Notes on the Theory of Automorphic Functions (continued). By A. C. DIXON. Received March 7th, 1900. Read March 8th, 1900. Received, in revised form, November 1st, 1900.

The present paper is supplementary to one published in the Society's *Proceedings* (Vol. XXXI., pp. 297–314), the object of which was to derive the fundamental theorems in Riemann's theory of the Abelian integrals from the properties of Poincaré's theta-Fuchsian series. It appeared to me that, if our knowledge of the properties of those series were complete, we should be able to deduce these theorems from them. At one point (pp. 307, 308) I had to fall back on the older methods to cover a want of rigour in the argument, and my purpose now is to supply this defect.

The results to be proved are numerical; there are three numbers involved:—

p, where $p+1$ is the least number of poles arbitrarily assigned that a Fuchsian function can have, if it is to have no more;

q_1, the number of Abelian integrals of the first kind, or of theta-Fuchsian functions of index 1 without poles;

g, the number of irreducible closed circuits on the closed surface formed from the polygon by fastening together corresponding sides.*

It is to be shown that $p = q_1 = \frac{1}{2}g$.

§§ 1–13 are taken up in proving that $p \ngtr q_1$. For this it is necessary to discuss the formation, by means of Poincaré's series, of theta-Fuchsian and Fuchsian functions with assigned poles. Particular consideration has to be given to the case when poles are assigned at the vertices of the polygon, and account has to be taken of the special conventions as to the orders of poles and zeroes at these points. The proof would be incomplete if these points were excluded from its scope.

The next result (§ 14) is that $q_1 \nless g-p$; this was established in my former paper.

In §§ 15–19 the well known bilinear relation connecting the moduli

* In my former paper this number was denoted by c.

of periodicity of two Abelian integrals of the first kind is used to show that $2q_1 \not> g$. The relation is proved in §§ 15, 16, and in §§ 17, 18 it is reduced to the ordinary canonical form which is convenient for the argument of § 19.

The results $p = q_1 = \frac{1}{2}g$ are thus established.

§ 20 deals with Poincaré's theorem that all theta-Fuchsian functions of integral index > 1 can be expressed by means of his series.

The paper is concerned almost altogether with functions of the first, second, and sixth families. The circular boundary of the functions is generally taken to have centre 0, radius 1.

1. Let $\phi_m(z, y)$ denote the function

$$\Sigma \frac{1}{z - sy} \left(\frac{dsy}{dy}\right)^m,$$

where s denotes one of the substitutions of a Fuchsian group of the first, second, or sixth family, and the summation extends to all substitutions of the group (compare *Acta Math.*, Vol. I., p. 242). Suppose that

$$sy = \frac{\alpha y + \beta}{\gamma y + \delta},$$

and that for the different substitutions of the group s, α, β, γ, δ are distinguished by suffixes. The index m is taken to be an integer greater than 1.

Then, giving y a fixed value c, we have for any substitution s_1 of the group

$$\phi_m(s_1 z, c) = \Sigma \frac{1}{s_1 z - sc} \left(\frac{dsc}{dc}\right)^m$$

$$= \Sigma \frac{1}{s_1 z - s_1 sc} \left(\frac{ds_1 sc}{dc}\right)^m$$

(by rearrangement of the terms)

$$= \Sigma \frac{(\gamma_1 z + \delta_1)(\gamma_1 sc + \delta_1)}{z - sc} \left(\frac{ds_1 sc}{dsc}\right)^m \left(\frac{dsc}{dc}\right)^m$$

$$= (\gamma_1 z + \delta_1) \Sigma \frac{1}{z - sc} (\gamma_1 sc + \delta_1)^{-2m+1} \left(\frac{dsc}{dc}\right)^m.$$

Thus $\phi_m(s_1 z, c)\left(\frac{ds_1 z}{dz}\right)^{1-m} - \phi_m(z,c)$

$$= \Sigma \frac{1}{z-sc}\left\{\left(\frac{\gamma_1 z+\delta_1}{\gamma_1 sc+\delta_1}\right)^{2m-1}-1\right\}\left(\frac{dsc}{dc}\right)^m$$

$$= C_0 + C_1 z + C_2 z^2 + \ldots + C_{2m-2} z^{2m-2},$$

where the coefficients C are theta-Fuchsian functions of c of index m, and without poles. Let q_m be the number of linearly independent theta-Fuchsian functions of index m, without poles, whether expressible as theta-Fuchsian series or not. Let $\theta_m^{(r)}$ denote one of these functions, r being one of the numbers $1, 2, \ldots, q_m$. Then, if any q_m+1 values $c_1, c_2, \ldots, c_{q_m+1}$ are taken within the fundamental polygon, it is possible to choose coefficients $A_1, A_2, \ldots, A_{q_m+1}$ so that

$$\sum_{i=1}^{q_m+1} A_i \theta_m^{(r)}(c_i) = 0 \quad (r = 1, 2, \ldots, q_m).$$

Now $C_0, C_1, \ldots, C_{2m-2}$ are linear combinations of $\theta_m^{(1)}(c), \theta_m^{(2)}(c), \ldots$.

Thus $$\sum_{i=1}^{q_m+1} A_i \phi_m(s_1 z, c_i) = \left(\frac{ds_1 z}{dz}\right)^{m-1} \sum_{i=1}^{q_m+1} A_i \phi_m(z, c_i).$$

This holds when s_1 is any substitution of the group, and therefore

$$\sum_{i=1}^{q_m+} A_i \phi_m(z, c_i)$$

is a theta-Fuchsian function of z of negative index $1-m$ having q_m+1 arbitrary poles $c_1, c_2, \ldots, c_{q_m+1}$, and no others.*

2. By means of the derivatives of the function $\phi_m(z, c)$ theta-Fuchsian functions of negative index with given multiple poles may be constructed, since $\frac{\partial^a}{\partial c^a}\phi_m(z, c)$ has a pole at c of order $a+1$, and since

$$\frac{\partial^a}{\partial c^a}\phi_m(s_1 z, c)\left(\frac{ds_1 z}{dz}\right)^{1-m} - \frac{\partial^a}{\partial c^a}\phi_m(z, c)$$

$$= \frac{d^a C_0}{dc^a} + z\frac{d^a C_1}{dc^a} + \ldots + z^{2m-2}\frac{d^a C_{2m-2}}{dc^a}.$$

If, for instance, there is to be a double pole at c_1 and a simple one

* Compare *Acta Math.*, Vol. I., pp. 215, 246.

at each of the other points $c_2, c_3, \ldots, c_{q_m}$, we must choose coefficients or residues $A_1, B_1, A_2, \ldots, A_q$ so that

$$B_1 \frac{d\theta^{(r)}(c_1)}{dc_1} + \sum_{i=1}^{q_m} A_i \theta^{(r)}(c_i) = 0 \quad (r = 1, 2, \ldots, q_m),$$

and then the function

$$B_1 \frac{\partial}{\partial c_1} \phi_m(z, c_1) + \sum_{i=1}^{q_m} A_i \phi_m(z, c_i)$$

will be a theta-Fuchsian function of z of index $1-m$ having its poles as desired.

For a multiple pole of order a at c, we must use the values when $y = c$ of $\phi_m(z, y)$ and its first $a-1$ derivatives with respect to y, or, in other words, the first a coefficients in the expansion of $\phi_m(z, y)$ in ascending powers of $y-c$.

In this way a theta-Fuchsian function of index $1-m$ can be constructed with poles arbitrarily assigned, if the sum of the orders of multiplicity of these poles is q_m+1. The q_m+1 residues must satisfy q_m homogeneous linear equations, so that in general their ratios will be determinate and the function will be defined save as to a constant factor. If the q_m equations are not all independent, there will be two or more functions of index $1-m$ with the assigned poles.

In particular cases one or more of the residues may come out with the value zero, so that some of the assigned poles may disappear or be of lower orders than were assigned.

Hence, *if poles within the polygon are arbitrarily assigned, the sum of whose orders of multiplicity is* q_m+1, *at least one theta-Fuchsian function of negative index* $1-m$ *can be constructed, having no poles other than those assigned, and having none of the assigned poles to a higher order than the assigned order.*

I shall call this *Poincaré's negative construction.*

4. Again, if z is supposed fixed within the polygon, $\phi_m(z, y)$ is a theta-Fuchsian function of y of positive index m, and it has a simple pole at z and no other within the polygon. The derivatives of $\phi_m(z, y)$ with respect to z or the functions

$$\Sigma \frac{1}{(z-sy)^a} \left(\frac{dsy}{dy}\right)^m,$$

where a is a positive integer, are theta-Fuchsian functions of y of index m, having multiple poles at z.

By means of these, *theta-Fuchsian functions of positive index* m (>1) *can be constructed with arbitrary poles within the polygon of any orders and with arbitrary residues.* When the poles and their orders are fixed, the number of arbitrary constant coefficients in such a function is $q_m +$ the sum of the orders of the poles, for q_m is the number of such functions without poles.

This may be called *Poincaré's positive construction.* In it we use the coefficients in a Taylor expansion of $\phi_m(z, y)$ considered as a function of z.

5. Both constructions need special investigation when one of the assigned poles is at a vertex of the polygon. Let c be this vertex, $2\pi/\lambda$ the sum of the angles of the cycle to which it belongs, c' the inverse of c with respect to the fundamental circle.

Then we must discuss the behaviour of $\phi_m(z, y)$ when y or z approaches c. Now $\phi_m(z, y)$ is a theta-Fuchsian function of y of index m, and thus

$$\phi_m(z, y)(y-c)^m (y-c')^m$$

is a uniform function of $\left(\frac{y-c}{y-c'}\right)^\lambda$ (*Acta Math.*, Vol. I., p. 218).

Let
$$\frac{y-c}{y-c'} = \eta, \quad \frac{z-c}{z-c'} = \zeta;$$

then it follows that

$$\phi_m(z, y) = (1-\eta)^{2m}\eta^{-m}\psi(\zeta, \eta^\lambda),$$

where ψ denotes a uniform function.

Since $\phi_m(z, y)$ contains a term $\frac{1}{z-y}$, ψ must be infinite when $\eta^\lambda = \zeta^\lambda$, and must, in fact, behave like

$$\frac{\lambda}{c-c'}\,\frac{(1-\zeta)^{2-2m}\zeta^{\lambda+m-1}}{\zeta^\lambda - \eta^\lambda}$$

in the neighbourhood of this value of η^λ. Thus $\phi_m(z, y)$ behaves like

$$\frac{\lambda}{c-c'}(1-\eta)^{2m}(1-\zeta)^{2-2m}\frac{\eta^{a\lambda-m}\zeta^{\lambda-a\lambda+m-1}}{\zeta^\lambda-\eta^\lambda},$$

where a is any integer. Now $\phi_m(z, y)$ does not become infinite when z or y alone approaches c, and hence, if a is such that neither $a\lambda-m$ nor $\lambda-a\lambda+m-1$ is negative, we have here an expression from which $\phi_m(z, y)$ will differ by a finite quantity when z or y approaches c, or when both approach c. The necessary value of a is $\left\{\frac{m}{\lambda}\right\}+1$, where $\{x\}$ denotes the integer next below x.

Now for the negative construction we take the successive coefficients in a Taylor expansion of $\phi_m(z, y)$ as a function of y. Here there is an expansion of the form

$$\phi_m(z, y) = (1-\eta)^{2m}\eta^{a\lambda-m}\left[Z_0+\eta^{\lambda}Z_1+\eta^{2\lambda}Z_2+\ldots\right],$$

and the functions thus available are

$$Z_0,\ Z_1,\ Z_2,\ \ldots .$$

It is seen at once that these are equivalent to the series of derivatives of $\phi_m(z, y)$ with respect to y when $y=c$, but that in the series of derivatives $\lambda-1$ out of every consecutive λ are useless for our purpose, being either identically zero or else linear combinations of lower derivatives.

For a small value of ζ the function Z_n behaves like

$$\frac{\lambda}{c-c'}\,\zeta^{\lambda-a\lambda+m-1-(n+1)\lambda}\,(1-\zeta)^{2-2m},$$

and so appears to have a pole of order

$$(a+n)\,\lambda-m+1.$$

But, according to convention, this order must be divided by λ, as the pole is to be shared among all the polygons which meet in c. Thus the order is

$$a+n-\frac{m-1}{\lambda}$$

in any one polygon; allowance is made in this number for all the vertices of the cycle to which c belongs.

Now the only orders that a pole at these vertices can have for a theta-Fuchsian function of index $1-m$ (*Acta Math.*, Vol. I., p. 218) are the numbers

$$a+n-\frac{m-1}{\lambda}\quad (n=0,\ 1,\ 2,\ \ldots),$$

where

$$a=1+\left\{\frac{m}{\lambda}\right\}.$$

Hence, by means of the functions Z_0, Z_1, Z_2, ..., the negative construction is still possible when one of the assigned poles is at c. The order assigned to this pole must be "admissible," that is, it must be one of the numbers $a+n-\frac{m-1}{\lambda}$. The functions used in constructing a function with a pole of admissible order h at c will be

$$Z_0, Z_1, \ldots, Z_{\{h\}},$$

and others that are finite at c. Thus the number of residues at c is $\{h\}+1$. It will be convenient to call this the "rank" of the pole. The rank is the same as the order when the order is integral; otherwise the rank is the integer next above the order.

6. For the positive construction in this case we must take the coefficients in the expansion of $\phi_m(z, y)$ in ascending powers of ζ. Suppose

$$\phi_m(z, y) = (1-\eta)^{2m}\eta^{a\lambda-m}\left[Y_0+Y_1\zeta+Y_2\zeta^2+\ldots\right].$$

Then it is seen from the above work that the first of the series $Y_0, Y_1, Y_2, \ldots$ which becomes infinite when $\eta=0$ is

$$Y_{\lambda-a\lambda+m-1},$$

and that this behaves like $\quad \frac{\lambda}{c'-c}\eta^{-\lambda}.$

The first that is infinite to a higher order than this is

$$Y_{2\lambda-a\lambda+m-1},$$

and this contains a term $\frac{\lambda}{c'-c}\eta^{-2\lambda}$, and so on.

The infinite terms that occur are multiples of $\eta^{-\lambda}, \eta^{-2\lambda}, \eta^{-3\lambda}, \ldots$, and a term in $\eta^{-n\lambda}$ occurs first in $Y_{n\lambda-a\lambda+m-1}$.

Hence the functions available for the positive construction are the coefficients of

$$\zeta^{\lambda-a\lambda+m-1},\ \zeta^{2\lambda-a\lambda+m-1},\ \ldots$$

in the expansion of $\phi_m(z, y)$ in ascending powers of ζ. These have poles at c of the orders

$$\frac{m}{\lambda}-a+1,\quad \frac{m}{\lambda}-a+2,\quad \ldots,$$

the order being in each case the index of $\eta^{-\lambda}$ in the most important term; these are the only admissible orders for a pole. The ranks are 1, 2, 3, ..., respectively. It will be substantially the same thing to use the series

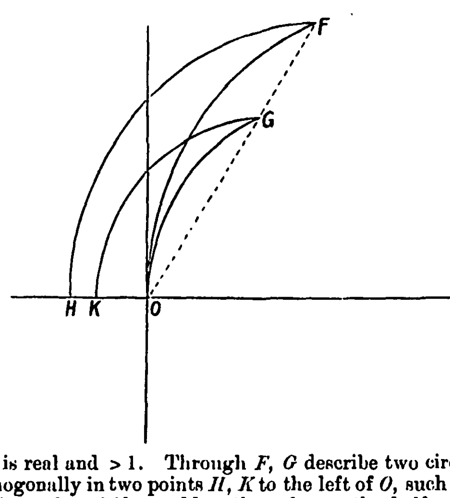
$$\Sigma\frac{1}{(sy-c)^{n\lambda-a\lambda+m}}\left(\frac{dsy}{dy}\right)^m \quad (n=1, 2, 3, \ldots).$$

7. When a proposed pole is at a vertex c which lies on the circular boundary there are special difficulties, since this point is an essential singularity of the functions.

The polygon may be altered, without affecting the group, so that c shall form a cycle by itself. Suppose this done; then the substitution by which one of the sides that meet in c is changed into the other must be parabolic; for, if it were hyperbolic, the vertex c could be abolished, and the polygon would have as a side part of the fundamental circle; so that the functions would no longer have the circular boundary.* Let t be this parabolic substitution, its actual

* On this point see Klein's paper (*Math. Ann.*, Vol. XL., pp. 130-139). The conclusion is that an automorphic group may quite well be generated by a polygon with a hyperbolic cycle, but that as a fundamental region this polygon is incomplete. Parts of the area within which the corresponding functions exist are not represented on the polygon, that is, cannot be brought into the polygon by any of the operations of the group.

Take, for instance, a Fuchsian group of real substitutions generated by a polygon, two of whose corresponding sides OF, OG touch at the origin O, F, G being collinear with O, as shown. The substitution that turns OG into OF is of the form



$z' = kz$, where k is real and > 1. Through F, G describe two circular arcs cutting the real axis orthogonally in two points H, K to the left of O, such that $OH = k.OK$. Then the group is unaltered if we add to the polygon the half-meniscus OFH and take away OGK.

Now it is clear that any point between OF and the imaginary axis is represented in the old fundamental polygon by a point in the curvilinear triangle OFG, and that therefore no point on the left of the imaginary axis is represented in the old polygon at all. Poincaré originally concluded from this that the imaginary axis was a natural boundary to the functions generated; but this is not the case, for in the new polygon any point in the second quadrant is represented by a point in the region bounded by HK, the arcs HF, KG, and the imaginary axis. Hence the functions exist in the whole space above the real axis, and, moreover, can be continued across HK, so that they exist in all the plane. In the original polygon the part representing the second quadrant had shrunk up into the point O.

equation being

$$\frac{1}{tz-c}=\frac{1}{z-c}+\frac{2\iota\pi\mu}{c},$$

where μ is real and positive.

Suppose the z-plane transformed by the substitution

$$\zeta=\frac{1}{2\mu}\,\frac{c+z}{c-z},$$

so that the inside of the fundamental circle becomes the half of the ζ-plane on the right of the imaginary axis. Let η, σ, τ correspond to y, s, t, so that

$$\eta=\frac{1}{2\mu}\,\frac{c+y}{c-y},\quad \sigma\zeta=\frac{1}{2\mu}\,\frac{c+sz}{c-sz},\quad \tau\zeta=\frac{1}{2\mu}\,\frac{c+tz}{c-tz}=\zeta-2\iota\pi,$$

$$\frac{dsy}{dy}=\frac{4c^2}{(y-c)^2}\,\frac{1}{(1+2\mu\sigma\eta)^2}\,\frac{d\sigma\eta}{d\eta}=\left(\frac{1+2\mu\eta}{1+2\mu\sigma\eta}\right)^2\frac{d\sigma\eta}{d\eta}.$$

Now the substitution τ gives a division of the ζ-half-plane into strips of breadth 2π parallel to the real axis, and each of these strips is further divided by the other operations of the group. One polygon in each strip reaches to infinity, and the sum of the areas of the rest must, therefore, be finite; so that the series

$$\Sigma'\left|\frac{d\sigma\eta}{d\eta}\right|^m$$

must be convergent if $m \not< 2$ and Σ' denotes a summation over those operations which turn the fundamental polygon into another belonging to the same strip. This follows by Poincaré's method, since the areal magnification is $\left|\dfrac{d\sigma\eta}{d\eta}\right|^2$.

Also $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$ is a convergent series. Thus the double series

$$\sum_{n=1}^{\infty}\Sigma'\frac{1}{n^2}\left|\frac{d\sigma\eta}{d\eta}\right|^m$$

is absolutely convergent, and its sum is the product of the sums of the former two.

Now let $f(\eta)$ denote a function of η, uniform, finite, and continuous over the right half-plane and its boundary, and such that

$$\operatorname*{Lt}_{\eta=\infty}\eta^2 f(\eta)$$

is finite or zero. Then, since all the points $\sigma\eta$ lie in the right half-plane, there is a superior limit to the quantities

$$n^2\,|f(\sigma\eta+2n\iota\pi)|,\quad n^2\,|f(\sigma\eta-\overline{2n-2}\,\iota\pi)|\,;$$

and therefore the series

$$\sum_{n=-\infty}^{n=\infty}\Sigma' f(\sigma\eta+2n\iota\pi)\left(\frac{d\sigma\eta}{d\eta}\right)^m,$$

that is,
$$\Sigma f(\sigma\eta)\left(\frac{d\sigma\eta}{d\eta}\right)^m,$$

is absolutely convergent. The convergency is clearly uniform in the domain of an ordinary value of η.

This series represents then a continuous uniform function of η, and, in fact, of $\exp\eta$, since its value is unaffected by the addition of $2\iota\pi$ to η. As η increases $\frac{d\sigma\eta}{d\eta}$ diminishes without limit, except for the identical substitution, and those of the form τ^n. Thus when η is infinite the series reduces to $\operatorname{Lt}_{\eta=\infty}\sum_{n=-\infty}^{n=\infty} f(\eta+2n\iota\pi)$, which is zero if the real part of η is made infinite. Hence when $\exp\eta$ is infinite the function

$$\Sigma f(\sigma\eta)\left(\frac{d\sigma\eta}{d\eta}\right)^m$$

vanishes, and is, in fact, of the same order as $\exp(-\eta)$, or some integral power of this.

Going back to the variable y, we find that

$$\Sigma F(sy)\left(\frac{dsy}{dy}\right)^m \quad\text{or}\quad \left(\frac{2c}{y-c}\right)^{2m}\Sigma\,(2\mu\sigma\eta+1)^{-2m}F\left(c\,\frac{2\mu\sigma\eta-1}{2\mu\sigma\eta+1}\right)\left(\frac{d\sigma\eta}{d\eta}\right)^m$$

represents a theta-Fuchsian function of y, even if the function $F(y)$ has a pole at c, so long as the order of the pole is not higher than $2m-2$. But this theta-Fuchsian function is zero at c to the same order as

$$(y-c)^{-2m}\exp\frac{n}{2\mu}\,\frac{y+c}{y-c},$$

where n is some positive integer; n is, in fact, the order of the zero according to the convention (*Acta Math.*, Vol. I., pp. 216, 217). Thus the positive construction in its ordinary form does not succeed. But it is now readily seen that

$$\Sigma \exp\left(\frac{n}{2\mu}\,\frac{c+sy}{c-sy}\right)\left(\frac{dsy}{dy}\right)^m$$

represents a function suitable for the purpose. For this series may be arranged in the form

$$(1+2\mu\eta)^{2m}\sum_{r=-\infty}^{r=\infty}\Sigma'\left\{1+2\mu\left(\sigma\eta+2r\iota\pi\right)\right\}^{-2m}\exp n\sigma\eta\left(\frac{d\sigma\eta}{d\eta}\right)^m,$$

and we may suppose the fundamental polygon in the η-plane to be one which reaches to infinity. Then there is a finite superior limit to $|\exp n\sigma\eta|$ (except when σ is the identical substitution) if η lies in the fundamental polygon, and hence as η increases indefinitely all the terms tend to zero in the aggregate except those of the series

$$(1+2\mu\eta)^{2m}\exp n\eta\sum_{r=-\infty}^{r=\infty}\left\{1+2\mu\left(\eta+2r\iota\pi\right)\right\}^{-2m}.$$

The sum of these terms is

$$-(1+2\mu\eta)^{2m}\exp n\eta\,\frac{(2\mu)^{-2m}}{(2m-1)!}\left(\frac{d}{d\eta}\right)^{2m-1}\frac{1}{\exp\left(\eta+\dfrac{1}{2\mu}\right)-1},$$

which is infinite with η to the same order as

$$\eta^{2m}\exp(n-1)\,\eta,$$

or

$$(y-c)^{-2m}\exp\frac{n-1}{2\mu}\,\frac{c+y}{c-y}.$$

Hence the function

$$\Sigma \exp\left(\frac{n}{2\mu}\,\frac{c+sy}{c-sy}\right)\left(\frac{dsy}{dy}\right)^m$$

is available for the positive construction. It has a pole at c whose order is $n-1$, for, according to convention, the order of a pole at c for a theta-Fuchsian function $\theta_m(y)$ of index m is the exponent of $\exp\eta$ in the most important part of $\theta_m(y)(y-c)^{2m}$.

If a pole of order $n-1$ at c is among those assigned to a theta-Fuchsian function of index m which is to be constructed, the functions available, besides those that are finite at c, are n in number, namely,

$$\Sigma \exp\left(\frac{r}{2\mu}\,\frac{c+sy}{c-sy}\right)\left(\frac{dsy}{dy}\right)^m \quad (r=1,\,2,\,\ldots,\,n).$$

Hence the rank of such a pole is n. This agrees with the general rule that a pole of order h is of rank $\{h\}+1$ if we remember that here properly the order is infinitesimally greater than $n-1$, the integer λ of §§ 5, 6 being infinite.

8. The negative construction is somewhat simpler. The expression $\phi_m(z, y)(y-c)^{2m}$ is a uniform function of $\exp\eta$, and has a simple pole when $y = z$, that is, when $\exp\eta = \exp\zeta$. Thus in the neighbourhood of this pole it behaves like

$$\frac{A}{\exp\eta - \exp\zeta},$$

where

$$A = \operatorname*{Lt}_{y=z} \frac{(y-c)^{2m}}{z-y}(\exp\eta - \exp\zeta)$$

$$= -\frac{c}{\mu}(z-c)^{2m-2}\exp\zeta.$$

Let $\phi_m(z, y)(y-c)^{2m}$ be expanded in descending powers of $\exp\eta$. The successive coefficients will be functions available for the negative construction. The coefficient of $\exp(-n\eta)$ will contain a term

$$-\frac{c}{\mu}(z-c)^{2m-2}\exp n\zeta;$$

so that this coefficient has at c a pole of order n, or, rather, infinitesimally below n.

It is now easily seen that the rank of a pole at c is equal to the order in the case of a theta-Fuchsian function of negative index. For the functions infinite at c that may be used in the negative construction when a pole of order n at c is among those assigned are the coefficients of $\exp(-\eta)$, $\exp(-2\eta)$, ..., $\exp(-n\eta)$, and are therefore n in number.

9. The results reached may now be stated as follows:—By the negative construction *it is possible to form a theta-Fuchsian function of given negative integral index* $-m$, *all of whose poles shall be included among certain points arbitrarily assigned with ranks respectively not exceeding certain positive integers arbitrarily assigned, whose sum is not less than* $q_{m+1}+1$. If this sum is $q_{m+1}+r$, then r such functions can be formed. By the positive construction *it is possible to form a theta-Fuchsian function of given positive integral index* >1 *with any given poles, of any admissible orders, with any assigned residues.*

The rank of a pole of order h is $\{h\}+1$, and it must be borne in mind that the order of a pole at a vertex lying on the circular boundary is infinitesimally above or below an integer according as the index of the function is positive or negative, since the integer λ is infinite for such a vertex. This convention will enable us to use the same numerical formulæ for elliptic and for parabolic vertices, although the proofs of these formulæ may not be the same.

10. With regard to theta-Fuchsian functions of index 1, it is to be noted that none of them can have one simple pole only. This will now be proved.

Let $\theta_1(z)$ be a theta-Fuchsian function of index 1, c one of its poles; then the corresponding residue is

$$\operatorname*{Lim}_{z=c}\,(z-c)\,\theta_1(z).$$

Any point sc, into which c is transformed by an operation of the group, is also a pole, and the residue corresponding to sc is

$$\operatorname*{Lim}_{z=sc}\,(z-sc)\,\theta_1(z),$$

or

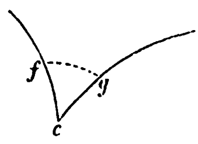
$$\operatorname*{Lim}_{z=c}\,(sz-sc)\,\theta_1(sz).$$

This is the same as the residue at c, since

$$\theta_1(sz)=\theta_1(z)\left(\frac{dsz}{dz}\right)^{-1}.$$

Also $\int\theta_1(z)dz$ taken along two corresponding sides of the generating polygon gives equal results and, therefore, when it is taken round the whole perimeter of the polygon, the result is zero; therefore, the sum of the residues of $\theta_1(z)$ at all its poles within the polygon is zero.

11. In the interpretation of this result, the residue at a multiple pole c is, as usual, to be taken as the coefficient of $\dfrac{1}{z-c}$ in the expansion in ascending powers of $z-c$; there are also special conventions relating to the vertices of the polygon when the sum of the angles in a cycle is not 2π. Take an elliptic cycle in which the sum of the angles is $2\pi/\lambda$, and suppose the polygon transformed, if necessary, so that the cycle shall consist of a single vertex c. The expansion of $\theta_1(z)(z-c)(z-c')$, where c' is the inverse of c with respect to the bounding circle, in powers of $\dfrac{z-c}{z-c'}$, contains only such powers as the $h\lambda$-th, where h is any whole number.



Let f, g be two corresponding points near to c on the two sides that meet at c. Suppose f, to be joined by an arc; then it will be necessary to take as the residue of a pole at c the value of

$$-\frac{1}{2\iota\pi}\int_f^g \theta_1(z)\,dz$$

taken along this arc, which must be so near c that the triangle fgc does not include any other pole. Since the angle fcg of the polygon is $2\pi/\lambda$, this residue is $C \div \lambda\,(c-c')$, where C is the absolute term in the expansion of $\theta_1(z)(z-c)(z-c')$ in ascending powers of $(z-c)/(z-c')$.

For, if $h \neq 0$,

$$\int \left(\frac{z-c}{z-c'}\right)^{h\lambda-1} \frac{dz}{(z-c')^2} = \frac{1}{h\lambda\,(c-c')}\left(\frac{z-c}{z-c'}\right)^{h\lambda},$$

which has the same value at f, g; when $h = 0$, we have

$$\int \left(\frac{z-c}{z-c'}\right)^{-1} \frac{dz}{(z-c')^2} = \frac{1}{c-c'}\log\frac{z-c}{z-c'},$$

which is greater at f than at g by $\dfrac{1}{c-c'}\,\dfrac{2\iota\pi}{\lambda}$.

If the vertex c is on the bounding circle and the substitution parabolic, let the substitution that turns cf into cg be

$$\frac{1}{z'-c} = \frac{1}{z-c} - \frac{2\iota\mu\pi}{c},$$

where μ is a real positive quantity. Then $\theta_1(z)(z-c)^2$ in the neighbourhood of c may be expanded in ascending integral powers of $\exp\dfrac{1}{2\mu}\,\dfrac{z+c}{z-c}$. Let the absolute term of this expansion be C. Then

$$-\frac{1}{2\iota\pi}\int_f^g \theta_1(z)\,dz = -\frac{\mu}{c}\,C,$$

since any integral power of $\exp\dfrac{1}{2\mu}\,\dfrac{z+c}{z-c}$ has the same value at f and g. Thus $-\mu C/c$ must be taken as the residue in this case.

With these special conventions the sum of the residues is zero, and therefore, if there is only a simple pole, its residue must vanish, or a theta-Fuchsian function of index 1 with only a simple pole cannot exist.* A pole of rank 1 is to be counted as simple.

12. It follows from the positive construction that the number of arbitrary coefficients in a theta-Fuchsian function of given index $m\ (>1)$ having given poles of assigned ranks is

$$q_m + \text{ the sum of the ranks.}$$

We may hence find an expression for $q_m - q_{m-1}$ when $m > 2$. Take a particular theta-Fuchsian function of index 1, η_1, and let θ_m denote the most general theta-Fuchsian function of index m without poles, ϑ_{m-1} the quotient θ_m/η_1, so that ϑ_{m-1} is a theta-Fuchsian function of index $m-1$. The number of arbitrary coefficients in ϑ_{m-1} is the same as in θ_m, that is, q_m. Now the zeroes and poles of η_1 are poles and zeroes of ϑ_{m-1} in general, but the vertices of the polygon again need special consideration. Suppose the vertex c to be a zero of order $h - \dfrac{1}{\lambda}$ or η_1, h being a positive integer. Then, since θ_m has a zero at c of order at least

$$\left\{\frac{m}{\lambda}\right\} + 1 - \frac{m}{\lambda},$$

ϑ_{m-1} has a pole of order

$$h + \frac{m-1}{\lambda} - \left\{\frac{m}{\lambda}\right\} - 1$$

at most; the rank of this pole is

$$h + \left\{\frac{m-1}{\lambda}\right\} - \left\{\frac{m}{\lambda}\right\},$$

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* We have here a reason for not expecting theta-Fuchsian series of index 1 in the first, second, or sixth family to converge absolutely. For, if they did, the function $\phi_1(c, z)$ of § 1 would exist, and would be a theta-Fuchsian function of z of index 1 with a simple pole at c and no other, in contravention of the theorem here proved. The same argument applies also to theta-Kleinian series when any of the regions within which the corresponding functions exist is only of finite extent. Another proof is given by Ritter (*Math. Ann.*, Vol. XLI., p. 58). Lindemann, on the other hand, has tried to prove the contrary (*Münchener Sitzungsberichte*, Vol. XXIX., pp. 423-454).

It is possible that such series may converge conditionally; for the two series

$$\phi_1(c, s_1 z)\frac{ds_1 z}{dz} \quad \text{and} \quad \phi_1(c, z)$$

consist of the same terms, but in different orders, and, if the convergency is conditional, we cannot conclude that their sums are equal.

which exceeds the order of the zero of η_1 by

$$\left\{\frac{m-1}{\lambda}\right\}-\left\{\frac{m}{\lambda}\right\}+\frac{1}{\lambda}.$$

This formula applies to an ordinary point by taking $\lambda=1$, and to a parabolic vertex by taking λ infinite, the result in either case being zero.

If, on the other hand, c is a pole of order $h+\frac{1}{\lambda}$, rank $h+1$, for η_1, then for ϑ_{m-1} it is a zero of order

$$h-\frac{m-1}{\lambda}+\left\{\frac{m}{\lambda}\right\}+1.$$

Now, for any theta-Fuchsian function of index $m-1$, it is, if not a pole, a zero of order at least

$$\left\{\frac{m-1}{\lambda}\right\}+1-\frac{m-1}{\lambda}.$$

The order of the zero being here greater than this by

$$h+\left\{\frac{m}{\lambda}\right\}-\left\{\frac{m-1}{\lambda}\right\},$$

the coefficients in ϑ_{m-1} are restricted by this number of conditions, which falls short of the order of the pole by

$$\left\{\frac{m-1}{\lambda}\right\}-\left\{\frac{m}{\lambda}\right\}+\frac{1}{\lambda}.$$

This is the same expression as before, and again it applies also to an ordinary point and to a parabolic vertex.

The number of zeroes of η_1 exceeds the number of its poles by

$$n-1-\Sigma\frac{1}{\lambda},$$

where $2n$ is the number of sides. The zeroes, generally speaking, are poles of ϑ_{m-1}, and increase the number of its arbitrary coefficients; the poles, on the other hand, are generally zeroes of ϑ_{m-1}, and decrease this number. It follows from the discussion just given that the net effect is to raise the number of arbitrary coefficients from q_{m-1}, which it would be if ϑ_{m-1} had no poles and no assigned zeroes, to

$$q_{m-1}+n-1-\Sigma\frac{1}{\lambda}+\Sigma\left[\left\{\frac{m-1}{\lambda}\right\}-\left\{\frac{m}{\lambda}\right\}+\frac{1}{\lambda}\right]$$

or

$$q_{m-1}+n-1+\Sigma\left\{\frac{m-1}{\lambda}\right\}-\Sigma\left\{\frac{m}{\lambda}\right\}.$$

The summations are taken over the different cycles of vertices. This number would have to be increased if any of the restrictions arising from the zeroes that must be assigned to ϑ_{m-1} were necessarily satisfied; but this cannot be, since it would imply that the function $\vartheta_{m-1}\eta_1$ or θ_m could not possibly have the corresponding poles with arbitrary residues, which we know from the positive construction to be untrue.

It follows then that when $m > 2$

$$q_m - q_{m-1} = n - 1 - \Sigma\left\{\frac{m}{\lambda}\right\} + \Sigma\left\{\frac{m-1}{\lambda}\right\}.$$

This ceases to hold when $m = 2$, since the positive construction is not available for the index 1. The result is, in fact, untrue. The number of arbitrary coefficients in a theta-Fuchsian function of index 1 with poles and ranks assigned is at most

$$q_1 - 1 + \text{the sum of the ranks},$$

since there cannot be just one pole of rank 1. Thus, by the above method, we find that

$$q_2 - q_1 \not> n - 2 - \Sigma\left\{\frac{2}{\lambda}\right\} + \Sigma\left\{\frac{1}{\lambda}\right\},$$

or, say,
$$q_2 - q_1 = n - 2 - \Sigma\left\{\frac{2}{\lambda}\right\} - \nu,$$

where ν is zero or a positive integer, for $\left\{\frac{1}{\lambda}\right\} = 0$ always.

It follows by summation that

$$q_m = q_1 - 1 + (m-1)(n-1) - \Sigma\left\{\frac{m}{\lambda}\right\} - \nu.$$

13. We can now discuss the formation of Fuchsian functions with assigned poles. Let ϑ_2 be a theta-Fuchsian series of index 2, having a pole of order $\frac{2}{\lambda} - \left\{\frac{3}{\lambda}\right\}$ for each cycle, and one other pole arbitrarily chosen. Thus it cannot be identically zero. It will have

$$2(n-1) - \Sigma\left\{\frac{3}{\lambda}\right\} + 1 \text{ zeroes.}$$

By the negative construction two theta-Fuchsian functions $\Theta_{-2}, \Theta'_{-2}$

can be formed of index -2 with q_3+2, that is,

$$q_1+2n-1-\Sigma\left\{\frac{3}{\lambda}\right\}-\nu \text{ arbitrary poles.}$$

The zeroes of Θ_{-2} or Θ'_{-2} or $A\Theta_{-2}+B\Theta'_{-2}$ will be

$$q_1+1-\Sigma\left\{\frac{3}{\lambda}\right\}+\Sigma\frac{2}{\lambda}-\nu$$

in number, each cycle contributing $\frac{2}{\lambda}-\left\{\frac{3}{\lambda}\right\}$; since $A\Theta_{-2}+B\Theta'_{-2}$ has an arbitrary zero as well as these at the vertices, it follows that $q_1-\nu$ is not negative, so that the poles of Θ_{-2}, Θ'_{-2} are at least as numerous as the zeroes of ϑ_2.

Suppose then that the zeroes of ϑ_2 are all included among the poles assigned to Θ_{-2}, Θ'_{-2}, and take Θ'_{-2} to be the reciprocal of ϑ_2, which is allowable. Then the product $\Theta_{-2}\vartheta_2$ will not be constant, but will be a Fuchsian function having $q_1-\nu+1$ arbitrary poles, namely, the arbitrary pole of ϑ_2 and the $q_1-\nu$ poles that are still to be assigned for Θ_{-2}, and no others.

In this result, which depends on a combination of the positive and negative constructions, the poles assigned may be at the vertices as well as anywhere else, the proof needing very little modification for this case. The rank of a pole of a Fuchsian function is the same as its order, the conventions as to the order being derived from those for a theta-Fuchsian function by taking the index of the function to be zero.

Now, as at *Proc. Lond. Math. Soc.*, Vol. XXXI., p. 307, let $p+1$ be the least number of arbitrary poles that can be assigned to a Fuchsian function which is to have no others, and let g be the number of irreducible circuits, so that

$$g=n-k+1,$$

where $2n$ is the number of sides of the generating polygon, k the number of cycles of vertices. Then, from what we have just proved,

$$q_1-\nu \nless p,$$

since a Fuchsian function with $q_1-\nu+1$ arbitrary poles and no others can be formed.*

* It may not be superfluous to point out that, if by some other process it should be possible to construct Fuchsian functions with a lower number of arbitrary poles, so that $p < q_1-\nu$, the vertices of the polygon would still behave like ordinary

14. Other inequalities to be satisfied by the numbers p, q_1, g may be found. Let u be an Abelian integral of the first kind; then $\frac{du}{dz}$ is a theta-Fuchsian function of index 1, and is finite everywhere in the polygon and on its boundary. Conversely, if $\theta_1(z)$ is any such theta-Fuchsian function, $\int \theta_1(z)\, dz$ will be a uniform function of z, finite everywhere within the polygon and on its boundary, and the values of this function at corresponding points in different polygons will only differ by multiples of certain moduli of periodicity, the multiples depending only on the particular polygons in question. Thus q_1, the number of theta-Fuchsian functions of index 1 without poles, is the number of Abelian integrals of the first kind, and the argument used (*Proc. Lond. Math. Soc.*, Vol. xxxi., p. 307) shows that this number does not fall below $g-p$. We have then

$$q_1 \nless g-p,$$

and, from § 13, $$q_1-\nu \nless p,$$

so that $$2q_1 \nless g+\nu,$$

We must now investigate a superior limit for q_1.

15. The moduli of any two Abelian integrals u, v of the first kind are connected by a well known bilinear relation found by evaluating $\int u\, dv$, taken round the perimeter of the generating polygon, as follows. Let ab, cd be two corresponding sides, so that the expression to be evaluated contains the two terms $\int_a^b u\, dv$, $\int_d^c u\, dv$. Denote the values of u, v at a, b, c, d by u_a, v_a, u_b, ... Then

$$u_a-u_c = u_b-u_d \text{ being a modulus for } u,$$

$$v_a-v_c = v_b-v_d \text{ being the corresponding modulus for } v,$$

$$\int_a^b u\, dv+\int_d^c u\, dv = \int_a^b (u_a-u_c)\, dv = (u_a-u_c)(v_b-v_a) = (u_a-u_c)(v_d-v_c).$$

points in respect to this process, regard being had to the conventions. For any vertex may be one of the $q_1-\nu+1$ arbitrary poles of the process in the text, and, if the other poles are ordinary points to which the new process applies, the function with $q_1-\nu+1$ poles can be reduced by subtraction of functions formed by the new process, so as to have only $p+1$ poles, one being the vertex in question and the rest arbitrary. The same argument will apply if the pole at the vertex is to be of a higher order, the different orders being treated successively.

The whole integral, being the sum of a set of expressions typified by this, is therefore a homogeneous linear function of the moduli of u. Also the term just written

$$= (u_b - u_d)\, v_b - (u_a - u_c)\, v_a$$

$$= (u_b - u_d)\, v_d - (u_a - u_c)\, v_c$$

$$= u_b\,(v_d - v_b) - u_a\,(v_c - v_a) + u_b v_b - u_a v_a - u_d v_d + u_c v_c.$$

Now the four last terms in this disappear on summation, and the expression sought is, therefore, linear and homogeneous in the moduli of v; in fact the effect of interchanging u, v is simply to change its sign.

Since the modulus of periodicity for an integral of the first or second kind is zero in the case of an elliptic or parabolic substitution,* the contribution of two sides connected by such a substitution to the integral just considered has been taken as zero. This needs no justification if the substitution is elliptic; but, if it is parabolic, there is a difficulty, as the vertex in which the sides meet is an essential singularity of the functions.

This difficulty may, however, be easily avoided. Take the notation used for such a case in §§ 7, 8, 11. We may replace the integral along fc, cg (Fig. § 11) by that along fg. Now u, v are both unchanged by the substitution t, and they are, therefore, both uniform functions of $\exp(-\zeta)$, or, say, Z. The path in the Z-plane corresponding to fg is a closed curve round the origin, and u, v are uniform, finite, and continuous in the domain of the Z-origin, so that $\int u\,dv$, taken round this closed curve in the Z-plane, that is, along fg, will vanish; which was to be proved. A cycle of vertices lying on the circular boundary may be reduced to a single parabolic vertex, and will, therefore, now cause no difficulty.

Thus the value of $\int u\,dv$ round the perimeter of the polygon is a skew-symmetrical bilinear expression in the moduli of periodicity of u, v. This is true when the Abelian integrals u, v are of the first or second kind; but, when both are of the first kind, we have the result that this bilinear expression must vanish, since the uniform function $u\dfrac{dv}{dz}$ has no pole within the contour of integration.

* *Proc. Lond. Math. Soc.*, Vol. XXXI., pp. 305, 307.

16. It should further be shown that the bilinear relation thus found between the moduli of u, v is not illusory. Now, if there are irreducible circuits on the closed surface into which the polygon is deformed by joining together corresponding sides, there will be at least two pairs of corresponding sides which separate each other; that is, if ab, cd are one pair, and ef, gh the other, the order in which these four sides are met with in going round the perimeter will be, say, ab, ef, dc, hg.* Now cut the polygon in two by a line from b to d, and subtract the part $b \dots ef \dots d$, adding the corresponding part of the polygon adjoining along hg. Thus, a, b, d, c are four consecutive vertices of the new figure; ab still corresponds to cd; suppose kl to be the side corresponding to bd, the polygon being thus $abdc \dots h \dots lk \dots g \dots$. Cut this in two by a line from d to l and take off the piece $dc \dots h \dots l$, adding on the corresponding part of the polygon adjoining along ab; let m be the vertex of this polygon which corresponds to l; then, in the new polygon, m, b, d, l, k are five consecutive vertices, and mb, ld correspond, as also do bd, kl.

Let S, T be the operations that turn mb into ld, bd into kl, respectively, and let us examine the way in which S, T enter into the relations connecting the fundamental substitutions. The five points m, l, d, b, k belong to the same cycle of vertices, and the sequence $T S^{-1} T^{-1} S$ will occur in the relation arising from this cycle (see *Acta Math.*, Vol. I., pp. 45–7); also the operations S, T will not occur in any other of the relations. Thus the restrictions upon moduli of periodicity in general (*Proc. Lond. Math. Soc.*, Vol. XXXI., p. 305, note) are obeyed if we take arbitrarily the moduli corresponding to S, T and make all the other moduli zero. This will secure the isomorphism referred to in the passage cited last.

Now, if this were the case with the functions u, v,† the value of $\int u\,dv$ round the rest of the perimeter would be zero, and from mb and dl we should have the contribution

$$(u_d - u_b)(v_l - v_d),$$

* It is easily seen that if no two pairs of conjugate sides separate each other, there must be sides adjacent to their conjugates. These pairs being fastened together, the resulting surface must again have sides adjacent to their conjugates. Fastening these together, and continuing the process, we arrive in the end at a simply connected closed surface.

† They would not of course be integrals of the first kind; but for the present purpose that does not matter, the object being to show that a relation which must be satisfied when they are of the first kind is not generally satisfied when they are not of the first kind.

from bd and lk, $(u_l - u_d)(v_b - v_d)$.
The sum of these does not vanish identically, and hence the condition cannot be illusory.

17. If there are other pairs of corresponding sides that separate each other, this reduction may be carried further so as to bring the polygon to a canonical form. Suppose $a'b'$, $c'd'$ to be a pair of corresponding sides, and $e'f'$, $g'h'$ another, the order of the vertices being

$$a'b' \dots e'f' \dots mbdlk \dots d'c' \dots h'g' \dots .$$

Cut off the part $k \dots d'c'$ and add on the corresponding part of the polygon adjoining along $a'b'$; let k' be the vertex of this polygon corresponding to k; then the order of letters in the new perimeter is

$$a'k' \dots b' \dots e'f' \dots mbdlkc' \dots h'g' \dots ,$$

and $a'k'$, $c'k$ are corresponding sides.

Now let the process of § 16 be used, the sides $a'k'$, $e'f'$, kc', $h'g'$ taking the place of ab, ef, dc, hg. The result will be to bring together two sets of four sides, in each of which the first and third correspond, as also the second and fourth.

If two more pairs of sides separate each other, the same process may be applied again, and so on, until all such pairs are exhausted. Then the perimeter will consist of a series of sets of four sides, in each of which the first corresponds to the third, the second to the fourth, followed, possibly, by a series in which no two pairs of corresponding sides separate each other. If this latter series exists, it must include some pair of corresponding sides adjacent to each other, their common end forming a cycle by itself. Let $abcd \dots$ denote the part of the perimeter now being considered, ef, gf the first pair of adjacent corresponding sides. Cut off the part $a \dots ef$ and add on a corresponding piece of the polygon adjoining along fg; then the number of sides is not increased, but the first two have been made to correspond. In the same way the next two may be made to correspond, and so on till every side in this part of the perimeter is adjacent to the corresponding side. The order of the vertices will now be, with changed notation,

$$a_1 b_1 c_1 d_1 a_2 b_2 c_2 d_2 \dots , \quad a_\kappa b_\kappa c_\kappa d_\kappa e_1 f_1 e_2 f_2 \dots , \quad e_\rho f_\rho ,$$

corresponding sides being

$$a_i b_i,\ c_i d_i;\ b_i c_i,\ d_i a_{i+1} \quad (i = 1, 2, \dots, \kappa - 1);$$

$$a_\kappa b_\kappa,\ c_\kappa d_\kappa;\ b_\kappa c_\kappa,\ d_\kappa e_1;\ e_i f_i,\ f_i e_{i+1} \ (i = 1, 2, \dots, \rho - 1);\ e_\rho f_\rho,\ f_\rho a_1.$$

(Compare Klein, *Math. Annalen*, Vol. XXI., p. 184.)

18. Let S_i, T_i, U_i denote the substitution by which $a_i b_i$, $b_i c_i$, $e_i f_i$ are transformed into the sides corresponding to them respectively. Then the restrictions on moduli of periodicity for Abelian integrals of the first two kinds are simply that those corresponding to

$$U_1,\ U_2,\ \ldots,\ U_p$$

shall vanish; the others are arbitrary.

Let A_i, B_i be the moduli corresponding to S_i for the functions u, v respectively, and A_i', B_i' those corresponding to T_i. Then the bilinear relation to be satisfied if u, v are everywhere finite is

$$\sum_{i=1}^{i=\kappa} (A_i B_i' - A_i' B_i) = 0;$$

this is the well known form.

19. Thus, if any number j of Abelian integrals of the first kind are known, the moduli of periodicity for each of these, as also for any others, must satisfy j linear equations, which are all independent; if they were not, the systems of moduli for the j known functions would not be linearly independent, and, therefore, the j functions themselves would not be. This is made very clear by the form in which the bilinear relation was written at the end of § 18.

Thus the q_1 sets of moduli belonging to the q_1 Abelian integrals of the first kind are common solutions of q_1 linear equations. Since the number of moduli in each set is g, it follows that

$$g - q_1 \nless q_1$$

or

$$2q_1 \ngtr g.$$

But now

$$2q_1 \nless g + \nu,$$

where ν is zero or positive. It follows that $\nu = 0$, and

$$q_1 = \tfrac{1}{2} g = \tfrac{1}{2}(n - k + 1);$$

also $\quad p \ngtr q_1 - \nu \quad$ and $\quad \nless g - q_1, \quad$ so that $\quad p = q_1$;

these are the desired results.

We have further, when $m > 1$,

$$q_m = \tfrac{1}{2}(n - k - 1) + (m-1)(n-1) - \Sigma \left\{ \frac{m}{\lambda} \right\}.$$

(Compare *Acta Math.*, Vol. I., p. 266.)

20. Lastly, it must be possible to express all theta-Fuchsian functions of index >1 and without poles as theta-Fuchsian series (*Acta Math.*, Vol. I., pp. 244-246, 285). For, if of the q_m such functions of index m not more than q_m-1 can be expressed as series, the coefficients $C_0, C_1, \ldots$ of § 1 must be linear combinations of these q_m-1, and thus by the negative construction a theta-Fuchsian function of index $1-m$ can be formed with q_m assigned poles only. More generally $r+1$ can be formed with q_m+r given poles and no others, and two of these, say θ_{1-m} and θ'_{1-m}, will have $r-1$ assigned zeroes.

Let these $r-1$ zeroes be the poles of any particular function θ_{m-1} of index $m-1$, and assign the zeroes of θ_{m-1} among the q_m+r poles of $\theta_{1-m}, \theta'_{1-m}$. The sum of the orders of the poles thus assigned is

$$r-1+(m-1)\left(n-1-\Sigma\frac{1}{\lambda}\right),$$

and the sum of their ranks is easily found to be

$$r-1+(m-1)(n-1)-\Sigma\left\{\frac{m}{\lambda}\right\}.$$

Thus
$$q_m+1-(m-1)(n-1)+\Sigma\left\{\frac{m}{\lambda}\right\} \text{ or } q_1$$

of the poles of $\theta_{1-m}, \theta'_{1-m}$ are still at our disposal. Now we may take the reciprocal of θ_{m-1} to be θ'_{1-m}, and thus the product $\theta_{m-1}\theta_{1-m}$ is not a constant, but is a Fuchsian function having q_1 *arbitrary* poles and no more. This is impossible, since $q_1<p+1$.

Thus every theta-Fuchsian function without poles, of index >1, can be expressed as a theta-Fuchsian series. If a theta-Fuchsian function of index >1 has poles, a theta-Fuchsian series can be formed by the positive construction with the same poles and residues and the same index; the difference between this series and the function will be a theta-Fuchsian function without poles. Hence every theta-Fuchsian function of index >1, with or without poles, can be expressed as a series of Poincaré's form.

Also the product of a theta-Fuchsian and a Fuchsian function, or of two theta-Fuchsian functions, is a theta-Fuchsian function. Thus, any Fuchsian function or theta-Fuchsian function of index 1 can be expressed as the quotient of two theta-Fuchsian series.

[I am indebted to the referees for pointing out some flaws in this article as originally written. The effect of their suggestions has been a considerable increase in its length.]