectors,  $\rho_1, \rho_2, \rho_3, \rho_4$ , drawn from the same origin, will be coplanar if

$$p_1 \rho_1 + p_2 \rho_2 + p_3 \rho_3 + p_4 \rho_4 = 0,$$

where the scalars  $p_1, p_2, \dots$  satisfy the equation

$$p_1 + p_2 + p_3 + p_4 = 0.$$

But,  $Si \Pi$  being still denoted by  $-a$ , &c, (19) can be written

$$A_1 Si \Pi_1 + A_2 Si \Pi_2 + A_3 Si \Pi_3 - a \frac{I}{-a} = 0,$$

while (18) gives  $Si \Pi_1 + Si \Pi_2 + Si \Pi_3 - a = 0$ .

Hence the extremities of  $A_1, A_2$ , and  $A_3$  lie in a plane containing the point  $P_i$ , which has been shown to be the centre of a system of parallel forces obtained by resolving each force parallel to  $i$ .

The remaining equations show in the same way that these extremities lie in a plane containing  $P_j$  and  $P_k$ .

Hence the points of application of the three forces which astatically equilibrate the system lie in the plane of the three centres  $P_i, P_j, P_k$ ; that is, the plane containing the extremities of the vectors  $-\frac{I}{a}, -\frac{J}{b}, -\frac{K}{c}$ .

This plane we shall call the *Plane of Centres*.

The astatic reduction to three forces is therefore always possible.

The plane of centres can be easily shown to be unique; that is, if the forces be all resolved in any common direction, the centre of the system thus obtained will lie in the plane of  $P_i, P_j$ , and  $P_k$ . For, let the unit vector in the common direction be  $xi + yj + zk$ . Then the vector to the new centre is

$$\frac{x \sum a Si \varpi + y \sum a Sj \varpi + z \sum a Sk \varpi}{x Si \Pi + y Sj \Pi + z Sk \Pi}, \text{ or } -\frac{xI + yJ + zK}{ax + by + cz};$$

and by multiplying this vector by  $ax + by + cz$ , and the vectors  $-\frac{I}{a}, -\frac{J}{b}, -\frac{K}{c}$  by  $-ax, -by, -cz$  respectively, and adding, we get a

result which vanishes identically. Therefore the new centre is in the same plane as the others.

The equation of the plane of centres is, of course,

$$S(aVJK + bVKI + cVIJ)\rho = -SIJK \dots\dots\dots(22).$$

Operating on (19), (20), (21) with  $S \cdot VA_1A_2$ , we have

$$Si\Pi_1 \cdot SA_1A_2A_3 = -SIVA_2A_3,$$

$$Sj\Pi_1 \cdot SA_1A_2A_3 = -SJVA_3A_1,$$

$$Sk\Pi_1 \cdot SA_1A_2A_3 = -SKVA_3A_1;$$

which give

$$\Pi_1 = \frac{1}{SA_1A_2A_3} (iSIVA_2A_3 + jSJVA_3A_1 + kSKVA_3A_1);$$

and in the same way we obtain  $\Pi_2$  and  $\Pi_3$ , so that when their points of application are assumed the forces  $\Pi_1$ ,  $\Pi_2$ ,  $\Pi_3$  are thus completely known.

It may be observed that the origin of vectors can always be so chosen that the vectors  $I, J, K$  shall be a rectangular system, the vectors  $i, j, k$  remaining the same. For the points  $P_i, P_j, P_k$  depend only on the vectors  $i, j, k$ , and not on the origin of vectors; and, given three points,  $P_i, P_j, P_k$ , two other points,  $O$  and  $O'$ , can be found such that the lines  $OP_i, OP_j$ , and  $OP_k$  are a rectangular system, as are also the lines  $O'P_i, O'P_j$ , and  $O'P_k$ .

The points  $O$  and  $O'$  are the points common to three spheres described on the sides of the triangle  $P_iP_jP_k$  as diameters; they are equidistant from this plane on opposite sides, and lie on the perpendicular to it drawn through the orthocentre of the triangle.

We may suppose either of these points taken as origin of vectors, and treat  $I, J, K$  as a rectangular system.

For simplicity, denoting the vectors  $VA_2A_3, VA_3A_1, VA_1A_2$  by  $\epsilon_1, \epsilon_2, \epsilon_3$ , we have

$$\Pi_1 = \frac{1}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}} (iSI\epsilon_1 + jSJ\epsilon_1 + kSK\epsilon_1) \dots\dots\dots(23),$$

$$\Pi_2 = \frac{1}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}} (iSI\epsilon_2 + jSJ\epsilon_2 + kSK\epsilon_2) \dots\dots\dots(24).$$

$$\Pi_3 = \frac{1}{\sqrt{-S\epsilon_1\epsilon_2\epsilon_3}} (iSI\epsilon_3 + jSJ\epsilon_3 + kSK\epsilon_3) \dots\dots\dots(25).$$

It may be noticed that, for different systems of rectangular vectors  $i', j', k'$ , the vectors to the centres of the corresponding systems of parallel forces are in the directions of conjugate diameters of a certain ellipsoid. For if  $i' = xi + yj + zk$ ,  $j' = x'i + y'j + z'k$ ,  $k' = x''i + y''j + z''k$ , it is obvious that  $I', J', K'$  are  $xi + yJ + zK$ ,  $x'I + y'J + z'K$ ,

$x''I + y''J + z''K$  respectively. And since  $xx' + yy' + zz' = 0$ , &c., it follows that

$$\frac{SI'SIJ'}{I^4} + \frac{SJJ'SJJ'}{J^4} + \frac{SKI'SKJ'}{K^4} = 0, \text{ \&c.,}$$

showing that the vectors to new centres are in the directions of conjugate diameters of the ellipsoid

$$\frac{S^2Ip}{I^4} + \frac{S^2Jp}{J^4} + \frac{S^2Kp}{K^4} = 1.$$

The origin of vectors is now a fixed point in the body, and the points  $P_i, P_j, P_k$  are, of course, fixed points in the body; and, by the nature of astatic equilibrium, we may consider the body as being placed in any position whatever. Suppose, then, that it is so turned round the origin of vectors that  $I, J$ , and  $K$  coincide in directions with  $i, j$ , and  $k$  respectively. This may be regarded as a sort of *initial position* of the body. Let the tensors of  $I, J$ , and  $K$  be  $t_1, t_2$ , and  $t_3$  respectively. Then  $iSIe_1 + jSJ e_1 + kSK e_1$  becomes  $t_1 iS i e_1 + t_2 jS j e_1 + t_3 kS k e_1$ , that is, a self-conjugate linear vector function of  $e_1$ . Denote it by  $\phi e_1$ . Then (23), (24), and (25) give

$$\Pi_1 = \frac{\phi e_1}{\sqrt{-S e_1 e_1 e_3}}, \quad \Pi_2 = \frac{\phi e_2}{\sqrt{-S e_1 e_2 e_3}}, \quad \Pi_3 = \frac{\phi e_3}{\sqrt{-S e_1 e_2 e_3}} \dots (26).$$

But it is well known that in the ellipsoid  $S\rho\rho\rho = 1$ , the normal at the extremity of a vector  $a$  is parallel to  $\phi a$ .

Hence we have the following theorem:—

*The body being placed in the initial position, the forces applied at the extremities of any three vectors,  $A_1, A_2, A_3$ , drawn from the origin to points in the plane of centres, are in the directions of normals to the ellipsoid  $S\rho\rho\rho = 1$  at the points where its surface is intersected by the vectors  $VA_1A_2, VA_2A_3, VA_3A_1$ .*

If we wish the forces  $\Pi_1, \Pi_2, \Pi_3$  to be a mutually rectangular set, we must take  $S\phi e_1 S\phi e_2 = 0$ , &c., from which it is evident that the directions of  $e_1, e_2$ , and  $e_3$  must be conjugate diameters of the ellipsoid

$$S\rho\phi^3\rho = 1.$$

Of course the points  $P_i, P_j, P_k$  may be taken as those at which the forces are applied. The forces will then be in the directions  $i, j, k$  respectively, and in the initial position they will meet in a point. Their magnitudes are obviously  $a, b, c$ .

A case in which the reduction to three forces fails deserves to be noticed.

Suppose the direction of the vector  $i$  to be chosen so as to coincide with that of the Resultant of Translation. Then  $j$  and  $k$  are perpendicular to this direction, and therefore  $b$  and  $c$ , the sums of the resolved parts of the forces in directions perpendicular to the Resultant of

Translation, are each zero. Hence  $P_j$  and  $P_k$  are at infinity, while  $P_i$  is, of course, the centre of the plane of centres. The vectors  $J$  and  $K$  of course remain, and it is easy to see that the directions of  $j$  and  $k$  may be so chosen that  $J$  and  $K$  are perpendicular to each other. For if a new vector  $j'$  makes an angle  $\theta$  with  $j$ , we have

$$j' = j \cos \theta + k \sin \theta, \quad k' = -j \sin \theta + k \cos \theta;$$

and since  $J' = \Sigma a S j' \omega$ ,  $K' = \Sigma a S k' \omega$ , it is clear that

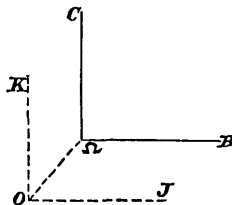
$$J' = J \cos \theta + K \sin \theta,$$

$$K' = -J \sin \theta + K \cos \theta,$$

so that  $J'$  and  $K'$  will be perpendicular if  $\tan 2\theta = \frac{2SJK}{K^2 - J^2}$ .

Let  $\Omega$  in the figure be the centre of the plane of centres;  $OJ$  and  $OK$  the vectors  $J$  and  $K$  drawn through the origin of vectors,  $O$ ; and let  $\Omega B$  and  $\Omega C$  be drawn (in the plane of centres) parallel to  $J$  and  $K$ .

Then, if  $\Omega\Omega$  be the perpendicular to this plane at  $\Omega$ , it is clear that the points  $P_i$ ,  $P_j$ ,  $P_k$  subtend right angles in pairs at  $O$ , the latter two being at infinity on  $\Omega B$  and  $\Omega C$  respectively.



Suppose that the force  $\Pi_1$  is applied at  $\Omega$ , i.e.,  $A_1 = \overline{O\Omega} = -\frac{I}{a}$ . Then,

the body being in its initial position, the axes of the ellipsoids  $S\rho\phi\rho=1$  and  $S\rho\phi^2\rho=1$  are in the directions  $O\Omega$ ,  $OJ$ , and  $OK$ . Hence the vectors  $\epsilon_1$  and  $\epsilon_3$  are in the plane  $JOK$ , and (the applied forces being supposed a rectangular system)  $\epsilon_1$  coincides in direction with  $A_1$ ; therefore the vectors  $A_2$  and  $A_3$  are in the plane  $JOK$ , and their extremities are therefore at infinity. The forces  $\Pi_2$  and  $\Pi_3$  are then applied at infinity, and we can see that their magnitudes are each zero. For, since  $-S\epsilon_1\epsilon_2\epsilon_3 = S^2 A_1 A_2 A_3$ , it is very easy to show in equations (26)

$$\text{that} \quad T\Pi_1 = \frac{1}{TA_1 \cdot \sin p_1} \sqrt{t_1^2 \cos^2 \theta_1 + t_2^2 \cos^2 \phi_1 + t_3^2 \cos^2 \psi_1},$$

where  $p_1$  is the angle between the direction of  $A_1$  and the plane of  $A_2 A_3$ , and  $\theta_1$ ,  $\phi_1$ ,  $\psi_1$  are the angles between  $\epsilon_1$  and the directions of  $I$ ,  $J$ ,  $K$ . Similar values are obtained for  $T\Pi_2$  and  $T\Pi_3$ .

Now, in the present case,  $TA_2 = TA_3 = \infty$ ; therefore  $\Pi_2$  and  $\Pi_3$  are zero forces applied at infinity.

This result, of course, indicates a new mode of reduction—namely, to a force and two couples; and this is the mode of reduction adopted in all cases by Moigno.

Let the centre of the plane of centres be taken as origin of vectors, and suppose a force  $\Pi_2$  applied at the extremity of a vector  $\gamma$ , while a force  $-\Pi_2$  is applied at the extremity of  $\gamma'$ . Then in equa-

tion (19) we shall have the term  $\gamma Si\Pi_2 - \gamma' Si\Pi_2$ , or  $(\gamma - \gamma') Si\Pi_2$ . Denote  $\gamma - \gamma'$  by  $\mu$ ; then  $\mu$  is the vector joining the points at which the forces  $(\Pi_2, -\Pi_2)$  constituting the couple act. Similarly, let  $\mu'$  be the vector joining the points at which the forces  $(\Pi_3, -\Pi_3)$  act. We may, for shortness, call  $\mu$  and  $\mu'$  the *vector arms* of the couples. Then our equations are ( $I$  and  $A_1$  being zero)

$$\begin{aligned}\Pi_1 + \Pi &= 0, \\ \mu Si\Pi_2 + \mu' Si\Pi_3 &= 0, \\ \mu Sj\Pi_2 + \mu' Sj\Pi_3 &= -J, \\ \mu Sk\Pi_2 + \mu' Sk\Pi_3 &= -K.\end{aligned}$$

The second requires  $Si\Pi_2 = 0$ ,  $Si\Pi_3 = 0$ ; i.e., the forces of the couples are in a plane perpendicular to the Resultant of Translation.

Suppose the body placed in the initial position; then  $J = t_2 j$ ,  $K = t_3 k$ , and  $i$  is the unit vector perpendicular to the plane of centres. Hence we have

$$\begin{aligned}\mu Sj\Pi_2 + \mu' Sj\Pi_3 &= -t_2 j, \\ \mu Sk\Pi_2 + \mu' Sk\Pi_3 &= -t_3 k.\end{aligned}$$

Operate on these with  $S \cdot \nabla i\mu'$ , and observe that

$$S\mu \nabla i\mu' = -Si\mu\mu' = -Si\nabla\mu\mu' = TV\mu\mu';$$

also

$$Sj \nabla i\mu' = -Sk\mu';$$

then we have  $Sj\Pi_2 = \frac{t_2 Sk\mu'}{TV\mu\mu'}$ ,  $Sk\Pi_3 = -\frac{t_3 Sj\mu'}{TV\mu\mu'}$ .

Hence  $\Pi_2 = \frac{1}{TV\mu\mu'} (-t_2 j Sk\mu' + t_3 k Sj\mu') \dots\dots\dots(27),$

$$\Pi_3 = \frac{1}{TV\mu\mu'} (-t_2 j Sk\mu + t_3 k Sj\mu) \dots\dots\dots(28).$$

Let  $\Pi_1$  and  $\Pi_3$  be perpendicular to each other. Then

$$\frac{Sj\mu Sj\mu'}{t_2^2} + \frac{Sk\mu Sk\mu'}{t_3^2} = 0,$$

which shows that the directions of  $\mu$  and  $\mu'$  are those of a pair of conjugate diameters of the ellipse

$$Sp \phi p = 1 \dots\dots\dots(29),$$

where  $\phi p \equiv f^2 \left( \frac{j}{t_2^2} Sj p + \frac{k}{t_3^2} Sk p \right)$ , and  $f$  (denoting any constant force magnitude) is introduced for homogeneity.

Assume the arms to be represented, not only in directions, but also in magnitudes, by a pair of semi-conjugate diameters. Then  $TV\mu\mu'$  is constant and equal to  $\frac{t_2 t_3}{f^2}$ , the product of the semiaxes.

Hence, from (27) and (28),

$$\Pi_1 = f^2 \left( -j \frac{Sk\mu'}{t_1} + k \frac{Sj\mu'}{t_1} \right) = f^2 \left( j \frac{Sj\mu}{t_1} + k \frac{Sk\mu}{t_1} \right),$$

$$\Pi_2 = f^2 \left( -j \frac{Sk\mu}{t_2} + k \frac{Sj\mu}{t_2} \right) = f^2 \left( j \frac{Sj\mu'}{t_2} + k \frac{Sk\mu'}{t_2} \right).$$

Now the ellipse (29) can be written  $T\psi\rho = 1$ , where

$$\psi\rho = f \left( j \frac{Sj\rho}{t_1} + k \frac{Sk\rho}{t_2} \right),$$

and  $\psi\rho$  obviously denotes the vector to the corresponding point on the circumscribed circle. Hence we have simply

$$\Pi_1 = f \cdot \psi\mu, \quad \Pi_2 = f \cdot \psi\mu' \dots\dots\dots (30),$$

and we arrive at the following result:—

*The body having been placed so that the plane of centres is perpendicular to the Resultant of Translation, and the vectors  $J$  and  $K$ , fixed in the body, coincide with the corresponding vectors  $j$  and  $k$  fixed in space, the system may be astatically equilibrated by a single force acting at the centre of the plane of centres, equal and opposite to the Resultant of Translation, together with two couples in this plane, the forces of these couples acting in two rectangular directions at the extremities of any pair of semi-conjugate diameters of a certain ellipse, their forces being all equal and of constant magnitude whatever pair of diameters be chosen, and the forces at the extremities of each semi-diameter of the ellipse being parallel to the corresponding semi-diameter of its circumscribed circle.*

The force and two couples which astatically equilibrate the system will in certain positions reduce to a single resultant force, and we propose to show that, when this is the case, the line of action of this force intersects two conics fixed in the body.

Choose the arms of the couples in the directions of the axes of the ellipse just mentioned, and let their forces be applied at the extremities of the semi-axes. Then, the body being in any position, the vectors  $j$  and  $k$  will not coincide with  $J$  and  $K$ , and we must put

$$A_1 = \frac{J}{f}, \quad A_2 = \frac{K}{f}, \quad \Pi_1 = fj, \quad \Pi_2 = fk, \quad \Pi_3 = ai.$$

Hence, if  $G$  is the vector axis of the resultant couple,

$$G = V(Jj + Kk),$$

and, for a single resultant,  $SGG = 0$ , or

$$SJK = SKJ \dots\dots\dots (31).$$

Again, Poinso't's Axis will coincide with the line of action of the single resultant, and its equation is

$$\rho = xi + \frac{1}{a} (jSiJ + kSiK),$$

(by 7,) or it can evidently be written

$$\rho = \lambda i + \frac{1}{a} V (Jk - Kj) \dots \dots \dots (32),$$

where  $\lambda$  is an arbitrary scalar.

If we denote  $VJK$  by  $I$ , and  $TJ$ ,  $TK$ , as before, by  $t_1$  and  $t_2$ , and also if the direction angles of the vectors  $I$ ,  $J$ ,  $K$  with respect to  $i, j, k$  be represented by the following scheme :—

Equation (31) gives

	$i$	$j$	$k$
$I$	$\alpha_1$	$\beta_1$	$\gamma_1$
$J$	$\alpha_2$	$\beta_2$	$\gamma_2$
$K$	$\alpha_3$	$\beta_3$	$\gamma_3$

$$t_2 \cos \gamma_2 = t_3 \cos \beta_3 \dots \dots \dots (33).$$

Also, operating on (32) successively with  $S.I$ ,  $S.J$ , and  $S.K$ , and denoting  $SI\rho$  by  $-t_1 t_3 x$ ,  $SJ\rho$  by  $-t_2 y$ ,  $SK\rho$  by  $-t_3 z$ , we have

$$x = \lambda \cos \alpha_1 + \frac{1}{a} (t_3 \cos \beta_3 + t_2 \cos \gamma_2),$$

$$y = \lambda \cos \alpha_2 - \frac{t_3}{a} \cos \beta_1,$$

$$z = \lambda \cos \alpha_3 - \frac{t_2}{a} \cos \gamma_1.$$

To find the point where Poinso't's Axis meets the plane of  $IJ$ , put  $z=0$ ; therefore  $\lambda = \frac{t_2 \cos \gamma_1}{a \cos \alpha_3}$ . Substituting this in the values of  $x$  and  $y$ , and remembering that  $I, J, K$  and  $i, j, k$  are two rectangular systems, we shall find, with the aid of (33),

$$\frac{x^2}{t_3^2} + \frac{y^2}{t_3^2 - t_2^2} = \frac{1}{a^2}.$$

Similarly, for the point in which Poinso't's Axis intersects the plane  $KI$ ,

$$\frac{x^2}{t_2^2} + \frac{z^2}{t_2^2 - t_3^2} = \frac{1}{a^2}.$$

Now these are the focal conics of the quadric

$$\frac{x^2}{t_2^2 + t_3^2} + \frac{y^2}{t_3^2} + \frac{z^2}{t_2^2} = \frac{1}{a^2}.$$



Hence they are both intersected by the line of action of the single resultant to which the system of forces reduces in certain positions of the body.

This is Minding's Theorem.

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*On certain Extensions of Frullani's Theorem.*

By C. LEUDES DORF, M.A.

[Read April 11th, 1878.]

The following investigation is intended to be supplementary to two papers by Mr. Elliott, on "Certain Multiple Definite Integrals," published in No. 106 and No. 113 of the *Proceedings*. Starting with an amended form of the theorem generally called Frullani's, the author deduces certain elegant extensions of this theorem to the cases of integrals of the 2nd, 3rd, ... 6th order; and in the second paper (§ 5) he writes down (though without proof) the theorem for the general case of the  $n$ -tuple integral to which he is led by his results in the cases investigated. These results are shewn to be true when certain conditions hold good among the involved constants; but these conditions are unsymmetrical in form, and their number in any given case seems subject to no law. For the integrals of order 1, 2, ... 6, the number of constants involved is 2, 4, ... 12, while the number of conditions found is 0, 1, 2, 4, 5, 7. There is thus no means of determining, for the general case, how many conditions must hold among the  $2n$  constants, nor yet what these conditions are; so that there is no guarantee that it shall be possible to satisfy the conditions with the disposable constants. It is however difficult, on reading Mr. Elliott's papers, to avoid concluding that he is correct in asserting that the general theorem *does* hold, and the following proof of it may perhaps be of interest, as the subject is considered from rather a different point of view from that taken in the two papers mentioned; while the conditions found are symmetrical in form and less than the number of disposable constants, being  $n-1$  in number when there are  $2n$  constants.

As in the second paper, let  $S(p, q, r \dots)$  denote any symmetric function of  $p, q, r \dots$  which does not become infinite for any positive values of  $p, q, r \dots$  from zero to infinity, both inclusive. Denote

$$\int_0^\infty \frac{\phi(ax)}{x} dx \text{ by } [a], \quad \int_0^\infty \int_0^\infty S(ax, by) \frac{dx dy}{xy} \text{ by } [ab],$$

$$\int_0^\infty \int_0^\infty \int_0^\infty S(ax, by, cz) \frac{dx dy dz}{xyz} \text{ by } [abc], \text{ \&c.}$$