

On Elliptic Function Formulæ connected with the Transformation of Rectangular Coordinates. By Rev. M. M. U. WILKINSON.

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1. We define the nine quantities

$$\beta, \varpi, \phi, \gamma, \kappa, \chi, \delta, \tau, \theta$$

by the following equations :

$$\begin{aligned} \beta^2 &= k^2 \operatorname{cn}(q-t) \operatorname{dn}(q-t) & \div \\ \varpi^2 &= k^2 \operatorname{cn}^2 p \operatorname{dn}(q-t) \operatorname{cn}(t-p) \operatorname{cn}(p-q) & \div \\ \phi^2 &= -\operatorname{dn}^2 p \operatorname{cn}(q-t) \operatorname{dn}(t-p) \operatorname{dn}(p-q) & \div \dots (1). \\ \text{denominator} &= k^2 k^2 \operatorname{sn}(t-p) \operatorname{sn}(p-q) \{ \operatorname{sn}^2(q-t) - \operatorname{sn}^2 p \} \\ \gamma^2 &= k^2 \operatorname{cn}(t-p) \operatorname{dn}(t-p) & \div \\ \kappa^2 &= k^2 \operatorname{cn}^2 q \operatorname{dn}(t-p) \operatorname{cn}(p-q) \operatorname{cn}(q-t) & \div \\ \chi^2 &= -\operatorname{dn}^2 q \operatorname{cn}(t-p) \operatorname{dn}(p-q) \operatorname{dn}(q-t) & \div \dots (2). \\ \text{denominator} &= k^2 k^2 \operatorname{sn}(p-q) \operatorname{sn}(q-t) \{ \operatorname{sn}^2(t-p) - \operatorname{sn}^2 q \} \\ \delta^2 &= k^2 \operatorname{cn}(p-q) \operatorname{dn}(p-q) & \div \\ \tau^2 &= k^2 \operatorname{cn}^2 t \operatorname{dn}(p-q) \operatorname{cn}(q-t) \operatorname{cn}(t-p) & \div \\ \theta^2 &= -\operatorname{dn}^2 t \operatorname{cn}(p-q) \operatorname{dn}(q-t) \operatorname{dn}(t-p) & \div \dots (3). \\ \text{denominator} &= k^2 k^2 \operatorname{sn}(q-t) \operatorname{sn}(t-p) \{ \operatorname{sn}^2(p-q) - \operatorname{sn}^2 t \} \end{aligned}$$

When writing at full length, we shall write $\beta(p, q, t, k)$ for β , and so for the others. Thus we have

$$\left. \begin{aligned} \gamma(p, q, t, k) &= \beta(q, t, p, k) \\ \delta(p, q, t, k) &= \beta(t, p, q, k) \\ \kappa(p, q, t, k) &= \varpi(q, t, p, k) \\ \tau(p, q, t, k) &= \varpi(t, p, q, k) \\ \chi(p, q, t, k) &= \phi(q, t, p, k) \\ \theta(p, q, t, k) &= \phi(t, p, q, k) \end{aligned} \right\} \dots \dots \dots (4).$$

The cyclical character of these formulæ is obvious. It will be convenient to express them all (as γ, δ are here done) as β functions. We shall show that this can be done. We may call β, ϖ, ϕ, p labials, γ, κ, χ, q gutturals, and δ, τ, θ, t dentals; also we may call β, γ, δ smooths, ϖ, κ, τ roughs, and ϕ, χ, θ aspirates. And by arguments we mean p, q, t ; by functions, β, γ , &c.

We may notice that β , &c. can be expressed factorially by means of Θ functions.

2. Since

$$\begin{aligned} k^2 \operatorname{cn}(q-t) \operatorname{dn}(q-t) &= (\operatorname{dn}^2 p - k^2 \operatorname{cn}^2 p) \operatorname{cn}(q-t) \operatorname{dn}(q-t), \\ k^2 \operatorname{cn}^2 p \operatorname{dn}(q-t) \operatorname{cn}(t-p) \operatorname{cn}(p-q) &= k^2 \operatorname{cn}^2 p \operatorname{cn}(q-t) \operatorname{dn}(q-t) \\ &\quad + k^2 \operatorname{sn}(t-p) \operatorname{sn}(p-q) \operatorname{cn}^2 p \operatorname{dn}^2(q-t) \\ -\operatorname{dn}^2 p \operatorname{cn}(q-t) \operatorname{dn}(t-p) \operatorname{dn}(p-q) &= -\operatorname{dn}^2 p \operatorname{cn}(q-t) \operatorname{dn}(q-t) \\ &\quad - k^2 \operatorname{sn}(t-p) \operatorname{sn}(p-q) \operatorname{dn}^2 p \operatorname{cn}^2(q-t), \\ \operatorname{cn}^2 p \operatorname{dn}^2(q-t) - \operatorname{dn}^2 p \operatorname{cn}^2(q-t) &= k^2 \{ \operatorname{sn}^2(q-t) - \operatorname{sn}^2 p \}, \end{aligned}$$

we have,

$$\left. \begin{aligned} \beta^2 + \varpi^2 + \phi^2 &= 1 \\ \gamma^2 + \kappa^2 + \chi^2 &= 1 \\ \delta^2 + r^2 + \theta^2 &= 1 \end{aligned} \right\} \dots\dots\dots (5).$$

3. Our definitions are as yet ambiguous as to the signs of β , &c.

But we have

$$\frac{\varpi^2 \kappa^2}{\beta^2 \gamma^2} = \frac{k^4 \operatorname{cn}^2 p \operatorname{cn}^2 q \operatorname{cn}^2 (p-q)}{k'^4},$$

$$\frac{\phi^2 \chi^2}{\beta^2 \gamma^2} = \frac{\operatorname{dn}^2 p \operatorname{dn}^2 q \operatorname{dn}^2 (p-q)}{k'^4}.$$

We shall always take

$$\left. \begin{aligned} \frac{\varpi \kappa}{\beta \gamma} &= \frac{k^2 \operatorname{cn} p \operatorname{cn} q \operatorname{cn} (p-q)}{k'^2} \\ \frac{\phi \chi}{\beta \gamma} &= - \frac{\operatorname{dn} p \operatorname{dn} q \operatorname{dn} (p-q)}{k'^2} \end{aligned} \right\} \dots\dots\dots (6).$$

And similarly for the others. Thus we have

$$\left. \begin{aligned} \beta \gamma + \varpi \kappa + \phi \chi &= 0 \\ \gamma \delta + \kappa r + \chi \theta &= 0 \\ \delta \beta + r \varpi + \theta \phi &= 0 \end{aligned} \right\} \dots\dots\dots (7).$$

Equations (5) and (6) show that our elliptic function formulæ express the nine quantities which are used in the Transformation of Rectangular Coordinates.

4. A system of β functions will be called positive or negative according as $+1$ or -1 is the value of the determinant

$$\begin{vmatrix} \beta, & \gamma, & \delta \\ \varpi, & \kappa, & r \\ \phi, & \chi, & \theta \end{vmatrix}.$$

When nothing is said to the contrary, it will be assumed that the system is a positive system.

5. The β functions lead to several interesting formulæ. We will give some of the simplest.

$$\left. \begin{aligned} \frac{\varpi \chi}{\phi \kappa} &= \frac{\operatorname{cn} p \operatorname{dn} q \operatorname{cn} (t-p) \operatorname{dn} (q-t)}{\operatorname{dn} p \operatorname{cn} q \operatorname{cn} (q-t) \operatorname{dn} (t-p)} \\ \frac{\phi \gamma}{\beta \chi} &= \frac{\operatorname{dn} p \operatorname{dn} (t-p)}{\operatorname{dn} q \operatorname{dn} (q-t)} \\ \frac{\beta \kappa}{\varpi \gamma} &= \frac{\operatorname{cn} q \operatorname{cn} (q-t)}{\operatorname{cn} p \operatorname{cn} (t-p)} \end{aligned} \right\} \dots\dots\dots (8).$$

Since we have

$$\begin{aligned} \operatorname{dn} q \operatorname{cn} t \operatorname{cn} (t-p) \operatorname{dn} (p-q) - \operatorname{cn} q \operatorname{dn} t \operatorname{cn} (p-q) \operatorname{dn} (t-p) \\ = k^2 \{ \operatorname{sn} t \operatorname{sn} (t-p) + \operatorname{sn} q \operatorname{sn} (p-q) \}, \end{aligned}$$

$$\begin{aligned} \operatorname{cn} t \operatorname{cn} (t-p) - \operatorname{cn} q \operatorname{cn} (p-q) &= -\operatorname{dn} p \{ \operatorname{sn} t \operatorname{sn} (t-p) + \operatorname{sn} q \operatorname{sn} (p-q) \}, \\ \operatorname{dn} t \operatorname{dn} (t-p) - \operatorname{dn} q \operatorname{dn} (p-q) &= -k^2 \operatorname{cn} p \{ \operatorname{sn} t \operatorname{sn} (t-p) + \operatorname{sn} q \operatorname{sn} (p-q) \}, \end{aligned}$$

we may write

$$\left. \begin{aligned} \frac{\beta}{\chi r} &= \frac{k^2 \{ \operatorname{sn} q \operatorname{sn} (p-q) + \operatorname{sn} t \operatorname{sn} (t-p) \}}{\operatorname{dn} q \operatorname{cn} t \operatorname{dn} (p-q) \operatorname{cn} (t-p)} \\ \frac{\varpi}{\gamma \theta} &= \frac{k^2 \operatorname{cn} p \{ \operatorname{sn} q \operatorname{sn} (p-q) + \operatorname{sn} t \operatorname{sn} (t-p) \}}{\operatorname{dn} t \operatorname{dn} (t-p)} \\ \frac{\phi}{\kappa \delta} &= \frac{-\operatorname{dn} p \{ \operatorname{sn} q \operatorname{sn} (p-q) + \operatorname{sn} t \operatorname{sn} (t-p) \}}{\operatorname{cn} q \operatorname{cn} (p-q)} \end{aligned} \right\} \dots\dots\dots (9).$$

Let

$$\begin{aligned} Q &= \{ \operatorname{sn} q \operatorname{sn} (p-q) + \operatorname{sn} t \operatorname{sn} (t-p) \} \{ \operatorname{sn} t \operatorname{sn} (q-t) + \operatorname{sn} p \operatorname{sn} (p-q) \} \\ &\quad \times \{ \operatorname{sn} p \operatorname{sn} (t-p) + \operatorname{sn} q \operatorname{sn} (q-t) \} \dots\dots\dots (10). \end{aligned}$$

Then shall

$$\left. \begin{aligned} \beta \chi r &= -\operatorname{cn} t \operatorname{dn} q \operatorname{cn} (t-p) \operatorname{cn} (q-t) \operatorname{dn} (p-q) \operatorname{dn} (q-t) \div \\ \beta \kappa \theta &= -\operatorname{cn} q \operatorname{dn} t \operatorname{cn} (p-q) \operatorname{cn} (q-t) \operatorname{dn} (t-p) \operatorname{dn} (q-t) \div \\ \text{denominator} &= k^2 k'^2 Q \end{aligned} \right\} \dots\dots\dots (11).$$

$$\begin{aligned} Q^2 &= \{ \operatorname{sn}^2 p - \operatorname{sn}^2 (q-t) \} \{ \operatorname{sn}^2 q - \operatorname{sn}^2 (t-p) \} \{ \operatorname{sn}^2 t - \operatorname{sn}^2 (p-q) \} \\ &\quad \times \operatorname{sn}^2 (q-t) \operatorname{sn}^2 (t-p) \operatorname{sn}^2 (p-q) \dots\dots\dots (12). \end{aligned}$$

It is obvious that three more formulæ, of almost equal interest, can be obtained by increasing p, q, t , in Q, Q^2 , by K, iK' , and $K+iK'$.

6. In equations (4) we express, by a cyclical change of the arguments, guttural and dental functions by labial functions. We shall now show how to express rough and aspirate functions by smooth functions. We shall thus be able to express any of the nine functions in a set by a β function.

The change of k into $\frac{1}{k'}$, $\frac{1}{k'}$ into $-\frac{ik'}{k}$, and $-\frac{ik'}{k}$ into k , is a cyclical change. In order to change the modulus from k into $\frac{1}{k'}$ or from k into $-\frac{ik'}{k}$, we require the formulæ

$$\left. \begin{aligned} \operatorname{sn} (u, k) &= -i \operatorname{sn} \left(ik' u, \frac{1}{k'} \right) \div \\ \operatorname{cn} (u, k) &= k' \div \\ \operatorname{dn} (u, k) &= k' \operatorname{cn} \left(ik' u, \frac{1}{k'} \right) \div \\ \text{where denominator} &= k' \operatorname{dn} \left(ik' u, \frac{1}{k'} \right) \end{aligned} \right\} \dots\dots\dots (13),$$

$$\left. \begin{aligned} \operatorname{sn}(u, k) &= -i \operatorname{sn}\left(iku, \frac{-ik'}{k}\right) \div \\ \operatorname{cn}(u, k) &= k \operatorname{dn}\left(iku, -\frac{ik'}{k}\right) \div \\ \operatorname{dn}(u, k) &= k \div \\ \text{where denominator} &= k \operatorname{cn}\left(iku, -\frac{ik'}{k}\right) \end{aligned} \right\} \dots\dots\dots(14).$$

When the modulus is sufficiently indicated by the coefficient of the argument, we shall not think it necessary always to express it.

7. In the two expressions,

$$\beta(p+K, q+K, t+K, k),$$

$$\beta\left(p+K, q+K, t+K, \frac{1}{k'}\right),$$

K has a different meaning, meaning $K(k)$ in the first, and $K\left(\frac{1}{k'}\right)$ in the second.

The equations which we shall want for transforming K, K' from one modulus to any of the other five moduli connected with it, are, to a sign *près*,

$$\left. \begin{aligned} K(k) &= K'(k') = \frac{1}{k'} K'\left(\frac{1}{k'}\right) = \frac{1}{k'} K\left(\frac{ik}{k'}\right) \\ &= \frac{1}{k} K\left(\frac{1}{k}\right) + \frac{i}{k} K'\left(\frac{1}{k}\right) = \frac{1}{k} K'\left(\frac{ik}{k}\right) + \frac{i}{k} K\left(\frac{ik}{k}\right) \\ K'(k) &= K(k') = \frac{1}{k} K'\left(\frac{1}{k}\right) = \frac{1}{k} K\left(\frac{ik'}{k}\right) \\ &= \frac{1}{k'} K\left(\frac{1}{k'}\right) + \frac{i}{k'} K'\left(\frac{1}{k'}\right) = \frac{1}{k'} K'\left(\frac{ik}{k'}\right) + \frac{i}{k'} K\left(\frac{ik}{k'}\right) \\ K\left(\frac{1}{k'}\right) &= K'\left(\frac{ik}{k'}\right) = k' K(k') + ik' K'(k') = \frac{ik'}{k} K'\left(\frac{ik'}{k}\right) \\ &= \frac{ik'}{k} K\left(\frac{1}{k}\right) = k' K'(k) + ik' K(k) \\ K'\left(\frac{1}{k'}\right) &= k' K(k) = k' K'(k') = \frac{k'}{k} K\left(\frac{1}{k}\right) + \frac{ik'}{k} K'\left(\frac{1}{k}\right) \\ &= K\left(\frac{ik}{k'}\right) = \frac{k'}{k} K\left(\frac{ik'}{k}\right) + \frac{ik'}{k} K\left(\frac{ik'}{k}\right) \end{aligned} \right\} \dots(15),$$

and so on: consistency of sign in these equations is not, at present, of any importance.

8. We have then, the modulus in every case being sufficiently indicated by the coefficient of the argument,

$$k^3 k'^3 \operatorname{sn}(t-p) \operatorname{sn}(p-q) \{\operatorname{sn}^2(q-t) - \operatorname{sn}^2 p\}$$

$$= \frac{k^3 \operatorname{sn} ik'(t-p) \operatorname{sn} ik'(p-q) \{\operatorname{sn}^3 ik'(q-t) - \operatorname{sn}^3 ik'p\}}{k'^3 \operatorname{dn} ik'(t-p) \operatorname{dn} ik'(p-q) \operatorname{dn}^2 ik'p \operatorname{dn}^2 ik'(q-t)};$$

$$k^3 \operatorname{cn}^2 p \operatorname{dn} (q-t) \operatorname{cn} (t-p) \operatorname{cn} (p-q) \\ = \frac{k^3}{\operatorname{dn}^3 ik'p} \cdot \frac{\operatorname{cn} ik' (q-t)}{\operatorname{dn} ik' (q-t) \operatorname{dn} ik' (t-p) \operatorname{dn} ik' (p-q)};$$

therefore

$$\varpi^2 (p, q, t, k) = \beta^3 \left(ik'p, ik'q, ik't, \frac{1}{k'} \right) \dots \dots \dots (16).$$

Again,

$$k^3 k'^3 \operatorname{sn} (t-p) \operatorname{sn} (p-q) \{ \operatorname{sn}^2 (q-t) - \operatorname{sn}^2 p \} \\ = \frac{k^3 \operatorname{sn} ik (t-p) \operatorname{sn} ik (p-q) \{ \operatorname{sn}^2 ik (q-t) - \operatorname{sn}^2 ikp \}}{k^3 \operatorname{cn}^2 ikp \operatorname{cn}^2 ik (q-t) \operatorname{cn} ik (t-p) \operatorname{cn} ik (p-q)} \\ - \operatorname{dn}^3 p \operatorname{cn} (q-t) \operatorname{dn} (t-p) \operatorname{dn} (p-q) \\ = - \frac{1}{\operatorname{cn}^2 ikp} \cdot \frac{\operatorname{dn} ik (q-t)}{\operatorname{cn} ik (q-t) \operatorname{cn} ik (t-p) \operatorname{cn} ik (p-q)};$$

$$\text{therefore} \quad \phi^3 (p, q, t, k) = \beta^3 \left(ikp, ikq, ikt, \frac{ik'}{k} \right) \dots \dots \dots (17).$$

9. As we have

$$\beta^3 = -k^3 \left(1 - \frac{1}{k^2} \right) \operatorname{dn} \left(kq - kt, \frac{1}{k} \right) \operatorname{cn} \left(kq - kt, \frac{1}{k} \right) \div \\ \varpi^3 = k^3 \operatorname{dn}^2 \left(kp, \frac{1}{k} \right) \operatorname{cn} \left(kq - kt, \frac{1}{k} \right) \operatorname{dn} \left(kt - kp, \frac{1}{k} \right) \operatorname{dn} \left(kp - kq, \frac{1}{k} \right) \div \\ \phi^3 = -\operatorname{cn}^2 \left(kp, \frac{1}{k} \right) \operatorname{dn} \left(kq - kt, \frac{1}{k} \right) \operatorname{cn} \left(kt - kp, \frac{1}{k} \right) \operatorname{dn} \left(kp - kq, \frac{1}{k} \right) \div \\ \text{denominator} = \frac{1-k^3}{k^2} \operatorname{sn} \left(kt - kp, \frac{1}{k} \right) \operatorname{sn} \left(kp - kq, \frac{1}{k} \right) \\ \times \left\{ \operatorname{sn}^2 \left(kq - kt, \frac{1}{k} \right) - \operatorname{sn}^2 \left(kp, \frac{1}{k} \right) \right\}.$$

These equations show that

$$\left. \begin{aligned} \beta^3 (p, q, t, k) &= \beta^3 \left(kp, kq, kt, \frac{1}{k} \right) \\ \varpi^3 (p, q, t, k) &= \varpi^3 \left(kp, kq, kt, \frac{1}{k} \right) \\ \phi^3 (p, q, t, k) &= \phi^3 \left(kp, kq, kt, \frac{1}{k} \right) \end{aligned} \right\} \dots \dots \dots (18).$$

10. The sets of formulæ obtained by increasing p, q, t each by K , iK' , $K+iK'$, are as follows. We may distinguish them by writing $\beta (K)$, $\beta (iK')$, $\beta (K+iK')$, &c.

$$\left. \begin{aligned} \beta^3 (K) &= -\operatorname{dn}^2 p \operatorname{cn} (q-t) \operatorname{dn} (q-t) \div \\ \varpi^3 (K) &= -k^3 \operatorname{sn}^2 p \operatorname{dn} (q-t) \operatorname{cn} (t-p) \operatorname{cn} (p-q) \div \\ \phi^3 (K) &= \operatorname{cn} (q-t) \operatorname{dn} (t-p) \operatorname{dn} (p-q) \div \\ \text{denominator} &= k^2 \operatorname{sn} (t-p) \operatorname{sn} (p-q) \{ \operatorname{cn}^2 p \operatorname{cn}^2 (q-t) \\ &\quad - k'^2 \operatorname{sn}^2 p \operatorname{sn}^2 (q-t) \} \end{aligned} \right\} \dots \dots \dots (19),$$

&c.

$$\left. \begin{aligned} \beta^3 (iK') &= -k^2 \operatorname{sn}^2 p \operatorname{cn} (q-t) \operatorname{dn} (q-t) \\ \varpi^3 (iK') &= \operatorname{dn}^2 p \operatorname{dn} (q-t) \operatorname{cn} (t-p) \operatorname{cn} (p-q) \\ \phi^3 (iK') &= -\operatorname{cn}^2 p \operatorname{cn} (q-t) \operatorname{dn} (t-p) \operatorname{dn} (p-q) \\ \text{denominator} &= k^2 \operatorname{sn} (t-p) \operatorname{sn} (p-q) \{1 - k^2 \operatorname{sn}^2 p \operatorname{sn}^2 (q-t)\} \\ &\&c. \end{aligned} \right\} \dots (20),$$

$$\left. \begin{aligned} \beta^3 (K+iK') &= -\operatorname{cn}^2 p \operatorname{cn} (q-t) \operatorname{dn} (q-t) \\ \varpi^3 (K+iK') &= \operatorname{dn} (q-t) \operatorname{cn} (t-p) \operatorname{cn} (p-q) \\ \phi^3 (K+iK') &= -\operatorname{sn}^2 p \operatorname{cn} (q-t) \operatorname{dn} (t-p) \operatorname{dn} (p-q) \\ \text{denominator} &= \operatorname{sn} (t-p) \operatorname{sn} (p-q) \{k^2 + k^2 \operatorname{cn}^2 p \operatorname{cn}^2 (q-t)\} \\ &\&c. \end{aligned} \right\} \dots (21).$$

11. Since there are four quantities p, q, t, k , they cannot be expressed in terms of the nine quantities $\beta, \&c.$ which are connected by the six equations (5) and (7). The question which naturally presents itself is to express them in terms of $\beta, \&c.$ and an arbitrary parameter. We will consider two cases: one, in which k is given; the other, in which another set of functions, $\beta', \varpi', \&c.$ are introduced, and defined by the equations

$$\left. \begin{aligned} \beta' (p, q, t, k) &= \beta (p+r, q+r, t+r, k) \\ \varpi' (p, q, t, k) &= \varpi (p+r, q+r, t+r, k) \\ &\&c. \end{aligned} \right\} \dots (22).$$

And we shall, for convenience, put

$$\left. \begin{aligned} \operatorname{cn} (p-q) \operatorname{cn} (q-t) \operatorname{cn} (t-p) &= m^2 \\ \operatorname{dn} (p-q) \operatorname{dn} (q-t) \operatorname{dn} (t-p) &= n^2 \end{aligned} \right\} \dots (23).$$

So that

$$\left. \begin{aligned} n^2 &= k^2 m^2 + k'^2 \\ k^2 (1-m^2) &= 1-n^2 \\ k'^2 (1-m^2) &= n^2 - m^2 \end{aligned} \right\} \dots (24).$$

There will be one relation connecting the $\beta', \&c.$ functions with the $\beta, \&c.$ functions; and p, q, t, r, k can be determined in terms of $\beta, \gamma, \dots \beta', \gamma', \dots$

12. *Formation of the equations for m^2, n^2 , when k is given.*

$$\text{We have } \varpi^2 k^2 - \beta^2 k^2 m^2 = \frac{\operatorname{dn} (q-t) \operatorname{cn} (t-p) \operatorname{cn} (p-q)}{\operatorname{sn} (t-p) \operatorname{sn} (p-q)} \dots (25);$$

$$\beta^2 k^2 m^2 + (1 - \varpi^2) k^2 = - \frac{\operatorname{cn} (q-t) \operatorname{dn} (t-p) \operatorname{dn} (p-q)}{\operatorname{sn} (t-p) \operatorname{sn} (p-q)} \dots (26).$$

Whence the equation

$$\frac{m^2}{k^2 m^2 + k'^2} = \frac{(\varpi^2 k^2 - \beta^2 k^2 m^2)(k^2 k'^2 - \gamma^2 k^2 m^2)(\tau^2 k^2 - \delta^2 k^2 m^2)}{(\varpi^2 k^2 - \beta^2 k^2 m^2 - k'^2)(k^2 k'^2 - \gamma^2 k^2 m^2 - k'^2)(\tau^2 k^2 - \delta^2 k^2 m^2 - k'^2)} \dots (27),$$

an equation which may be written in the somewhat simpler form :

$$\frac{m^2}{1-m^2} = - \frac{(\varpi^2 k^2 - \beta^2 k^2 m^2)(\kappa^2 k^2 - \gamma^2 k^2 m^2)(\tau^2 k^2 - \delta^2 k^2 m^2)}{\left\{ (\varpi^2 \kappa^2 + \kappa^2 \tau^2 + \tau^2 \varpi^2) k'^4 + (\beta^2 \varpi^2 + \gamma^2 \kappa^2 + \delta^2 \tau^2) k^2 k^2 m^2 \right.} \dots (28)$$

$$\left. + (\beta^2 \gamma^2 + \gamma^2 \delta^2 + \delta^2 \beta^2) k^4 m^4 \right\}$$

And, in like manner, we have,

$$\frac{k^2 n^2}{n^2 - k^2} = \frac{(\phi^2 k^2 + \beta^2 n^2)(\chi^2 k^2 + \gamma^2 n^2)(\theta^2 k^2 + \delta^2 n^2)}{(\phi^2 k^2 + \beta^2 n^2 - k^2)(\chi^2 k^2 + \gamma^2 n^2 - k^2)(\theta^2 k^2 + \delta^2 n^2 - k^2)} \dots (29),$$

which we may write,

$$\frac{n^2}{1-n^2} = \frac{(\phi^2 k^2 + \beta^2 n^2)(\chi^2 k^2 + \gamma^2 n^2)(\theta^2 k^2 + \delta^2 n^2)}{\left\{ (\phi^2 \chi^2 + \chi^2 \theta^2 + \theta^2 \phi^2) k'^4 - (\beta^2 \phi^2 + \gamma^2 \chi^2 + \delta^2 \theta^2) k^2 n^2 \right.} \dots (30).$$

$$\left. + (\beta^2 \gamma^2 + \gamma^2 \delta^2 + \delta^2 \beta^2) n^4 \right\}$$

13. To express p, q, t, r, k in terms of the β and β' functions, and to obtain the relation connecting these functions.

We may obviously take the signs of m and n , so that

$$\left. \begin{aligned} \frac{m}{\operatorname{cn}(q-t)} &= \frac{k'}{k \operatorname{cn} p} \frac{\varpi}{\beta} = \frac{k'}{k \operatorname{cn}(p+r)} \frac{\varpi'}{\beta'} \\ \frac{n}{\operatorname{dn}(q-t)} &= \frac{ik'}{\operatorname{dn} p} \frac{\phi}{\beta} = \frac{ik'}{\operatorname{dn}(p+r)} \frac{\phi'}{\beta'} \end{aligned} \right\} \dots (31).$$

Equation (25) gives

$$\left. \begin{aligned} \frac{k^2 m^2}{k^2} &= \frac{\varpi^2 - \varpi'^2}{\beta'^2 - \beta^2} = \frac{\kappa^2 - \kappa'^2}{\gamma'^2 - \gamma^2} = \frac{\tau^2 - \tau'^2}{\delta'^2 - \delta^2} \\ \text{or} \quad - \frac{n^2}{k^2} &= \frac{\phi^2 - \phi'^2}{\beta'^2 - \beta^2} = \frac{\chi^2 - \chi'^2}{\gamma'^2 - \gamma^2} = \frac{\theta^2 - \theta'^2}{\delta'^2 - \delta^2} \end{aligned} \right\} \dots (32);$$

expressing the one relation connecting the two sets of functions.

From equation (25) we find

$$\begin{aligned} \varpi^2 k^2 + (1 - \beta^2) k^2 m^2 &= \frac{\operatorname{cn}(t-p) \operatorname{cn}(p-q) \operatorname{dn}(t-p) \operatorname{dn}(p-q)}{\operatorname{sn}(t-p) \operatorname{sn}(p-q)}, \\ \frac{\operatorname{cn}(t-p) \operatorname{cn}(p-q)}{\operatorname{cn}(q-t)} &= - \frac{\varpi^2 k^2 + (1 - \beta^2) k^2 m^2}{\phi^2 k^2 + \beta^2 n^2} \\ &= - \frac{\varpi^2 (\beta'^2 - \beta^2) + (1 - \beta^2) (\varpi'^2 - \varpi^2)}{\phi^2 (\beta'^2 - \beta^2) - \beta'^2 (\phi'^2 - \phi^2)} \\ &= \frac{\varpi^2 \phi'^2 - \phi^2 \varpi'^2}{\phi^2 \beta'^2 - \beta^2 \phi'^2} \dots (33); \end{aligned}$$

and, in like manner,

$$\begin{aligned} \frac{\operatorname{dn}(t-p) \operatorname{dn}(p-q)}{\operatorname{dn}(q-t)} &= \frac{\varpi^2 (\beta'^2 - \beta^2) + (1 - \beta^2) (\varpi'^2 - \varpi^2)}{\varpi^2 (\beta'^2 - \beta^2) - \beta'^2 (\varpi'^2 - \varpi^2)} \\ &= \frac{\phi^2 \varpi'^2 - \varpi^2 \phi'^2}{\varpi^2 \beta'^2 - \beta^2 \varpi'^2} \dots (34); \end{aligned}$$

and so we obtain

$$\operatorname{cn}^2(q-t) = \frac{(\kappa^2 \chi'^2 - \chi^2 \kappa'^2)(\tau^2 \theta'^2 - \theta^2 \tau'^2)}{(\chi^2 \gamma'^2 - \gamma^2 \chi'^2)(\theta^2 \delta'^2 - \delta^2 \theta'^2)} \dots\dots\dots (35),$$

$$\operatorname{dn}^2(q-t) = \frac{(\kappa^2 \chi'^2 - \chi^2 \kappa'^2)(\tau^2 \theta'^2 - \theta^2 \tau'^2)}{(\kappa^2 \gamma'^2 - \gamma^2 \kappa'^2)(\tau^2 \delta'^2 - \delta^2 \tau'^2)} \dots\dots\dots (36);$$

$$\operatorname{cn}^2 p = \frac{\varpi^2 (\beta'^2 - \beta^2)(\kappa^2 \chi'^2 - \chi^2 \kappa'^2)(\tau^2 \theta'^2 - \theta^2 \tau'^2)}{\beta^2 (\varpi'^2 - \varpi^2)(\chi^2 \gamma'^2 - \gamma^2 \chi'^2)(\theta^2 \delta'^2 - \delta^2 \theta'^2)} \dots\dots\dots (37),$$

$$\operatorname{dn}^2 p = \frac{\phi^2 (\beta'^2 - \beta^2)(\kappa^2 \chi'^2 - \chi^2 \kappa'^2)(\tau^2 \theta'^2 - \theta^2 \tau'^2)}{\beta^2 (\phi'^2 - \phi^2)(\kappa^2 \gamma'^2 - \gamma^2 \kappa'^2)(\tau^2 \delta'^2 - \delta^2 \tau'^2)} \dots\dots\dots (38);$$

other formulæ are

$$\frac{k^2 \operatorname{sn}(t-p) \operatorname{sn}(p-q) \operatorname{cn}(q-t)}{\operatorname{dn}(q-t)} = \frac{\varpi'^2 - \varpi^2}{\varpi^2 \beta'^2 - \beta^2 \varpi'^2} \dots\dots\dots (39),$$

$$\frac{\operatorname{sn}(t-p) \operatorname{sn}(p-q) \operatorname{dn}(q-t)}{\operatorname{cn}(q-t)} = \frac{\phi'^2 - \phi^2}{\phi^2 \beta'^2 - \beta^2 \phi'^2} \dots\dots\dots (40),$$

$$k^2 = \frac{(\varpi'^2 - \varpi^2)(\phi^2 \beta'^2 - \beta^2 \phi'^2)(\chi^2 \gamma'^2 - \gamma^2 \chi'^2)(\theta^2 \delta'^2 - \delta^2 \theta'^2)}{(\phi'^2 - \phi^2)(\varpi^2 \beta'^2 - \beta^2 \varpi'^2)(\kappa^2 \gamma'^2 - \gamma^2 \kappa'^2)(\tau^2 \delta'^2 - \delta^2 \tau'^2)} \dots\dots\dots (41),$$

$$m^2 = \frac{(\varpi^3 \phi'^2 - \phi^2 \varpi'^2)(\kappa^2 \chi'^2 - \chi^2 \kappa'^2)(\tau^2 \theta'^2 - \theta^2 \tau'^2)}{(\phi^2 \beta'^2 - \beta^2 \phi'^2)(\chi^2 \gamma'^2 - \gamma^2 \chi'^2)(\theta^2 \delta'^2 - \delta^2 \theta'^2)} \dots\dots\dots (42),$$

$$n^2 = - \frac{(\varpi^3 \phi'^2 - \phi^2 \varpi'^2)(\kappa^2 \chi'^2 - \chi^2 \kappa'^2)(\tau^2 \theta'^2 - \theta^2 \tau'^2)}{(\varpi^2 \beta'^2 - \beta^2 \varpi'^2)(\kappa^2 \gamma'^2 - \gamma^2 \kappa'^2)(\tau^2 \delta'^2 - \delta^2 \tau'^2)} \dots\dots\dots (43),$$

$$k'^2 = \frac{(\beta'^2 - \beta^2)(\varpi^2 \phi'^2 - \phi^2 \varpi'^2)(\kappa^2 \chi'^2 - \chi^2 \kappa'^2)(\tau^2 \theta'^2 - \theta^2 \tau'^2)}{(\phi'^2 - \phi^2)(\varpi^2 \beta'^2 - \beta^2 \varpi'^2)(\kappa^2 \gamma'^2 - \gamma^2 \kappa'^2)(\tau^2 \delta'^2 - \delta^2 \tau'^2)} \dots\dots\dots (44),$$

$$\frac{k^2 \operatorname{cn}^2(q-t)}{k'^2} = \frac{(\varpi'^2 - \varpi^2)(\phi^2 \beta'^2 - \beta^2 \phi'^2)}{(\beta'^2 - \beta^2)(\varpi^2 \phi'^2 - \phi^2 \varpi'^2)} \dots\dots\dots (45).$$

We have, too,

$$\begin{aligned} \frac{\operatorname{cn}^2(q-t) \operatorname{dn}^2(q-t)}{\operatorname{sn}^2(q-t)} &= \frac{\{\kappa^2 k'^2 + (1-\gamma^2) k^2 m^2\} \{\tau^2 k'^2 + (1-\delta^2) k^2 m^2\}}{\varpi^2 k'^2 + (1-\beta^2) k^2 m^2} \\ &= \frac{k'^2 \{\kappa^2 (\gamma'^2 - \gamma^2) + (1-\gamma^2)(\kappa'^2 - \kappa^2)\} \{\tau^2 (\delta'^2 - \delta^2) + (1-\delta^2)(\tau'^2 - \tau^2)\} (\beta'^2 - \beta^2)}{(\gamma'^2 - \gamma^2)(\delta'^2 - \delta^2) \{\varpi^2 (\beta'^2 - \beta^2) + (1-\beta^2)(\varpi'^2 - \varpi^2)\}} \\ &= \frac{k'^2 (\beta'^2 - \beta^2) (\chi^2 \kappa'^2 - \kappa^2 \chi'^2) (\theta^2 \tau'^2 - \tau^2 \theta'^2)}{(\gamma'^2 - \gamma^2)(\delta'^2 - \delta^2) (\phi^2 \varpi'^2 - \varpi^2 \phi'^2)} \dots\dots\dots (46), \end{aligned}$$

$$\begin{aligned} \operatorname{sn}^3(q-t) &= \frac{(\phi'^2 - \phi^2)(\gamma'^2 - \gamma^2)(\delta'^2 - \delta^2)(\beta^2 \varpi'^2 - \varpi^2 \beta'^2)}{(\beta'^2 - \beta^2)^3 (\chi^2 \gamma'^2 - \gamma^2 \chi'^2) (\theta^2 \delta'^2 - \delta^2 \theta'^2)} \\ &= \frac{(\chi'^2 - \chi^2)(\theta'^2 - \theta^2)(\beta^2 \varpi'^2 - \varpi^2 \beta'^2)}{(\phi'^2 - \phi^2)(\chi^2 \gamma'^2 - \gamma^2 \chi'^2)(\theta^2 \delta'^2 - \delta^2 \theta'^2)} \dots\dots\dots (47), \end{aligned}$$

$$k^2 \operatorname{sn}^2(q-t) = \frac{(\kappa'^2 - \kappa^2)(\tau'^2 - \tau^2)(\beta^2 \phi'^2 - \phi^2 \beta'^2)}{(\varpi'^2 - \varpi^2)(\kappa^2 \gamma'^2 - \gamma^2 \kappa'^2)(\tau^2 \delta'^2 - \delta^2 \tau'^2)} \dots\dots\dots (48).$$

14. Differentiating with respect to r , we have

$$\left. \begin{aligned} \frac{1}{\beta'} \frac{d\beta'}{dr} &= \frac{\operatorname{sn}(p+r) \operatorname{cn}(p+r) \operatorname{dn}(p+r)}{\operatorname{sn}^2(q-t) - \operatorname{sn}^2(p+r)} \\ \frac{1}{\varpi'} \frac{d\varpi'}{dr} &= \frac{\operatorname{sn}(p+r) \operatorname{dn}(p+r) \operatorname{cn}^2(q-t)}{\operatorname{cn}(p+r) \{ \operatorname{sn}^2(q-t) - \operatorname{sn}^2(p+r) \}} \\ \frac{1}{\phi'} \frac{d\phi'}{dr} &= \frac{\operatorname{sn}(p+r) \operatorname{cn}(p+r) \operatorname{dn}^2(q-t)}{\operatorname{dn}(p+r) \{ \operatorname{sn}^2(q-t) - \operatorname{sn}^2(p+r) \}} \end{aligned} \right\} \dots\dots\dots(49).$$

15. Examination of case when $\beta^2 = \kappa^2 = \theta^2 = 1$.

We have, from equations (25), (26),

$$\frac{k^2 m^2}{n^2} = \frac{\operatorname{dn}(q-t) \operatorname{cn}(t-p) \operatorname{cn}(p-q)}{\operatorname{cn}(q-t) \operatorname{dn}(t-p) \operatorname{dn}(p-q)},$$

or

$$k^2 = \frac{\operatorname{dn}^2(q-t)}{\operatorname{cn}^2(q-t)},$$

showing that

$$\operatorname{sn}(q-t) = \infty.$$

In like manner, we have

$$0 = \frac{\operatorname{cn}(t-p) \operatorname{dn}(p-q) \operatorname{dn}(q-t)}{\operatorname{dn}(t-p) \operatorname{cn}(p-q) \operatorname{cn}(q-t)},$$

$$0 = \frac{\operatorname{dn}(p-q) \operatorname{cn}(q-t) \operatorname{cn}(t-p)}{\operatorname{cn}(p-q) \operatorname{dn}(q-t) \operatorname{dn}(t-p)},$$

showing that

$$\operatorname{cn}(t-p) = 0,$$

$$\operatorname{dn}(p-q) = 0.$$

No further relation is required. We must exclude the values of p which satisfy $\operatorname{sn} p = \infty$. So we may take

$$q = p + (2n+1)K + (2m+1)iK',$$

$$t = p + (2n'+1)K + 2m'iK',$$

p having any value not included in the formula

$$2nK + 2n'iK' + iK'.$$

In like manner, if we were to take

$$q = p + (2n+1)K + 2miK',$$

$$t = p + (2n'+1)K + (2m'+1)iK',$$

we should find

$$\beta^2 = r^2 = \chi^2 = 1.$$

16. A special interest attaches to the case when

$$kk' = 1.$$

We will take

$$k = -i\rho, \quad k' = i\rho^2,$$

where

$$\rho = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3},$$

also

$$\sqrt{\rho-1} = \frac{3^{\frac{1}{2}}\rho(1-i)}{2^{\frac{1}{2}}},$$

$$\sqrt{\rho^2-1} = \frac{3^{\frac{1}{2}}\rho^2(1+i)}{2^{\frac{1}{2}}};$$

having the real part positive. Now let

$$y^3 = \frac{\operatorname{sn}^3 \sqrt{\rho-1} u}{\rho-1 + \operatorname{sn}^2 \sqrt{\rho-1} u} \dots\dots\dots (50).$$

Then

$$\left. \begin{aligned} \operatorname{sn}^3 \sqrt{\rho-1} u &= (\rho-1) y^3 & \div \\ \operatorname{cn}^3 \sqrt{\rho-1} u &= 1-\rho y^3 & \div \\ \operatorname{dn}^3 \sqrt{\rho-1} u &= 1-\rho^2 y^3 & \div \\ \text{denominator} &= 1-y^3 & \end{aligned} \right\} \dots\dots\dots (51).$$

We suppose that, initially, $y = u$.

We easily find

$$\begin{aligned} -\sqrt{\rho-1} \operatorname{cn} \cdot \operatorname{sn} \cdot \operatorname{dn} &= \frac{(1-\rho)y}{(1-y^3)^2} \frac{dy}{du}, \\ \frac{du}{dy} &= \frac{1}{\sqrt{1-y^6}} \dots\dots\dots (52). \end{aligned}$$

As y increases from zero, passing through real values, u increases from zero, passing through real values also, until $y=1$. The integral $\int_0^1 \frac{dy}{\sqrt{1-y^6}}$ is evaluated in Legendre's *Exercices*.

Also, y being always real, cn^2 and dn^2 in equations (51) are always imaginary, except when $y=0$. So that, since $\operatorname{cn} 0 = 1$, $\operatorname{dn} 0 = 1$, in extracting the square root, such sign must be given to cn , dn as will make the real part of the square root positive. We may, in constructing the Table given on pages 214 and 215, first define $\frac{1}{3}\rho^2 K$ by

$$\left. \begin{aligned} \operatorname{sn}^2 \frac{1}{3}\rho^2 K &= 2^{\frac{1}{2}}-1 & \div \\ \operatorname{cn}^2 \frac{1}{3}\rho^2 K &= \rho^2-2^{\frac{1}{2}} & \div \\ \operatorname{dn}^2 \frac{1}{3}\rho^2 K &= 2^{\frac{1}{2}}\rho^2-1 & \div \\ \text{denominator} &= \rho^2-1 & \end{aligned} \right\} \dots\dots\dots (53).$$

Then we must have

$$\operatorname{cn} \frac{1}{3}\rho^2 K = \frac{3^{\frac{1}{2}}(1+i)}{2^{\frac{1}{2}}(2^{\frac{1}{2}}+\rho)}, \quad \operatorname{dn} \frac{1}{3}\rho^2 K = \frac{3^{\frac{1}{2}}(1-i)}{2^{\frac{1}{2}}(2^{\frac{1}{2}}+\rho^2)} \dots\dots\dots (54).$$

For these have their real parts positive. It is to be understood that $\frac{1}{3}\rho^2 K$ is the value obtained by $\sqrt{\rho-1} u$, by changing continuously as y changes continuously in increasing from zero to $\frac{2^{\frac{1}{2}}-1}{2^{\frac{1}{2}}}$. To fix the

sign of $\text{sn } \frac{1}{3}\rho^2 K$, we take

$$\text{sn } \frac{1}{3}\rho^2 K = -\frac{3^{\frac{1}{2}}(1-i)\rho}{2^{\frac{1}{2}}(2^{\frac{1}{2}}+1)} \dots\dots\dots(55).$$

We can now construct the Table by means of the addition equations, and the equations (13), (14), which now become

$$\left. \begin{aligned} \text{sn } \rho u &= \frac{\rho \text{sn } u}{\text{dn } u}, & \text{cn } \rho u &= \frac{1}{\text{dn } u}, & \text{dn } \rho u &= \frac{\text{cn } u}{\text{dn } u} \\ \text{sn } \rho^2 u &= \frac{\rho^2 \text{sn } u}{\text{cn } u}, & \text{cn } \rho^2 u &= \frac{\text{dn } u}{\text{sn } u}, & \text{dn } \rho^2 u &= \frac{1}{\text{cn } u} \end{aligned} \right\} \dots\dots(56).$$

The following identities will be useful

$$\left. \begin{aligned} (2^{\frac{1}{2}}-1)(2^{\frac{1}{2}}+1)^2 &= 3 \\ (2^{\frac{1}{2}}-\rho^2)(2^{\frac{1}{2}}\rho+\rho^2)^2 &= 3 \\ (2^{\frac{1}{2}}-\rho)(2^{\frac{1}{2}}\rho^2+\rho)^2 &= 3 \\ \rho-1 &= -3^{\frac{1}{2}}i\rho^2 \\ \rho^2-1 &= 3^{\frac{1}{2}}i\rho \end{aligned} \right\} \dots\dots\dots(57).$$

It will be obvious that K has its usual meaning, and that

$$iK' = \rho^2 K \dots\dots\dots(58).$$

The Table could also be constructed by solving equations given by the triplication formulæ.

17. *Expressions for $\text{sn } (\rho-\rho^2)u$, &c., when $k = -ip$.*

By means of equations (56), and the ordinary addition equations, we can easily find

$$\left. \begin{aligned} \text{sn } (\rho-\rho^2)u &= \text{sn } u (\text{sn}^2 u + \rho - 1) \div \\ \text{cn } (\rho-\rho^2)u &= \text{cn } u (\text{sn}^2 u - \rho^2) \div \\ \text{dn } (\rho-\rho^2)u &= -\text{dn } u (\text{sn}^2 u + \rho^2) \div \\ \text{denominator} &= (\rho^2 - \rho) \text{sn}^2 u - \rho^2 \end{aligned} \right\} \dots\dots\dots(59).$$

$$\left. \begin{aligned} \text{sn } (1-\rho)u &= \rho^2 \text{sn } u (\text{sn}^2 u + \rho - 1) \div \\ \text{cn } (1-\rho)u &= -\text{dn } u (\text{sn}^2 u + \rho^2) \div \\ \text{dn } (1-\rho)u &= (\rho^2 - \rho) \text{sn}^2 u - \rho^2 \div \\ \text{denominator} &= \text{cn } u (\text{sn}^2 u - \rho^2) \end{aligned} \right\} \dots\dots\dots(60).$$

$$\left. \begin{aligned} \text{sn } (\rho^2-1)u &= -\rho \text{sn } u (\text{sn}^2 u + \rho - 1) \div \\ \text{cn } (\rho^2-1)u &= (\rho - \rho^2) \text{sn}^2 u + \rho^2 \div \\ \text{dn } (\rho^2-1)u &= -\text{cn } u (\text{sn}^2 u - \rho^2) \div \\ \text{denominator} &= \text{dn } u (\text{sn}^2 u + \rho^2) \end{aligned} \right\} \dots\dots\dots(61).$$

18. *Elliptic Function Formulæ, when $k = -ip$.*

Besides the formulæ of Art. 17, there are numerous other formulæ of great interest. Thus equations (56) enable us to express rationally, in sn 's only, the ordinary addition equations. As instances of formulæ

$k = -i\rho.$ A TABLE FOR sn , cn , dn , sn^2 , cn^2 , dn^2 .

$u =$	$\text{sn } u =$	$\text{cn } u =$	$\text{dn } u =$	$\text{sn}^2 u =$	$\text{cn}^2 u =$	$\text{dn}^2 u =$
$\frac{1}{2}\rho K$	$-\frac{3^{\frac{1}{2}}(1-i)\rho}{2^{\frac{1}{2}}(2^{\frac{1}{2}}+1)}$	$\frac{3^{\frac{1}{2}}(1+i)}{2^{\frac{1}{2}}(2^{\frac{1}{2}}+\rho)}$	$\frac{3^{\frac{1}{2}}(1-i)}{2^{\frac{1}{2}}(2^{\frac{1}{2}}+\rho^2)}$	$\frac{2^{\frac{1}{2}}-1}{\rho^2-1}$	$\frac{\rho^2-2^{\frac{1}{2}}}{\rho^2-1}$	$\frac{2^{\frac{1}{2}}-1}{\rho^2-1}$
$\frac{1}{2}K$	$-\frac{2^{\frac{1}{2}}\rho^2+\rho}{2^{\frac{1}{2}}+1}$	$\frac{2^{\frac{1}{2}}(2^{\frac{1}{2}}+\rho^2)}{3^{\frac{1}{2}}(1-i)}$	$\frac{i\rho(2^{\frac{1}{2}}-1)}{2^{\frac{1}{2}}+1}$	$\frac{2^{\frac{1}{2}}-1}{2^{\frac{1}{2}}-\rho}$	$\frac{1-\rho}{2^{\frac{1}{2}}-\rho}$	$\frac{1-2^{\frac{1}{2}}\rho}{2^{\frac{1}{2}}-\rho}$
$\frac{1}{2}\rho K$	$\frac{i(2^{\frac{1}{2}}+\rho)}{2^{\frac{1}{2}}+1}$	$-\frac{i\rho(2^{\frac{1}{2}}-\rho)}{2^{\frac{1}{2}}+1}$	$\frac{2^{\frac{1}{2}}(2^{\frac{1}{2}}+\rho)}{3^{\frac{1}{2}}(1+i)}$	$-\frac{(2^{\frac{1}{2}}-1)\rho}{2^{\frac{1}{2}}-\rho^2}$	$\frac{1-2^{\frac{1}{2}}\rho^2}{2^{\frac{1}{2}}-\rho^2}$	$\frac{1-\rho^2}{2^{\frac{1}{2}}-\rho^2}$
$\frac{1}{2}(1-\rho)K$	$-\frac{\rho(1+i)}{2^{\frac{1}{2}}\cdot 3^{\frac{1}{2}}}$	$-\frac{(1-i)\rho^2}{2^{\frac{1}{2}}\cdot 3^{\frac{1}{2}}}$	$-\frac{(1+i)\rho}{2^{\frac{1}{2}}\cdot 3^{\frac{1}{2}}}$	$\frac{1}{1-\rho^2}$	$\frac{\rho^2}{\rho^2-1}$	$\frac{1}{1-\rho^2}$
$\frac{1}{2}(\rho-\rho^2)K$	ρ	$-\frac{3^{\frac{1}{2}}(1-i)\rho^2}{2^{\frac{1}{2}}}$	$-i\rho$	ρ^2	$1-\rho^2$	$-\rho^2$
$\frac{1}{2}(\rho^2-1)K$	$i\rho$	$i\rho^2$	$-\frac{3^{\frac{1}{2}}(1+i)\rho}{2^{\frac{1}{2}}}$	$-\rho^2$	$-\rho$	$1-\rho$
$\frac{2}{3}\rho K$	$-\frac{2^{\frac{1}{2}}\rho(2^{\frac{1}{2}}+1)}{3^{\frac{1}{2}}(1+i)}$	$-\frac{(2^{\frac{1}{2}}+1)\rho}{2^{\frac{1}{2}}+\rho^2}$	$-\frac{(2^{\frac{1}{2}}+1)\rho^2}{2^{\frac{1}{2}}+\rho}$	$\frac{\rho-1}{2^{\frac{1}{2}}-1}$	$\frac{2^{\frac{1}{2}}-\rho}{2^{\frac{1}{2}}-1}$	$\frac{2^{\frac{1}{2}}-\rho^2}{2^{\frac{1}{2}}-1}$
$\frac{2}{3}K$	$\frac{2^{\frac{1}{2}}(2^{\frac{1}{2}}+\rho)}{3^{\frac{1}{2}}(1+i)}$	$-\frac{(2^{\frac{1}{2}}+\rho)\rho}{2^{\frac{1}{2}}+1}$	$\frac{(2^{\frac{1}{2}}+\rho)\rho^2}{2^{\frac{1}{2}}+\rho^2}$	$\frac{1-\rho^2}{2^{\frac{1}{2}}-\rho^2}$	$\frac{2^{\frac{1}{2}}-1}{2^{\frac{1}{2}}-\rho^2}$	$\frac{2^{\frac{1}{2}}-\rho}{2^{\frac{1}{2}}-\rho^2}$
$\frac{2}{3}\rho K$	$\frac{2^{\frac{1}{2}}\rho^2(2^{\frac{1}{2}}+\rho^2)}{3^{\frac{1}{2}}(1+i)}$	$\frac{(2^{\frac{1}{2}}+\rho^2)\rho}{2^{\frac{1}{2}}+\rho}$	$-\frac{(2^{\frac{1}{2}}+\rho^2)\rho^2}{2^{\frac{1}{2}}+1}$	$\frac{\rho^2-\rho}{2^{\frac{1}{2}}-\rho}$	$\frac{2^{\frac{1}{2}}-\rho^2}{2^{\frac{1}{2}}-\rho}$	$\frac{2^{\frac{1}{2}}-1}{2^{\frac{1}{2}}-\rho}$

K	1	0	$i\rho^3$	1	0	$-p$
ρK	$-i\rho^3$	$-i\rho$	0	$-p$	$-\rho^3$	0
$\rho^3 K$	∞	$i\infty$	$\rho\infty$	∞^3	$-\infty^3$	$\rho^3\infty^3$
$\frac{2}{3}(1-\rho)K$	$-\frac{3^{\frac{1}{2}}(1+i)\rho}{2^{\frac{1}{2}}}$	$-\rho^3$	$-p$	$1-p$	p	ρ^3
$\frac{2}{3}(\rho-\rho^3)K$	$\frac{3^{\frac{1}{2}}(1+i)\rho}{2^{\frac{1}{2}}}$	$-\rho^3$	p	$1-p$	p	ρ^3
$\frac{2}{3}(\rho^3-1)K$	$\frac{3^{\frac{1}{2}}(1+i)\rho}{2^{\frac{1}{2}}}$	ρ^3	$-p$	$1-p$	p	ρ^3
$(\rho-\rho^3)K$	-1	0	$-i\rho^3$	1	0	$-p$
$(\rho^3-1)K$	$-i\rho^3$	$i\rho$	0	$-p$	$-\rho^3$	0
$(1-\rho)K$	∞	$i\infty$	$-\rho\infty$	∞^3	$-\infty^3$	$\rho^3\infty^3$
$\frac{4}{3}K$	$\frac{2^{\frac{1}{2}}(2^{\frac{1}{2}}+1)\rho}{3^{\frac{1}{2}}(1+i)}$	$\frac{(2^{\frac{1}{2}}+1)\rho}{2^{\frac{1}{2}}+\rho^3}$	$\frac{(2^{\frac{1}{2}}+1)\rho^2}{2^{\frac{1}{2}}+p}$	$\frac{\rho-1}{2^{\frac{1}{2}}-1}$	$\frac{2^{\frac{1}{2}}-p}{2^{\frac{1}{2}}-1}$	$\frac{2^{\frac{1}{2}}-\rho^3}{2^{\frac{1}{2}}-1}$
$\frac{4}{3}\rho K$	$\frac{2^{\frac{1}{2}}\rho^3(2^{\frac{1}{2}}+\rho^3)}{3^{\frac{1}{2}}(1+i)}$	$\frac{(2^{\frac{1}{2}}+\rho^3)\rho}{2^{\frac{1}{2}}+p}$	$\frac{(2^{\frac{1}{2}}+\rho^3)\rho^3}{2^{\frac{1}{2}}+1}$	$\frac{\rho^2-p}{2^{\frac{1}{2}}-p}$	$\frac{2^{\frac{1}{2}}-\rho^3}{2^{\frac{1}{2}}-p}$	$\frac{2^{\frac{1}{2}}-1}{2^{\frac{1}{2}}-p}$
$\frac{4}{3}K$	$\frac{2^{\frac{1}{2}}(2^{\frac{1}{2}}+\rho)}{3^{\frac{1}{2}}(1+i)}$	$\frac{(2^{\frac{1}{2}}+\rho)\rho}{2^{\frac{1}{2}}+1}$	$\frac{(2^{\frac{1}{2}}+\rho)\rho^3}{2^{\frac{1}{2}}+\rho^3}$	$\frac{1-\rho^3}{2^{\frac{1}{2}}-\rho^3}$	$\frac{2^{\frac{1}{2}}-1}{2^{\frac{1}{2}}-\rho^3}$	$\frac{2^{\frac{1}{2}}-p}{2^{\frac{1}{2}}-\rho^3}$

readily verified, we may notice

$$\left. \begin{aligned} \operatorname{cn} u \operatorname{cn} \rho u \operatorname{cn} \rho^3 u &= 1 \\ \operatorname{dn} u \operatorname{dn} \rho u \operatorname{dn} \rho^3 u &= 1 \end{aligned} \right\} \dots\dots\dots (62).$$

$$\rho^2 \operatorname{sn}^2 u - \operatorname{sn}^2 \rho u = \rho^2 \operatorname{sn}^2 u \operatorname{sn}^2 \rho u \dots\dots\dots (63).$$

$$\left. \begin{aligned} \operatorname{sn}(u+v) &= \frac{\operatorname{sn} \rho u \operatorname{sn} \rho^2 u \operatorname{sn} \rho v \operatorname{sn} \rho^2 v (\operatorname{sn}^2 u - \operatorname{sn}^2 v)}{\operatorname{sn} u \operatorname{sn} v (\operatorname{sn} v \operatorname{sn} \rho u \operatorname{sn} \rho^3 u - \operatorname{sn} u \operatorname{sn} \rho v \operatorname{sn} \rho^3 v)} \\ &= \frac{\operatorname{sn} u \operatorname{sn} v (\operatorname{sn} v \operatorname{sn} \rho u \operatorname{sn} \rho^3 u + \operatorname{sn} u \operatorname{sn} \rho v \operatorname{sn} \rho^3 v)}{\operatorname{sn} \rho u \operatorname{sn} \rho^2 u \operatorname{sn} \rho v \operatorname{sn} \rho^2 v (1 + \rho^2 \operatorname{sn}^2 u \operatorname{sn}^2 v)} \\ \operatorname{cn}(u+v) &= \frac{\operatorname{sn} u \operatorname{sn} \rho u \operatorname{sn} \rho^2 v - \operatorname{sn} v \operatorname{sn} \rho v \operatorname{sn} \rho^2 u}{\operatorname{sn} v \operatorname{sn} \rho u \operatorname{sn} \rho^3 u - \operatorname{sn} u \operatorname{sn} \rho v \operatorname{sn} \rho^3 v} \\ \operatorname{dn}(u+v) &= \frac{\operatorname{sn} u \operatorname{sn} \rho^2 u \operatorname{sn} \rho v - \operatorname{sn} v \operatorname{sn} \rho^3 v \operatorname{sn} \rho u}{\operatorname{sn} v \operatorname{sn} \rho u \operatorname{sn} \rho^3 u - \operatorname{sn} u \operatorname{sn} \rho v \operatorname{sn} \rho^3 v} \end{aligned} \right\} \dots\dots\dots (64).$$

$$\frac{\operatorname{sn}^2 u - \operatorname{sn}^2 v}{\operatorname{sn}^2 u \operatorname{sn}^2 v} = \frac{\operatorname{sn}^2 \rho u - \operatorname{sn}^2 \rho v}{\rho \operatorname{sn}^2 \rho u \operatorname{sn}^2 \rho v} \dots\dots\dots (65).$$

And the five formulæ given by the late Professor H. J. S. Smith in *Proceedings of the London Mathematical Society*, Vol. x., p. 97, give the following:—

$$\begin{aligned} \text{When } x_1 + x_2 + x_3 + x_4 &= 0, \\ \rho^3 &= \frac{1}{\operatorname{sn} x_1 \operatorname{sn} x_2 \operatorname{sn} x_3 \operatorname{sn} x_4} + \frac{1}{\operatorname{sn} \rho x_1 \operatorname{sn} \rho x_2 \operatorname{sn} \rho x_3 \operatorname{sn} \rho x_4} \\ &\quad + \frac{1}{\operatorname{sn} \rho^2 x_1 \operatorname{sn} \rho^3 x_2 \operatorname{sn} \rho^2 x_3 \operatorname{sn} \rho^2 x_4} \dots\dots\dots (66). \end{aligned}$$

$$\begin{aligned} \frac{1}{\operatorname{sn} \rho^2 x_1 \operatorname{sn} \rho^3 x_2} - \frac{1}{\operatorname{sn} \rho^3 x_3 \operatorname{sn} \rho^2 x_4} + \frac{\rho^3}{\operatorname{sn} x_1 \operatorname{sn} x_2 \operatorname{sn} \rho x_3 \operatorname{sn} \rho x_4} \\ - \frac{\rho^3}{\operatorname{sn} \rho x_1 \operatorname{sn} \rho x_2 \operatorname{sn} x_3 \operatorname{sn} x_4} = 0 \dots\dots\dots (67), \end{aligned}$$

$$\begin{aligned} \frac{1}{\operatorname{sn} x_2 \operatorname{sn} \rho x_3 \operatorname{sn} \rho^2 x_4} + \frac{1}{\operatorname{sn} x_3 \operatorname{sn} \rho x_2 \operatorname{sn} \rho^2 x_1} + \frac{1}{\operatorname{sn} x_4 \operatorname{sn} \rho x_1 \operatorname{sn} \rho^2 x_3} \\ + \frac{1}{\operatorname{sn} x_1 \operatorname{sn} \rho x_4 \operatorname{sn} \rho^2 x_2} = 0 \dots\dots\dots (68). \end{aligned}$$

The six formulæ of this type will give various simple expressions, by determinants, for $\operatorname{sn} x_1$ in terms of sn 's of $x_2, x_3, x_4, \rho x_2$, &c.

19. When $k = -ip$, our original equations, (1), (2), (3), become

$$\left. \begin{aligned} \beta^2 &= \frac{\rho \operatorname{cn}(q-t) \operatorname{dn}(q-t)}{\operatorname{sn}(t-p) \operatorname{sn}(p-q) \{\operatorname{sn}^2 p - \operatorname{sn}^2(q-t)\}} \\ \omega^2 &= \frac{\rho \operatorname{cn} \rho^2(q-t) \operatorname{dn} \rho^2(q-t)}{\operatorname{sn} \rho^2(t-p) \operatorname{sn} \rho^2(p-q) \{\operatorname{sn}^2 \rho^2 p - \operatorname{sn}^2 \rho^2(q-t)\}} \\ \phi^2 &= \frac{\rho \operatorname{cn} \rho(q-t) \operatorname{dn} \rho(q-t)}{\operatorname{sn} \rho(t-p) \operatorname{sn} \rho(p-q) \{\operatorname{sn}^2 \rho p - \operatorname{sn}^2 \rho(q-t)\}} \end{aligned} \right\} \dots\dots\dots (69);$$

the other six formulæ being obtained, as in Art. 1, by a cyclical permutation of p, q, t .

20. The formulæ of Art. 5 give, when $k = -ip$, formulæ worthy of separate notice. Thus we have

$$\left. \begin{aligned} \frac{\beta\kappa\theta}{\gamma r\phi} &= \frac{\text{cn } p p \text{ cn } q \text{ cn } \rho^3 t}{\text{cn } \rho (q-t) \text{ cn } (t-p) \text{ cn } \rho^3 (p-q)} \\ &= \frac{\text{dn } (q-t) \text{ dn } \rho^3 (t-p) \text{ dn } \rho (p-q)}{\text{dn } p \text{ dn } \rho^3 q \text{ dn } p t} \\ \frac{\beta r\chi}{\gamma \omega\theta} &= \frac{\text{dn } \rho^3 p \text{ dn } q \text{ dn } p t}{\text{dn } \rho^3 (q-t) \text{ dn } (t-p) \text{ dn } \rho (p-q)} \\ &= \frac{\text{cn } (q-t) \text{ cn } \rho (t-p) \text{ cn } \rho^3 (p-q)}{\text{cn } p \text{ cn } p q \text{ cn } \rho^3 t} \end{aligned} \right\} \dots\dots\dots(70).$$

$$\begin{aligned} \frac{\beta\kappa\theta \text{ sn } p \text{ sn } \rho^3 q \text{ sn } p t}{\text{sn } (q-t) \text{ sn } \rho^3 (t-p) \text{ sn } \rho (p-q)} &= \frac{\gamma r\phi \text{ sn } p p \text{ sn } q \text{ sn } \rho^3 t}{\text{sn } \rho (q-t) \text{ sn } (t-p) \text{ sn } \rho^3 (p-q)} \\ &= \frac{\delta \omega\chi \text{ sn } \rho^3 p \text{ sn } p q \text{ sn } t}{\text{sn } \rho^3 (q-t) \text{ sn } \rho (t-p) \text{ sn } (p-q)} = \frac{\beta r\chi \text{ sn } p \text{ sn } p q \text{ sn } \rho^3 t}{\text{sn } (q-t) \text{ sn } \rho (t-p) \text{ sn } \rho^3 (p-q)} \\ &= \frac{\gamma \omega\theta \text{ sn } \rho^3 p \text{ sn } q \text{ sn } p t}{\text{sn } \rho^3 (q-t) \text{ sn } (t-p) \text{ sn } \rho (p-q)} = \frac{\delta \kappa\phi \text{ sn } p p \text{ sn } \rho^3 q \text{ sn } t}{\text{sn } \rho (q-t) \text{ sn } \rho^3 (t-p) \text{ sn } (p-q)} \end{aligned} \dots\dots\dots(71).$$

21. Examination of case when

$$q = p p, \quad t = \rho^3 p, \quad k = -ip.$$

In this case we shall find

$$\left. \begin{aligned} \beta^3(p, p p, \rho^3 p, -ip) &= \frac{(\rho - \text{sn}^4 p)^3}{4 \text{sn}^4 p (1 - \rho - \text{sn}^3 p)^3} \\ &= \kappa^3 = \theta^3 \\ \gamma^3(p, p p, \rho^3 p, -ip) &= \frac{(\rho - \text{sn}^4 p p)^3}{4 \text{sn}^4 p p (1 - \rho - \text{sn}^3 p p)^3} \\ &= r^3 = \phi^3 \\ \delta^3(p, p p, \rho^3 p, -ip) &= \frac{(\rho - \text{sn}^4 \rho^3 p)^3}{4 \text{sn}^4 \rho^3 p (1 - \rho - \text{sn}^3 \rho^3 p)^3} \\ &= \omega^3 = \chi^3 \end{aligned} \right\} \dots\dots\dots(72).$$

For the details of the investigation, we have, from equations (59), (60), (61),

$$\begin{aligned} \text{sn}^3 p - \text{sn}^3 (\rho - \rho^3) p &= \text{sn}^2 p \left\{ 1 - \frac{\text{sn}^4 p + 2(\rho - 1) \text{sn}^2 p - 3\rho}{\rho - 2(\rho - 1) \text{sn}^2 p - 3 \text{sn}^4 p} \right\} \\ &= \frac{4\rho \text{sn}^3 p \text{cn}^3 p \text{dn}^3 p}{\{(\rho^3 - \rho) \text{sn}^2 p - \rho^3\}^3}; \end{aligned}$$

$$\frac{\text{cn} (\rho - \rho^3) p \text{dn} (\rho - \rho^3) p}{\text{sn} (\rho^3 - 1) p \text{sn} (1 - \rho) p} = \frac{\text{cn}^3 p (\text{sn}^4 p - \rho)^3 \text{dn}^3 p}{\{(\rho^3 - \rho) \text{sn}^2 p - \rho^3\}^3 \text{sn}^3 p \{\text{sn}^3 p + \rho - 1\}^3},$$

whence the expression for β^3 .

To express γ^2, δ^2 in terms of $\text{sn } p$, we have

$$\begin{aligned}\rho - \text{sn}^4 \rho p &= \left(\rho^2 - \frac{\rho^2 \text{sn}^2 p}{1 + \rho^2 \text{sn}^2 p} \right) \left(\rho^2 + \frac{\rho^2 \text{sn}^2 p}{1 + \rho^2 \text{sn}^2 p} \right) \\ &= \frac{\rho \{1 + (\rho^2 - 1) \text{sn}^2 p\} \{1 - \rho \text{sn}^2 p\}}{(1 + \rho^2 \text{sn}^2 p)^2},\end{aligned}$$

$$\text{sn}^2 \rho p = \frac{\rho^2 \text{sn}^2 p}{1 + \rho^2 \text{sn}^2 p},$$

$$1 - \rho - \text{sn}^2 \rho p = \frac{1 - \rho - \text{sn}^2 p}{1 + \rho^2 \text{sn}^2 p};$$

so that we have

$$\gamma^2(p, \rho p, \rho^2 p, -ip) = \frac{\{1 + (\rho^2 - 1) \text{sn}^2 p\}^2 (\rho^2 - \text{sn}^2 p)^2}{4 \text{sn}^4 p (1 - \rho - \text{sn}^2 p)^2} \dots\dots (73),$$

$$\begin{aligned}\rho - \text{sn}^4 \rho^2 p &= \left(\rho^2 - \frac{\rho \text{sn}^2 p}{1 - \text{sn}^2 p} \right) \left(\rho^2 + \frac{\rho \text{sn}^2 p}{1 - \text{sn}^2 p} \right) \\ &= \frac{\rho^2 (\rho^2 + \text{sn}^2 p) \{1 + (\rho^2 - 1) \text{sn}^2 p\}}{(1 - \text{sn}^2 p)^2},\end{aligned}$$

$$\text{sn}^2 \rho^2 p = \frac{\rho \text{sn}^2 p}{1 - \text{sn}^2 p},$$

$$1 - \rho - \text{sn}^2 \rho^2 p = \frac{1 - \rho - \text{sn}^2 p}{1 - \text{sn}^2 p},$$

so that we have

$$\delta^2(p, \rho p, \rho^2 p, -ip) = \frac{(1 + \rho \text{sn}^2 p)^2 \{1 + (\rho^2 - 1) \text{sn}^2 p\}^2}{4 \text{sn}^4 p (1 - \rho - \text{sn}^2 p)^2} \dots\dots (74).$$

We are as yet at liberty to give what sign we please to β . If we take

$$\beta(p, \rho p, \rho^2 p, -ip) = \frac{\rho - \text{sn}^4 p}{2 \text{sn}^2 p (\text{sn}^2 p - 1 + \rho)} \dots\dots\dots (75),$$

we must take

$$\beta = \kappa = \theta$$

and

$$\left. \begin{aligned}\gamma &= \tau = \phi = \frac{\rho - \text{sn}^4 \rho p}{2 \text{sn}^3 \rho p (\text{sn}^2 \rho p - 1 + \rho)} \\ \delta &= \varpi = \chi = \frac{\rho - \text{sn}^4 \rho^2 p}{2 \text{sn}^3 \rho^2 p (\text{sn}^2 \rho^2 p - 1 + \rho)}\end{aligned} \right\} \dots\dots\dots (76),$$

and we have

$$\beta + \tau + \chi = 1,$$

$$\kappa \theta - \chi \tau = \beta^2 - \chi \tau = \beta (\beta + \tau + \chi),$$

showing that the system is a positive system.

We may notice the formulæ

$$\left. \begin{aligned}\beta \text{cn } p &= \rho \delta \text{cn } (\rho - \rho^2) p \\ \gamma \text{cn } \rho p &= \rho \beta \text{cn } (\rho^2 - 1) p \\ \delta \text{cn } \rho^2 p &= \rho \gamma \text{cn } (1 - \rho) p\end{aligned} \right\} \dots\dots\dots (77),$$

and

$$\left. \begin{aligned} \beta \operatorname{dn} p &= \rho^3 \gamma \operatorname{dn} (\rho - \rho^3) p \\ \gamma \operatorname{dn} \rho p &= \rho^3 \delta \operatorname{dn} (\rho^3 - 1) p \\ \delta \operatorname{dn} \rho^3 p &= \rho^3 \beta \operatorname{dn} (1 - \rho) p \end{aligned} \right\} \dots\dots\dots (78).$$

Had we taken β with the other sign, the sign of all the rest would have had to be changed, and we should have had a negative system.

22. Examination of case when

$$q = \rho^3 p, \quad t = \rho p, \quad k = -ip.$$

As the substitution of q for t and t for q changes the sign of Q in (10), it changes the sign of $\beta\kappa\theta$, $\beta\chi r$, &c. So we must take

$$\beta(p, \rho^3 p, \rho p, -ip) = -\beta(p, \rho p, \rho^3 p, -ip) \dots\dots\dots (79).$$

We observe, *en passant*, that, if we put $q = t$ in (1), we have

$$\beta^3(p, t, t, k) = \frac{1}{k^2 \operatorname{sn}^2 p \operatorname{sn}^2 (p - t)} \dots\dots\dots (80),$$

showing that, as we approach by continuous changes $\beta(p, t, t)$ in different directions, we get different signs.

$$\left. \begin{aligned} \text{We shall get } \beta &= \chi = r = \frac{\operatorname{sn}^4 p - \rho}{2 \operatorname{sn}^3 p (\operatorname{sn}^3 p - 1 + \rho)} \\ \gamma &= \varpi = \theta = \frac{\operatorname{sn}^4 \rho^3 p - \rho}{2 \operatorname{sn}^3 \rho^3 p (\operatorname{sn}^3 \rho^3 p - 1 + \rho)} \\ \delta &= \kappa = \phi = \frac{\operatorname{sn}^4 \rho p - \rho}{2 \operatorname{sn}^3 \rho p (\operatorname{sn}^3 \rho p - 1 + \rho)} \end{aligned} \right\} \dots\dots\dots (81),$$

and

$$\beta + \kappa + \theta = -1, \quad \kappa\theta - \chi r = \beta,$$

showing that the system is still a positive system.

This case may be considered as identical with the case considered in Art. 21, changing the sign of every function, and interchanging gutturals and dentals.

23. Examination of case when

$$p + \rho q + \rho^2 t = 0; \quad k = -ip.$$

We shall have, see equations (70),

$$\operatorname{on} \rho^3 (q - t) \operatorname{on} (t - p) \operatorname{on} \rho (p - q) = \operatorname{on} \rho^3 (q - t) \operatorname{cn} \rho (q - t) \operatorname{on} (q - t) = 1,$$

$$\operatorname{dn} \rho (q - t) \operatorname{dn} \rho^3 (t - p) \operatorname{dn} (p - q) = 1;$$

$$\frac{\gamma \varpi^3}{\delta \kappa \phi} = \operatorname{dn} \rho p \operatorname{dn} \rho^2 q \operatorname{dn} t,$$

$$\begin{aligned} \frac{\gamma \varpi \theta + \rho^2 \beta r \chi}{\delta \kappa \phi} &= \operatorname{dn} \rho p \operatorname{dn} \rho^2 q \operatorname{dn} t + \rho^2 \operatorname{cn} \rho p \operatorname{cn} \rho^2 q \operatorname{cn} t \\ &= -\rho, \end{aligned}$$

$$\rho^3 (\beta r \chi - \gamma \varpi \theta) + \rho (\delta \kappa \phi - \gamma \varpi \theta) = 0,$$

$$\rho^3 (\theta^2 - \beta^2) + \rho (\theta^2 - \kappa^2) = 0,$$

$$\beta^3 + \rho^3 \kappa^2 + \rho \theta^3 = 0.$$

24. *Examination of case when*

$$p + \rho^2 q + \rho t = 0, \quad k = -i\rho.$$

This case is precisely similar.

$$\text{on } \rho^3 (q - t) \text{ on } \rho (t - p) \text{ on } (p - q) = 1,$$

$$\frac{\gamma \tau \phi}{\delta \varpi \chi} = \text{on } \rho^2 p \text{ on } \rho q \text{ on } t,$$

$$\frac{\beta \kappa \theta}{\delta \varpi \chi} = \text{dn } \rho^2 p \text{ dn } \rho q \text{ dn } t,$$

and

$$\beta \kappa \theta + \rho^2 \gamma \tau \phi + \rho \delta \varpi \chi = 0,$$

$$\beta^3 + \rho^2 r^2 + \rho \chi^2 = 0.$$

25. The system of equations (1), (2), (3) suggested itself to me in this way. I supposed β , γ , &c. to satisfy equations (5) and (7), and to be functions of η , ϵ , a , where

$$\beta^3 + \rho^2 \kappa^2 + \rho \theta^3 = \eta, \quad \beta^3 + \rho^2 r^2 + \rho \chi^2 = \epsilon \dots \dots \dots (82).$$

Then, differentiating, regarding η , ϵ as constant, I found

$$\left. \begin{aligned} \frac{d\beta}{da} &= \rho A \varpi \phi (\kappa \tau - \rho \gamma \delta) \\ \frac{d\gamma}{da} &= \rho A \kappa \chi (\tau \varpi - \rho \delta \beta) \\ \frac{d\delta}{da} &= \rho A \tau \theta (\varpi \kappa - \rho \beta \gamma) \\ \frac{d\varpi}{da} &= A \phi \beta (\chi \theta - \rho \kappa \tau) \\ \frac{d\kappa}{da} &= A \chi \gamma (\theta \phi - \rho \tau \varpi) \\ \frac{d\tau}{da} &= A \theta \delta (\phi \chi - \rho \varpi \kappa) \\ \frac{d\phi}{da} &= \rho^3 A \beta \varpi (\gamma \delta - \rho \chi \theta) \\ \frac{d\chi}{da} &= \rho^3 A \gamma \kappa (\delta \beta - \rho \theta \phi) \\ \frac{d\theta}{da} &= \rho^3 A \delta \tau (\beta \gamma - \rho \phi \chi) \end{aligned} \right\} \dots \dots \dots (83).$$

To represent the solution of this system of differential equations, the

cases when (5) and (7) are satisfied alone being considered, I took

$$\left. \begin{aligned} \operatorname{sn}^2 \psi &= \frac{\beta^2 (\phi^2 - \rho \varpi^2)}{\phi^2 (\beta^2 - \rho^2 \varpi^2)} \\ \operatorname{dn}^2 \psi &= \frac{(\rho \phi^2 - \beta^2) (\kappa \tau - \rho \gamma \delta)^2}{\phi^2 (\chi^2 - \rho \kappa^2) (\theta^2 - \rho \tau^2)} \end{aligned} \right\} \dots\dots\dots (84).$$

which give

$$\left. \begin{aligned} k^2 &= \frac{\rho^2 (\beta^2 - \rho^2 \varpi^2) (\gamma^2 - \rho^2 \kappa^2) (\delta^2 - \rho^2 \tau^2)}{(\phi^2 - \rho \varpi^2) (\chi^2 - \rho \kappa^2) (\theta^2 - \rho \tau^2)} \\ k'^2 &= \frac{\rho (\phi^2 - \rho^2 \beta^2) (\chi^2 - \rho^2 \gamma^2) (\theta^2 - \rho^2 \delta^2)}{(\phi^2 - \rho \varpi^2) (\chi^2 - \rho \kappa^2) (\theta^2 - \rho \tau^2)} \end{aligned} \right\} \dots\dots\dots (85);$$

so that we may, consistently, take

$$\left. \begin{aligned} \operatorname{sn}^2 \xi &= \frac{\gamma^2 (\chi^2 - \rho \kappa^2)}{\chi^2 (\gamma^2 - \rho^2 \kappa^2)} \\ \operatorname{dn}^2 \xi &= \frac{(\rho \chi^2 - \gamma^2) (\tau \varpi - \rho \delta \beta)^2}{\chi^2 (\theta^2 - \rho \tau^2) (\phi^2 - \rho \varpi^2)} \\ \operatorname{sn}^2 \zeta &= \frac{\delta^2 (\theta^2 - \rho \tau^2)}{\theta^2 (\delta^2 - \rho^2 \tau^2)} \\ \operatorname{dn}^2 \zeta &= \frac{(\rho \theta^2 - \delta^2) (\varpi \kappa - \rho \beta \gamma)^2}{\theta^2 (\phi^2 - \rho \varpi^2) (\chi^2 - \rho \kappa^2)} \end{aligned} \right\} \dots\dots\dots (86).$$

These equations lead to

$$\left. \begin{aligned} \left(\frac{d\xi}{da} \right)^2 &= -\rho^2 A^2 (\phi^2 - \rho \varpi^2) (\chi^2 - \rho \kappa^2) (\theta^2 - \rho \tau^2) \\ &= \left(\frac{d\zeta}{da} \right)^2 = \left(\frac{d\psi}{da} \right)^2 \end{aligned} \right\} \dots\dots\dots (87),$$

and several interesting formulæ, such as

$$\operatorname{sn}^2 (\psi - \xi) = \frac{\tau^2 - \rho^2 \theta^2}{(\beta^2 - \rho^2 \varpi^2) (\gamma^2 - \rho^2 \kappa^2)} \dots\dots\dots (88),$$

$$\operatorname{cn}^2 (\psi - \xi) = \frac{\rho^2 (\beta^2 - \rho \phi^2) (\gamma^2 - \rho \chi^2)}{(\beta^2 - \rho^2 \varpi^2) (\gamma^2 - \rho^2 \kappa^2)} \dots\dots\dots (89),$$

$$\operatorname{dn}^2 (\psi - \xi) = \frac{(\beta^2 - \rho \phi^2) (\gamma^2 - \rho \chi^2)}{(\varpi^2 - \rho^2 \phi^2) (\kappa^2 - \rho^2 \chi^2)} \dots\dots\dots (90),$$

$$\operatorname{cn} (\psi - \xi) \operatorname{cn} (\xi - \zeta) \operatorname{cn} (\zeta - \psi) = \frac{\rho k'^2}{k^2} \dots\dots\dots (91),$$

$$\operatorname{dn} (\psi - \xi) \operatorname{dn} (\xi - \zeta) \operatorname{dn} (\zeta - \psi) = -\rho^2 k'^2 \dots\dots\dots (92),$$

$$\left. \begin{aligned} \operatorname{sn} (\psi + \xi - \zeta) &= \frac{\rho}{k} \cdot \frac{\beta (\gamma \delta - \rho^2 \kappa \tau) - \rho^2 \varpi \phi}{\phi (\chi \theta - \rho \kappa \tau) + \rho \beta \varpi} \\ &= \frac{\rho}{k} \cdot \frac{\gamma (\delta \beta - \rho^2 \tau \varpi) + \rho^2 \kappa \chi}{\chi (\theta \phi - \rho \tau \varpi) - \rho \gamma \kappa} \end{aligned} \right\} \dots\dots\dots (93),$$

$$\begin{aligned} \operatorname{cn}(\psi + \xi - \zeta) &= \frac{i\rho^3 k'}{k} \cdot \frac{\kappa(\rho\delta\beta - \tau\varpi) + \rho\gamma\chi}{\chi(\theta\varphi - \rho^2\delta\beta) + \rho^2\gamma\kappa} \left\{ \dots\dots\dots(94), \right. \\ &= \frac{i\rho^3 k'}{k} \cdot \frac{\varpi(\rho\gamma\delta - \kappa\tau) - \rho\beta\phi}{\phi(\chi\theta - \rho^2\gamma\delta) - \rho^2\beta\varpi} \end{aligned}$$

$$\operatorname{dn}^2(\psi + \xi - \zeta) = \frac{(\rho\phi^2 - \beta^2)(\rho\chi^2 - \gamma^2)(\rho\theta^2 - \delta^2)}{[\chi(\rho^2\tau\varpi - \rho\theta\phi) + \rho^2\gamma\kappa]^2} \dots\dots\dots(95);$$

$$\operatorname{sn} \psi = \frac{\rho\beta}{\phi} \cdot \frac{1}{k} \frac{\operatorname{dn}(\xi - \zeta)}{\operatorname{cn}(\xi - \zeta)} \dots\dots\dots(96),$$

$$\operatorname{cn} \psi = \frac{i\rho^3\varpi}{\phi} \cdot \frac{k'}{k} \frac{1}{\operatorname{cn}(\xi - \zeta)} \dots\dots\dots(97),$$

$$\operatorname{dn} \psi = \frac{i\rho(\rho\chi\theta - \rho^2\kappa\tau)}{\phi} \frac{k' \operatorname{sn}(\xi - \zeta)}{\operatorname{cn}(\xi - \zeta)} \dots\dots\dots(98);$$

and, making use of (91), (92) to obtain formulæ from which ρ , ρ^2 should be absent, I got a system of equations resembling (21), that is

$$\begin{aligned} \text{to say,} \quad \frac{\operatorname{sn}^3 \psi}{\operatorname{cn}^3 \psi} &= \frac{\beta^2}{\varpi^2} \cdot \frac{\operatorname{dn}(\xi - \zeta)}{\operatorname{dn}(\psi - \xi) \operatorname{dn}(\zeta - \psi)}, \\ \operatorname{cn}^3 \psi &= - \frac{\varpi^2}{\phi^2} \cdot \frac{\operatorname{cn}(\psi - \xi) \operatorname{cn}(\zeta - \psi)}{\operatorname{cn}(\xi - \zeta)}, \end{aligned}$$

in which β , γ , &c. satisfy (5), (7), without any such condition being necessary as is contained in equations (91), (92). This suggested the whole investigation.

On Mr. Wilkinson's Rectangular Transformation.

By Professor CAYLEY, F.R.S.

[Read May 10th, 1883.]

Considering the three cones,

$$\begin{aligned} (p + \lambda) X^2 + (q + \lambda) Y^2 + (r + \lambda) Z^2 &= 0, \\ (p + \mu) X^2 + (q + \mu) Y^2 + (r + \mu) Z^2 &= 0, \\ (p + \nu) X^2 + (q + \nu) Y^2 + (r + \nu) Z^2 &= 0, \end{aligned}$$

where

$$p + q + r + \lambda + \mu + \nu = 0,$$

it is easy to see that these contain a singly infinite system of rectangular axes, viz., we have in each cone one axis of a rectangular system, and for one of the cones the axis may be any line at pleasure of the cone. In fact, taking for the three axes (x, y, z) , (x', y', z') ,